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ON BINDABILITY OF PRODUCTS AND JOINS OF CATEGORIES

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A category is called binding if it is concrete and every concrete category can be fully embedded into it.

(A full embedding $F : \mathcal{K} \to \mathcal{L}$ is a faithful functor [1] which maps $\mathcal{K}$ onto a full subcategory of $\mathcal{L}$.)

The existence of a binding category is proved in [1].

We investigate in this paper products and joins of categories from the point of view of the property "to be a binding category".

The product $\mathcal{K} \times \mathcal{L}$ of categories $\mathcal{K}, \mathcal{L}$ is defined as follows:

objects of $\mathcal{K} \times \mathcal{L}$ are all couples (X, Y) where X (Y respectively) is an object of $\mathcal{K}$ ($\mathcal{L}$ respectively),

morphisms of $\mathcal{K} \times \mathcal{L}$ from (X, Y) into (U, V) are all couples (f, g), where $f : X \to U$ ($g : Y \to V$ resp.) is a morphism of $\mathcal{K}$ ($\mathcal{L}$ resp.),

$(f, g)(h, j) = (fh, gj)$.

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1) F must not be one-to-one mapping of a class of objects of $\mathcal{K}$ into a class of objects of $\mathcal{L}$.

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The join $\mathbb{K} \vee \mathbb{L}$ of the categories $\mathbb{K}, \mathbb{L}$ is defined as follows:

objects of $\mathbb{K} \vee \mathbb{L}$ are all couples (X, i), where either X is an object of $\mathbb{K}$ and $i = 0$ or X is an object of $\mathbb{L}$ and $i = 1$,

morphisms of $\mathbb{K} \vee \mathbb{L}$ from (X, i) into (Y, j) are all couples (f, k), where either $i = j = k = 0$ and $f: X \to Y$ is a morphism of $\mathbb{K}$ or $i = j = k = 1$ and $f: X \to Y$ is a morphism of $\mathbb{L}$,

$(f, 0)(g, 0) = (fg, 0)$, $(f, 1)(g, 1) = (fg, 1)$.

We shall prove the following theorems:

Theorem 1. $\mathbb{K} \vee \mathbb{L}$ is binding if and only if either $\mathbb{K}$ or $\mathbb{L}$ is binding.

Theorem 2. If $\mathbb{K} \times \mathbb{L}$ is binding then both $\mathbb{K}$ and $\mathbb{L}$ have a rigid object (i.e. an object, only endomorphism of which is the identity).

Theorem 3. If $\mathbb{K}$ is binding and a concrete category $\mathbb{L}$ has a rigid object then $\mathbb{K} \times \mathbb{L}$ is binding.

Theorem 4. If $\mathbb{K} \times \mathbb{L}$ is binding and $\mathbb{L}$ is a thin category (i.e. there is at most one morphism from X into Y for every two objects X, Y of $\mathbb{L}$) then $\mathbb{K}$ is a binding category.

The general problem whether the bindability of $\mathbb{K} \times \mathbb{L}$ implies the bindability of either $\mathbb{K}$ or $\mathbb{L}$ is, as far as we know, unsolved.

This paper is divided into three paragraphs: in § 1 we shall prove Theorems 1,2,3. The proof of the theorem 4 (§ 3) is based upon a theorem on EO-embeddings and maximal categories which are defined and investigated in § 2.

§ 1. First we give three obvious lemmas:

Lemma 1. $K \times L$ is concrete if and only if both K and L are concrete.

Lemma 2. $K \vee L$ is concrete if and only if both K and L are concrete.

Lemma 3. If $F: K \to L$ is a full embedding, K is binding and L is a concrete category then L is binding.

Proof of Theorem 1. The functors $F: K \to K \vee L$ and $G: L \to K \vee L$ defined by

$$F(X) = (X, 0) \quad , \quad F(f) = (f, 0) \quad ,$$

$$G(X) = (X, 1) \quad , \quad G(f) = (g, 1)$$

are full embeddings. Therefore if either K or L is binding then $K \vee L$ is binding in view of Lemmas 2,3.

Let $K \vee L$ be a binding category. Let the category M be obtained from $K \vee L$ by a formal addition of an initial object 0. It follows that M is binding from Lemma 3.

Because $K \vee L$ is binding, there is a full embedding $F: M \to K \vee L$. If $F(0) \in K^o \times \{0\}$ then it is evident that F maps M^o into $K^o \times \{0\}$. Therefore $G: M \to K$ defined by

$G(X) = Y$ if and only if $F(X) = (Y, 0)$ is a full embedding.

This implies that K is binding by Lemma 3.

Similarly, if $F(0) \in L^o \times \{1\}$ then there is a full embedding from M into L, which implies that L is binding.

Proof of Theorem 2. It is evident that a binding category has a rigid object. If (X, Y) is a rigid object of $K \times L$ then X (Y resp.) is a rigid object of K (L resp.).

Proof of Theorem 3. Let Y be a rigid object of L. Then $F: K \longrightarrow K \times L$ defined by

$$F(X) = (X, Y), \quad F(f) = (f, id\, Y)$$

is a full embedding. Therefore $K \times L$ is binding by Lemma 3.

§ 2. In this paragraph we deal with EO-embeddings and maximal categories:

Definition. A functor $F: K \longrightarrow L$ is called an EO-embedding if F is a one-to-one mapping of $M_K(X, Y)$ onto $M_L(F(X), F(Y))$ for every two objects X, Y of K with $M_K(X, Y) \neq \emptyset$.

Next two lemmas are obvious:

Lemma 4. A composition of EO-embeddings is an EO-embedding.

Lemma 5. A full embedding is an EO-embedding.

Definition. A category K is called maximal if every EO-embedding $F: K \longrightarrow L$ is a full embedding.

The main result of this paper is

Theorem 5. Every concrete category is a full subcategory of a maximal concrete category.

Proof. Denote by $Set(0, 1)$ the following category: objects of $Set(0, 1)$ are all sets X such that $0, 1 \in X$, morphisms of $\mathcal{S}et\ (0, 1)$ from X into Y are all mappings $f : X \longrightarrow Y$ such that $f(0) = 0$, $f(1) = 1$, the composition of morphisms is the composition of mappings.

Let K be a concrete category. Since $\mathcal{S}et\ (0, 1)$ is isomorphic to the category of all sets and all their mappings we can suppose, without loss of generality, that K is a subcategory of $\mathcal{S}et\ (0, 1)$.

We shall construct a sequence $K_0, K_1, K_2, \dots$ of subcategories of $\mathcal{S}et\ (0, 1)$ as follows:

1) $K = K_0$.

2) If K_{i-1} is defined then

objects of K_i are all objects of K_{i-1} together with all sets $\{(X, Y), X, 0, 1\}$, where X, Y are objects of K_{i-1};

if M, N are objects of K_i then

$M_{K_i}(M, N)$ —

- $M_{K_{i-1}}(M, N)$ for $M, N \in K^o_{i-1}$,
- set of all one-to-one morphisms $f : M \longrightarrow N$ of $\mathcal{S}et\ (0, 1)$ for $M = N \notin K^o_{i-1}$,
- set of all morphisms $f : M \longrightarrow N$ of $\mathcal{S}et\ (0, 1)$ such that $f(M) \subset \subset \{0, 1\}$ and $f((X, Y)) \neq f(X)$ for $M = \{(X, Y), X, 0, 1\}$, where $X, Y, N \in K^o_{i-1}$ and $M_{K_{i-1}}(X, N) = \emptyset$,
- set of all morphisms $f : M \longrightarrow N$ for

$$M = \begin{cases} \{(X,Y),\ X,\ 0,\ 1\}, & \text{where } X, Y, N \in X^{o}_{i-1} \text{ and } M_{K_{i-1}}(X,N) \neq \emptyset, \\ \emptyset & \text{in the other cases.} \end{cases}$$

The composition of morphisms is the composition of mappings.

It is evident that all K_i are subcategories of $Set\,(0,1)$ and K_{i-1} is a full subcategory of K_i for every natural i.

Denote the union of the categories $K_0, K_1, \dots$ by L. L is a subcategory of $Set\,(0,1)$ and K is a full subcategory of L.

We shall prove that L is a maximal category:
Let $F: L \to M$ be an EO-embedding. Let X, Y be objects of L such that $M_L(X,Y) = \emptyset \neq M_M(F(X), F(Y))$.

There is a natural n such that $X, Y \in X^{o}_{n}$.

Let f be a morphism of M from $F(X)$ into $F(Y)$. A mapping $g: \{(X,Y), X, 0, 1\} \to X$ defined by $g((X,Y)) = g(X) = g(0) = 0$, $g(1) = 1$ is a morphism of K_{n+1}. Since there is a morphism of K_{n+1} from $\{(X,Y), X, 0, 1\}$ into Y there is a morphism $h: \{(X,Y), X, 0, 1\} \to Y$ of K_{n+1} such that $F(h) = f\,F(g)$.

Let m, n be morphisms of K_{n+1} from $\{(X,Y), X, 0, 1\}$ into itself defined by

$$m((X,Y)) = n(X) = (X,Y), \quad m(X) = n((X,Y)) = X.$$

Then it is $gm = gn$, and $hm \neq hn$ and the following inequality holds:

$F(hm) \neq F(hn) = F(h)F(n) = f\,F(g)\,F(n) =$

$= fF(gn) = fF(gm) = fF(g)F(m) = F(h)F(m) = F(hm)$.
This is a contradiction. Therefore F is a full embedding.

Thus we have proved that $\mathbb{L}$ is a maximal category.

As a corollary to the theorem 5, to Lemma 3 and to the existence of binding category we have

Theorem 6. There is a maximal binding category.

§ 3. The proof of Theorem 4 is based upon the next lemma:

Lemma 6. Let $\mathbb{K}$ be a category and $\mathbb{L}$ be a thin category. Then there is an EO-embedding from $\mathbb{K} \times \mathbb{L}$ into $\mathbb{K}$.

Proof. A functor $F : \mathbb{K} \times \mathbb{L} \longrightarrow \mathbb{K}$ defined by $F((X, Y)) = X, F((f, g)) = f$ is an EO-embedding, because if $(X, Y), (U, V)$ are objects of $\mathbb{K} \times \mathbb{L}$ then either $M_{\mathbb{L}}(Y, V) = \emptyset$ and $M_{\mathbb{K} \times \mathbb{L}}((X, Y), (U, V)) = \emptyset$ or $M_{\mathbb{L}}(Y, V)$ is a one-point set and F is a one-to-one correspondence between $M_{\mathbb{K} \times \mathbb{L}}((X, Y), (U, V)) = M_{\mathbb{K}}(X, U) \times M_{\mathbb{L}}(Y, V)$ and $M_{\mathbb{K}}(X, U)$.

Proof of Theorem 4. Let $\mathbb{M}$ be a maximal binding category. Since $\mathbb{K} \times \mathbb{L}$ is a binding category, there is a full embedding $F : \mathbb{M} \longrightarrow \mathbb{K} \times \mathbb{L}$. If $G : \mathbb{K} \times \mathbb{L} \longrightarrow \mathbb{K}$ is an EO-embedding then $GF : \mathbb{M} \longrightarrow \mathbb{K}$ is an EO-embedding. Since $\mathbb{M}$ is maximal, GF is a full embedding. Therefore $\mathbb{K}$ is a binding category.

References

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