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ON DESCRIPTIVE CLASSIFICATION OF SET-FUNCTORS II.

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The present paper is a continuation of [1]. In [1] the preservation of limits of various types of diagrams by set-functors is studied. Here, the dual questions, concerning coequalizers, push-out-diagrams, colimits up to m are investigated. The paper has three parts, numerated IX. to XI. In IX, the coequalizer-preserving set-functors are characterized. In X, the preservation of push-out-diagrams and colimits up to m is considered. We prove, for example, that every set-functor which preserves colimits of finite diagrams, preserves also colimits of countable diagrams. In XI, the set-functors preserving some types of limits and some types of colimits are investigated. For example, the functors that preserve pull-back-push-out diagrams are characterized.

The notation, all the conventions and some facts from [1] are used.

IX.

IX.1. Definition. Let H be a functor, $f, g : X \longrightarrow Y$ mappings, $\psi_1, \psi_2 \in H(Y)$. An n-tuple

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$\langle\langle s_1, x_1, t_1\rangle, \dots, \langle s_m, x_m, t_m\rangle\rangle$ will be called an (f, g) -chain from y_1 to y_2 in H iff

1) $\{s_i, t_i\} = \{f, g\}$ for $i = 1, \dots, m$;

2) $x_1, \dots, x_m \in H(X)$;

3) $[H(s_1)](x_1) = y_1$, $[H(t_m)](x_m) = y_2$;

4) $[H(t_i)](x_i) = [H(s_{i+1})](x_{i+1})$ for $i = 1, \dots$
$\dots, m-1$.

IX.2. Proposition. If $\mathcal{F}$ is a $\aleph_1$-complete ultrafilter on a set M , then $Q_{M,\mathcal{F}}$ preserves coequalizers.

Proof. Put $H = Q_{M,\mathcal{F}}$. Let $f, g: X \longrightarrow Y$ be mappings, $h = coeq\,(f, g)$. We prove that $H(h) = coeq\,(H(f), H(g))$. Let $\varkappa_1^+, \varkappa_2^+ \in H(Y)$, $[H(h)](\varkappa_1^+) = [H(h)](\varkappa_2^+)$. Then there exists an $F_0 \in \mathcal{F}$ such that $h \circ \varkappa_1(x) = h \circ \varkappa_2(x)$ for every $x \in F_0$. Consequently one can choose an (f, g)-chain $p^x = \langle\langle s_1^x, x_1^x, t_1^x\rangle, \dots, \langle s_{n_x}^x, x_{n_x}^x, t_{n_x}^x\rangle\rangle$ from $\varkappa_1(x)$ to $\varkappa_2(x)$ in I . Define an equivalence $\sim$ on F_0 by

$$x \sim x' \Longleftrightarrow \langle\langle s_1^x, t_1^x\rangle, \dots, \langle s_{n_x}^x, t_{n_x}^x\rangle\rangle = \\ = \langle\langle s_1^{x'}, t_1^{x'}\rangle, \dots, \langle s_{n_{x'}}^{x'}, t_{n_{x'}}^{x'}\rangle\rangle .$$

The decomposition of F_0 by means of $\sim$ is countable; let A be its element which is in $\mathcal{F}$. If $\alpha_i : M \longrightarrow X$ are mappings such that $\alpha_i(x) = x_i^x$ for all $x \in A$, then obviously $\langle\langle s_1^x, \alpha_1^+, t_1^x\rangle, \dots, \langle s_{n_x}^x, \alpha_{n_x}^+, t_{n_x}^x\rangle\rangle$

is an (f, g)-chain from r_1^+ to r_2^+ in H for any $z \in A$.

IX.3. Proposition. A factorfunctor of a coequalizer-preserving functor preserves coequalizers.

Proof. Let $\nu : H \longrightarrow G$ be an epitransformation. Let H preserve coequalizers, $f, g : X \longrightarrow Y$ be mappings, $h = coeq(f, g)$, $h : Y \longrightarrow Z$. Let $y, y' \in G(Y)$, $[G(h)](y) = [G(h)](y')$. We prove that there exists an (f, g)-chain from y to y' in G. Choose $x, x' \in H(Y)$ with $\nu_Y(x) = y$, $\nu_Y(x') = y'$. Let $[H(h)](x) = b$, $[H(h)](x') = b'$. Then $\nu_Z(b) = a = \nu_Z(b')$. Choose $\ell : Z \longrightarrow Y$ with $h \circ \ell = id_Z$ and put $c = [H(\ell)](b)$, $c' = [H(\ell)](b')$. Since $[H(h)](x) = b = [H(h)](c)$, there is an (f, g)-chain $\langle\langle s_1, x_1, t_1\rangle, \dots, \langle s_n, x_n, t_n\rangle\rangle$ from x to c in H. Analogously, there exists an (f, g)-chain $\langle\langle s'_1, x'_1, t'_1\rangle, \dots, \langle s'_{n'}, x'_{n'}, t'_{n'}\rangle\rangle$ from c' to x' in H. Since $\nu_Y(c) = [G(\ell)](a) = \nu_Y(c')$, $\langle\langle s_1, \nu_X(x_1), t_1\rangle, \dots, \langle s_n, \nu_X(x_n), t_n\rangle$, $\langle s'_1, \nu_X(x'_1), t'_1\rangle, \dots, \langle s'_{n'}, \nu_X(x'_{n'}), t'_{n'}\rangle\rangle$ is an (f, g)-chain from y to y' in G.

IX.4. Definition. Let m be an infinite cardinal. We recall that a functor H preserves unions up to m iff $H(X) = \bigcup_{\alpha \in A} H(X_\alpha)_X$ whenever $X = \bigcup_{\alpha \in A} X_\alpha$ and $card\ A < m$.

IX.5. <u>Lemma</u>. Let $\mathfrak{m}$ be an infinite cardinal, let H be a functor such that

if $\{X_\alpha ; \alpha \in A\}$ is a disjoint collection such that $card\ A < \mathfrak{m}$ and $card\ X_\alpha = card\ X_{\alpha'}$, for every α, α', then $H(X) = \bigcup_{\alpha \in A} H(X_\alpha)_X$ where $X = \bigcup_{\alpha \in A} X_\alpha$.

Then H preserves unions up to $\mathfrak{m}$.

<u>Proof</u>. 1) Let $\{Y_\alpha ; \alpha \in A\}$ be a disjoint collection of non-empty sets, $card\ A < \mathfrak{m}$. Choose a disjoint collection $\{X_\alpha ; \alpha \in A\}$ such that $Y_\alpha \subset X_\alpha$ and $card\ X_\alpha = \sup_{\beta \in A} card\ Y_\beta$ for all $\alpha \in A$. Put $X = \bigcup_{\alpha \in A} X_\alpha$, $Y = \bigcup_{\alpha \in A} Y_\alpha$. Then $H(X) = \bigcup_{\alpha \in A} H(X_\alpha)_X$. Since $Y_\alpha \neq \emptyset$, $Y_\alpha = X_\alpha \cap Y$, we have $H(Y_\alpha)_X = H(Y)_X \cap H(X_\alpha)_X$. Consequently

$$H(Y)_X = H(Y)_X \cap H(X) = H(Y)_X \cap$$
$$\cap \left(\bigcup_{\alpha \in A} H(X_\alpha)_X\right) = \bigcup_{\alpha \in A} H(Y_\alpha)_X .$$

Thus, $H(Y) = \bigcup_{\alpha \in A} H(Y_\alpha)_Y$.

2) Let $\{Y_\alpha ; \alpha \in A\}$ be a disjoint collection, $card\ A < \mathfrak{m}$, $Y = \bigcup_{\alpha \in A} Y_\alpha$. If all Y_α are empty, then $Y = \emptyset$ and then $H(Y) = \bigcup_{\alpha \in A} H(Y_\alpha)_Y$. If $B = \{\alpha \in A ; Y_\alpha \neq \emptyset\} \neq \emptyset$, then $Y = \bigcup_{\alpha \in B} Y_\alpha$ and $H(Y) = \bigcup_{\alpha \in B} H(Y_\alpha)_Y = \bigcup_{\alpha \in A} H(Y_\alpha)_Y$.

3) If $\{Y_\alpha ; \alpha \in A\}$ is an arbitrary collection with $card\ A < \mathfrak{m}$, take a well-ordering $<$ of A and put $Z_\alpha = Y_\alpha - \bigcup_{\beta < \alpha} Y_\beta$. Then

$Y = \bigcup_{\alpha \in A} Y_\alpha = \bigcup_{\alpha \in A} Z_\alpha$ and consequently $H(Y) = \bigcup_{\alpha \in A} H(Z_\alpha)_Y$. Since $H(Z_\alpha)_Y \subset$ $\subset H(Y_\alpha)_Y \subset H(Y)$, we have $H(Y) = \bigcup_{\alpha \in A} H(Y_\alpha)_Y$.

IX.6. Lemma. Let m be an infinite cardinal. A functor H preserves unions up to m iff for every set X and every $x \in H(X)$ either the pair $\langle x, X \rangle$ is distinguished or $H^{x,X}$ is an m -complete ultrafilter.

Proof. 1) Let H preserve unions up to m , let $x \in H(X)$, $\langle x, X \rangle$ be not distinguished. Let $\{X_\alpha ; \alpha \in A\}$ be a decomposition of X , $card\, A <$ $< m$. Since $x \in H(X_{\alpha_0})_X$ for some $\alpha_0 \in A$, we have $X_{\alpha_0} \in H^{x,X}$.

2) Let $H^{x,X}$ be an m -complete ultrafilter whenever $\langle x, X \rangle$ is not distinguished. Let $X = \bigcup_{\alpha \in A} X_\alpha$, $card\, A < m$. If $X = \emptyset$, then necessarily $H(X) =$ $= \bigcup_{\alpha \in A} H(X_\alpha)_X$. If $X \neq \emptyset$, then there exists an $\alpha_0 \in A$ such that $X_{\alpha_0} \neq \emptyset$. Then $x \in H(X_{\alpha_0})_X$ whenever $\langle x, X \rangle$ is distinguished. If $\langle x, X \rangle$ is not distinguished then $X_{\alpha_1} \in H^{x,X}$ for some $\alpha_1 \in A$. Thus, $H(X) = \bigcup_{\alpha \in A} H(X_\alpha)_X$.

IX.7. Lemma. Let H preserve coequalizers of all pairs of bijections. Then it preserves countable unions.

Proof. Let Z be the set of all integers. It is sufficient to prove (see IX.5) that H preserves unions of all disjoint collections $\{X_n ; n \in Z\}$, where all X_n have the same cardinality. Put $X = \bigcup_{n \in Z} X_n$ and denote by $i_n : X_n \longrightarrow X$ the inclusion. For every $n \in Z$ choose a bijection $\varphi_n : X_n \longrightarrow X_{n+1}$. Let $g : X \longrightarrow X$ be the mapping with $g \circ i_n = i_{n+1} \circ \varphi_n$ for all $n \in Z$. Put $T = \bigcup_{n \in Z} H(X_n)_X$. Then

$(*)$ $[H(g)](T) \subset T$, $[H(g)]^{-1} \subset T$.

Let $\alpha = coeq(id_X, g)$. We may suppose $\alpha : X \to \to X_0$, $\alpha \circ i_0 = id_{X_0}$. Let $x \in H(X)$. Put $c = [H(i_0 \circ \alpha)](x)$, then $[H(\alpha)](x) = [H(\alpha)](c)$.

Consequently there exists an (id_X, g)-chain from x to c in H. Then necessarily either $c = [H(g)]^k(x)$ for some natural k, or $x = [H(g)]^\ell(c)$ for some natural ℓ. Since $c \in T$, $(*)$ implies $x \in T$.

IX.8. Theorem. The following properties of a functor H are equivalent:

(i) H preserves coequalizers;

(ii) H preserves coequalizers of pairs of bijections;

(iii) H preserves countable unions;

(iv) for every set X and every $x \in H(X)$ either the pair $\langle x, X \rangle$ is distinguished or $H^{x,X}$ is a $\aleph_1$-complete ultrafilter.

Proof. (i) $\Longrightarrow$ (ii) is trivial, for (ii) $\Longrightarrow$ (iii) see IX.7, for (iii) $\Longrightarrow$ (iv) see IX.6.

(iv) $\Longrightarrow$ (i): Let H do not preserve coequalizers. Then there are $f, g : X \longrightarrow Y$ and $a, b \in H(Y)$ such that $[H(\alpha)](a) = [H(\alpha)](b)$, where

$\alpha = coeq(f, g)$, while there is no (f, g)-chain from a to b in H . Put $G = H_{\langle a, Y \rangle} \cup H_{\langle b, Y \rangle}$. Then G does not preserve coequalizers. Put $K_a = H_{\langle a, Y \rangle}$ if $\langle a, Y \rangle$ is distinguished, $K_a = Q_{Y, H^{a,Y}}$ otherwise. Put $K_b = H_{\langle b, Y \rangle}$ if $\langle b, Y \rangle$ is distinguished, $K_b = Q_{Y, H^{b,Y}}$ otherwise. Since G is a factorfunctor of $K_a \vee K_b$, either K_a or K_b does not preserve coequalizers (see IX.3). If K_a does not preserve coequalizers then $\langle a, Y \rangle$ is not distinguished and $H^{a,Y}$ is not an $\aleph_1$ -complete ultrafilter (see IX.2)

<u>Corollary</u>. Every subfunctor of a coequalizer-preserving functor preserves coequalizers.

X.

X.1. <u>Convention</u>. Denote by $\mathbf{P}$ the category of all pointed sets, i.e. $\mathbf{P}^{\sigma}$ is the class of all $\langle A, a \rangle$, where A is a set and $a \in A$; $f : \langle A, a \rangle \longrightarrow \langle B, b \rangle$ is a morphism of $\mathbf{P}$ iff it is a mapping $f : A \rightarrow B$ with $f(a) = b$. Denote by $\Box : \mathbf{P} \longrightarrow \mathbf{S}$ the obvious forgetful-functor.

X.2. <u>Lemma</u>. In the category $\mathbf{P}$ every diagram has a colimit and the functor $\Box : \mathbf{P} \longrightarrow \mathbf{S}$ preserves coequalizers and push-put-diagrams.

<u>Proof</u> is trivial.

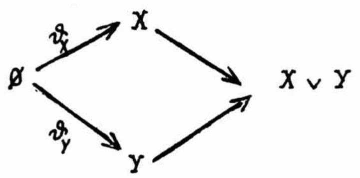
X.3. <u>Lemma</u>. Let $H : \mathbf{S} \longrightarrow \mathbf{S}$ be a connected functor with $card\, H(\emptyset) = 1$. Then there exists exactly one $\bar{H} : \mathbf{S} \longrightarrow \mathbf{P}$ with $\Box \circ \bar{H} = H$

Proof: It is evident.

Convention. If $H : \mathbf{S} \longrightarrow \mathbf{S}$ is a connected functor with $card\ H(\emptyset) = 1$, then $\bar{H}$ always denotes the functor from the lemma.

X.4. *Lemma*. Let $H : \mathbf{S} \longrightarrow \mathbf{S}$ be a connected functor with $card\ H(\emptyset) = 1$. If H preserves coequalizers or push-out-diagrams, then $\bar{H}$ preserves colimits of finite diagrams.

Proof. If H preserves coequalizers or push-out-diagrams, then $\bar{H}$ also preserves them. Consequently it is sufficient to prove that $\bar{H}$ preserves finite sums. If H preserves coequalizers, this follows from IX.7. If H preserves push-out-diagrams, use the diagram



X.5. *Proposition*. The following properties of a functor H are equivalent:

(i) H preserves push-out-diagrams;

(ii) H is regular and preserves coequalizers.

Proof. We may suppose H connected.

(i) $\Longrightarrow$ (ii): If H preserves push-out-diagrams, it is regular, clearly. If $H(\emptyset) = \emptyset$, then H preserves finite sums, consequently it preserves finite colimits, in particular coequalizers. If $H(\emptyset) \neq \emptyset$, consider a functor G with $G^* = H^*$, $card\ G(\emptyset) = 1$ and use X.4, X.2 for $\bar{G}$. Then G preserves coequalizers and so does H.

(ii) $\Longrightarrow$ (i): If $H(\emptyset) = \emptyset$, then H preserves finite sums (see IX.8 and II.4 in [1]), consequently it preserves finite colimits, in particular push-out-diagrams. If $H(\emptyset) \neq \emptyset$, consider a functor G with $G^* = H^*$, $card\, G(\emptyset) = 1$ and use X.4, X.2 again.

<u>Corollary</u>. Every regular subfunctor of a push-out-diagram-preserving functor preserves push-out-diagrams.

X.6. <u>Theorem</u>. The following properties of a functor H are equivalent:

(i) H preserves colimits of countable diagrams;

(ii) H preserves colimits of finite diagrams;

(iii) H is separating and preserves push-out-diagrams;

(iv) H is separating and preserves coequalizers of pairs of bijections;

(v) H is separating and for every $x \in H(X)$ the filter $H^{x,X}$ is an $\aleph_1$-complete ultrafilter;

(vi) H preserves countable sums.

<u>Proof</u>. The implications (i) $\Longrightarrow$ (ii), (ii) $\Longrightarrow$ (iii) are trivial. (iii) $\Longleftrightarrow$ (iv) follows from X.5 and IX.8, (iv) $\Longleftrightarrow$ (v) follows from IX.8, (v) $\Longrightarrow$ (vi) is easy. Clearly, ((vi) and (iii)) $\Longrightarrow$ (i).

X.7. <u>Theorem</u>. Let $m > \aleph_0$. The following properties of a functor H are equivalent:

(i) H preserves colimits of diagrams up to m;

(ii) H preserves sums up to m;

(iii) H is separating and for every $x \in H(X)$ the filter $H^{x,X}$ is an m-complete ultrafilter.

<u>Proof</u> is analogous to the previous one.

Corollary. Every subfunctor of a functor which preserves colimits of diagrams up to m also preserves colimits up to m .

X.8. Theorem. Every one from the following assertions is equivalent to the non-existence of measurable cardinal:

(1) The functors preserving colimits of finite diagrams are precisely $\simeq I \times C_M$.

(2) The functors preserving push-put-diagrams are precisely $\simeq (I \times C_M) \vee C_{T, t L}$, where $t : T \to L$ is a surjection.

(3) The functors preserving coequalizers are precisely $\simeq (I \times C_M) \vee C_{T, t, L}$.

Proof follows easily from X.6, X.5 and IX.8.

XI.

XI.1. Theorem. Every one of the following assertions is equivalent to the non-existence of a measurable cardinal:

(1) If a functor H preserves finite sums and countable products then either $H = C_0$ or $H \simeq I$.

(2) If a functor H preserves countable sums and finite products then either $H = C_0$ or $H \simeq I$.

(3) If a functor H preserves limits of finite diagrams and colimits of finite diagrams then either $H = C_0$ or $H \simeq I$.

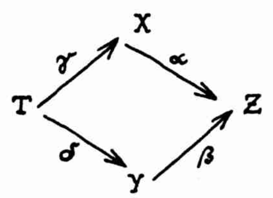
Proof follows easily from X.8.

XI.2. Proposition. If a functor preserves finite sums then it preserves proimages and sets of fixed points.

Proof. If a functor H preserves finite sums, then it is separating and $H^{x,X}$ is an ultrafilter for every X, $x \in H(X)$. The mappings $\varphi_X : H(X) \longrightarrow \beta(X)$ with $\varphi_X(x) = H^{x,X}$ form a natural transformation. Consequently H preserves proimages. Since β preserves sets of fixed points (see VI.8) one can prove easily that H preserves sets of fixed points.

XI.3. We recall that a pull-back-push-out diagram is called a <u>double-diagram</u>.

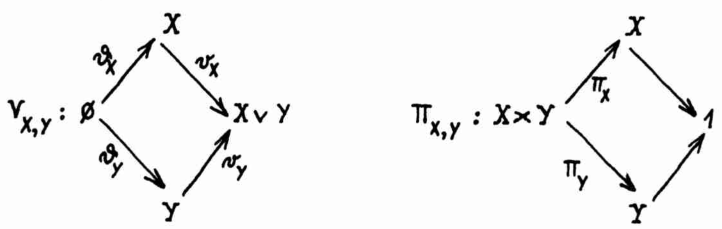
<u>Lemma</u>. Let



be a pull-back-diagram. Then it is a double-diagram iff $\alpha(X) \cup \beta(Y) = Z$ and $\alpha / X - \gamma(T)$, $\beta / Y - \delta(T)$ are injections.

<u>Proof</u> is easy.

XI.4. <u>Lemma</u>. For arbitrary sets X, Y the diagrams



are double-diagrams.

<u>Proof</u>: Well-known and evident.

XI.5. <u>Theorem</u>. The following properties of a functor H

are equivalent:

(i) H preserves double-diagrams;

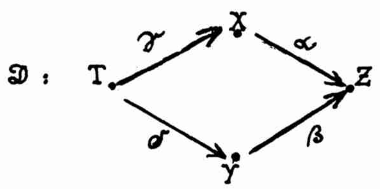
(ii) every component of H is either naturally equivalent to C_1 or preserves finite sums and finite products.

Proof. We may suppose H connected.

(i) $\Longrightarrow$ (ii): Since H preserves double-diagrams $\Pi_{X,Y}$, it preserves finite products. Consequently either $H \simeq C_1$ or H is separating (see IV.4 of [1], note that $C_{0,1}$ does not preserve double-diagrams).

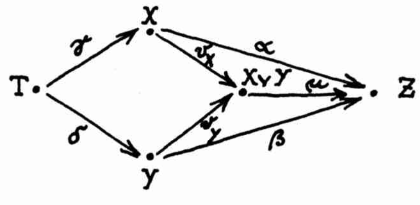
If a separating functor preserves double-diagrams $V_{X,Y}$, then it preserves finite sums.

(ii) $\Longrightarrow$ (i): Let H preserve finite sums and finite products. Then H is separating, consequently it preserves pull-back-diagrams (see VII.10). Let



be a double-diagram.

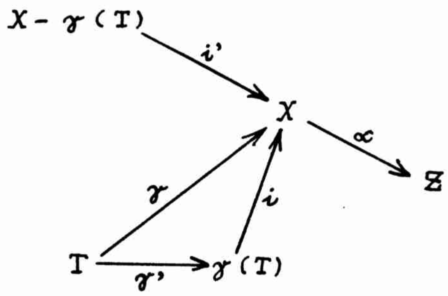
1) First, we prove that $[H(\alpha)](H(X)) \cup [H(\beta)](H(Y)) = H(Z)$. Consider the commutative diagram



(thus, $\mu = coeq(v_X \circ \gamma, v_Y \circ \delta)$). $H(\mu)$

is a surjection since μ is. Now, use $H(X \vee Y) = H(X) \vee H(Y)$.

2) Now we prove that $^{H(\alpha)}/H(X) - [H(\gamma)](H(T))$ is an injection. Consider the following commutative diagram:



where i, i' are the inclusions, γ' is a surjection. Since $\alpha \circ i'$ is an injection, $H(\alpha \circ i')$ is also an injection. Since $H(X) = H(\gamma(T)) \vee H(X - \gamma(T))$ and $H(\gamma(T))_X = [H(\gamma)](H(T))$, we have $H(X - \gamma(T))_X = H(X) - [H(\gamma)](H(T))$. Now, use XI.3.

R e f e r e n c e s

[1] V. TRNKOVÁ: On descriptive classification of set-functors I., Comment.Math.Univ.Carolinae 12(1971), 143-174.

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