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ON MINIMAL SEMIRIGID RELATIONS

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By a graph we mean a pair (X, R), where $X \neq \emptyset$ is a finite set and $R \subseteq X \times X$ is a binary relation on X. The outdegree of $x \in X$ is the number $od(x) = |\{y : y \neq x$ and $(x, y) \in R\}|$, the indegree of x is the number $id(x) = |\{y : y \neq x$ and $(y, x) \in R\}|$. If (X, R) and (Y, S) are graphs and f is a mapping of X into Y, we say that f is a compatible mapping of (X, R) into (Y, S) if $(f(x), f(y)) \in S$ whenever $(x, y) \in R$. If moreover f is 1-1 and also f^{-1} is compatible we say that f is an isomorphism. (X, R) and (Y, S) are called isomorphic if there exists an isomorphism f of (X, R) onto (Y, S); notation $(X, R) \cong (Y, S)$. A graph (X, C) is a cycle of length m (m is a positive integer) if $(X, C) \cong (\{1, 2, \dots, m\}, \{(i, i+1); i = 1, 2, \dots, m-1\} \cup \{(m, 1)\})$, the cycle of length 1 is also called trivial cycle. A reflexive cycle of length m is any graph isomorphic to $(\{1, 2, \dots, m\}, \{(i, i+1); i = 1, 2, \dots, m-1\} \cup \{(i, i); i = 1, 2, \dots, m\} \cup \{(m, 1)\})$. The set of all compatible mappings of (X, R) into itself forms a monoid (semigroup with a unit element) under composition, which

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is denoted $M(X, R)$. In the case $M(X, R)$ consists of the identity and all the constant mappings, the graph (X, R) is called a semirigid graph and R a semirigid relation on the set X. In [1], there are constructed semirigid relations on any finite set $X \neq \emptyset$ such that $|X| \neq 3, 4$ and proved there are no semirigid relations on sets of these powers. (As since the publishing of the paper [1] the terminology has stabilized so that the expression "rigid graph" is now exclusively used for a rather different notion (see [2],[3]), we use the term "semirigid graph" instead of "rigid graph" as used in [1].) The semirigid graphs given in [1] possess the following maximal property which is an obvious consequence of Lemma 2 in [1]: If S is an arbitrary semirigid relation on a finite set X, $|X| \neq 0, 3, 4$, then $|S| \leqq |X| + \binom{|X|}{2}$ and this bound is the best in an obvious sense. On the contrary, the aim of this note (besides of giving some further properties of semirigid graphs) is to construct for every set X, $|X| \neq 0, 3, 4$ a semirigid relation R on X such that for any semirigid relation S on X the inequality $|R| \leqq |S|$ holds. (In [3] similar questions were solved for rigid graphs.) We shall start with several lemmas.

<u>Lemma 1.</u> Any semirigid graph is connected.

<u>Proof</u>. Let (X, R) be a semirigid graph, assume that (X, R) is not connected. Denote $X_1, \dots, X_m$ its components, $m > 1$. Define a mapping $f: X \rightarrow X$ setting $f(x) = x$ for $x \in X_2 \cup X_3 \cup \dots \cup X_m$, $f(x) = a$

for $x \in X_1$, where a is an arbitrarily chosen element of X_2. From Lemma 1 of [1] it follows that f is compatible. Since f is neither identity nor a constant mapping we have got a contradiction.

Lemma 2. Let (X, R) be a semirigid graph, $|X| > 3$. Then there are at least two different elements x, y in X for which $od(x) \geqq 2$, $od(y) \geqq 2$ hold.

Proof: By Lemmas 3,4 of [1] there is an $x \in X$ for which $od(x) \geqq 2$. Assume there is no $y \in X$ such that $y \neq x$ and $od(y) \geqq 2$. From Lemma 3 of [1] it follows that $od(y) = 1$ for $y \in X - \{x\}$ and $id(y) \geqq 1$ for $y \in X$. Let $x_1, x_2, \dots, x_k$ be all the different points of X for which $(x, x_i) \in R$, $x \neq x_i$, $i = 1, 2, \dots, k$. Clearly $k = od(x) \geqq 2$. Define for $i = 1, 2, \dots, k$ the mappings $Q_i : X \longrightarrow X$ in the following way: $Q_i(x) = x_i$, $Q_i(y) = z$ for $y \neq x$, where z is the unique point of X for which $y \neq z$, $(y, z) \in R$ holds. By Lemma 1 for every $y \in X$ there is an i among $1, 2, \dots, k$ and a positive integer n such that $y = Q_i^n(x)$.

In the case $Q_i^m(x) \neq x$ for every integer $m > 1$, and for every $i = 1, 2, \dots, k$, put $f(x) = x$ and $f(y) = x_1$ for $y \neq x$. Obviously $f \in M(X, R)$.

In the case there is an integer i such that there exists an integer $p > 1$ satisfying the condition $Q_i^p(x) = x$, just the two following situations can occur[x]: (1) There is an i such that $m(i) = |X|$; (2) $m(i) < |X|$

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x) We denote by $m(i)$ the smallest p with $Q_i^p(x) = x$.

for every i , for which $m(i)$ is meaningful.

In the case (1) the mapping $g : X \longrightarrow X$ defined by $g(Q_i^s(x)) = x_j$ ($j \neq i$ is a fixed number from $1, 2, \dots, k$) for $s = 1, 2, 3, \dots, r$ (r is the smallest integer t for which $Q_i^t(x) = x_j$ holds) and $g(y) = y$ for $y \in X - \{Q_i^s(x) : s = 1, 2, 3, \dots, r\}$ is obviously a compatible mapping.

In the case (2) the two following situations will be distinguished: (i) x is a cut point of (X, R) and (ii) x is not a cut point of (X, R). (i) The mapping $h : X \rightarrow X$ carrying one of the components of the graph $(X - \{x\}, ([X - \{x\}] \times [X - \{x\}]) \cap R)$ into x and being identical for all other points of X is evidently compatible. (ii) In this case there exist i and $j, i \neq j$, $i, j \in \{1, 2, \dots, k\}$ and two integers $p, q, 1 < p \leqq q$ such that the following conditions are satisfied: (a) $Q_i^p(x) = Q_j^q(x)$; (b) $id(Q_i^r(x)) = 1$ for $1 < r < p$, and $id(Q_j^r(x)) = 1$ for $1 < r < q$. Define $e(Q_j^r(x)) = Q_i^r(x)$ for $r < p$, $e(Q_j^r(x)) = Q_j^q(x)$ for $r = p, p+1, \dots, q$ and $e(y) = y$ for $y \in X - \{Q_j^r(x) : r = 1, 2, \dots, q\}$. Evidently $e \in M(X, R)$.

As all the mappings f, g, h, e are also nonconstant and nonidentical, Lemma 2 is proved.

Since semirigid relations on sets containing more than one element are reflexive (see [1], Lemma 1) we get the following corollary:

<u>Corollary 1</u>. Let n be a positive integer, $n \neq 3, 4$. Denote $\varphi(n) = \min \{|R| ; R$ is a semirigid relation on a set of the cardinality $n\}$. Then $n > 3$ implies $\varphi(n) \geqq 2n + 2$.

<u>Lemma 3</u>. Let (X, R) be a semirigid graph, $|X| > 3$. Then (X, R) contains a cycle of length $n \geqq 3$.

<u>Proof</u>. According to Lemma 2 of [1] there is no cycle of length 2 contained in (X, R). Suppose there is no non-trivial cycle contained in (X, R). Let us choose $u_1 \in X$. If for an integer $p \geqq 1$ the point u_p is constructed, then there is at least one $x \in X - \{u_p\}$ such that $(u_p, x) \in R$. Denote by u_{p+1} one of these x's. Since no non-trivial cycle is contained in (X, R), the sequence $\{u_p\}_{p=1}^{\infty}$ of elements of the finite set X is univalent, which is a contradiction.

The easy proof of the following Lemma 4 which is useful for proving semirigidity of a relation, will be omitted.

<u>Lemma 4</u>. Let (C, R) be a reflexive cycle, f a compatible mapping of (C, R) onto (X, S). Then there exists $T \subseteq S$ such that (X, T) is a cycle.

<u>Theorem 1</u>. Let φ be the function defined in Corollary 1. Then $\varphi(1) = 0$, $\varphi(2) = 3$, $\varphi(5) = 13$ and $\varphi(n) = 2n + 2$ for $n \geqq 6$.

<u>Proof</u>. $\varphi(1) = 0$ since $(\{\emptyset\}, \emptyset)$ is semirigid. $\varphi(2) = 3$ follows immediately from Theorem 1 of [1] and from its proof. From Corollary 1 we have $\varphi(5) \geqq 12$. However, by constructing all relations of the power 12 on the set $\{1, 2, 3, 4, 5\}$ satisfying all

the well-known necessary conditions for semirigidity, we find out that none of them is semirigid (these considerations are omitted for technical reasons). Therefore $\varphi(5) \geqq 13$. As the graph $(\{1,2,3,4,5\}, \{(i,i);\ i=1,\dots,5\} \cup \{(i,i+1); i=1,\dots,4\} \cup \{(5,1),(3,1),(5,3),(4,2)\})$ is semirigid, we have $\varphi(5) = 13$. Let m be an odd number, $m \geqq 6$, i.e., $m = 2k+1$, $k \geqq 3$. Put $X = \{1,2,\dots,m\}$, $R = \{(i,i);\ i = 1,2,\dots,m\} \cup \{(i,i+1);\ i = 1,2,\dots,m-1\} \cup \{(m,1),(1,k+1),(k+3,2)\}$.

Let m be even, $m \geqq 6$, $m = 2p$, $p \geqq 3$. Put $Y = \{1,2,\dots,m\}$, $S = \{(i,i);\ i = 1,2,\dots,m\} \cup \{(i,i+1);\ i = 1,2,\dots,m-1\} \cup \{(m,1),(1,p+1),(p,m)\}$.

Clearly $|R| = 2m+2$, $|S| = 2m+2$, since both the graphs (X,R) and (Y,S) are semirigid, we have $\varphi(m) = 2m+2$ for $m \geqq 6$. The proofs of semirigidity of all relations given here are easy (they can be e.g. based upon Lemma 4) and therefore omitted.

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R e f e r e n c e s

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