Werk

Label: Article

Jahr: 1971

PURL: https://resolver.sub.uni-goettingen.de/purl?316342866_0012|log32

Kontakt/Contact

DIFFERENTIAL OPERATORS AND G-STRUCTURES OF HIGHER ORDER

Jarolím BUREŠ, Praha

0. Introduction. A differential operator D of order κ on the manifold M of dimension m (manifolds and maps are assumed to be $C^{-\infty}$-differentiable) can be expressed in every admissible coordinate system (c.s.) $(x^1 \ldots x^n)$ on M in the form:

$$D = \sum_{i=1}^{m} \alpha_i(x) \frac{\partial}{\partial x^i} + \ldots + \sum_{i_1,\ldots,i_k} \alpha_{i_1 \ldots i_k} \frac{\partial^k}{\partial x^{i_1} \ldots \partial x^{i_k}}$$

where $\alpha_i , \alpha_{i_1 i_2} , \ldots$ are differentiable functions on M. Changing the coordinate system, we obtain the expression of D using another functions $\alpha'_{i \ldots j}$ We ask if the coordinates can be changed in such a way that the functions α_i expressed in the new c.s. are constant. We try to solve this problem only locally. The coefficients of the differential operator of order κ are transformed in such a way that we use the derivatives up to the order κ of the change of coordinates only. Thus the differential operator of order κ determines a geometrical object of order κ and determines a mapping of the principal fibre bundle of κ-frames $H^\kappa(M)$ into the standard κ-tangent space $F^\kappa = T_0^\kappa(R^n)$.

The differential operator having in some c.s. constant coefficients,

(i) it maps $H^k(M)$ onto an orbit of the group L_m^k in F^k (see §3),

(ii) it determines a class of conjugate G -structures of order k on M,

(iii) the G -structures are integrable (see § 4) on the coordinate domain.

Conversely, if the conditions (i) - (iii) are fulfilled, the considered differential operator has constant coefficients.

Guillemin and Singer in [3] solved this problem for coefficients at the highest derivatives transforming it into a problem of integrability of G -structures of order 1. This paper generalizes the solution on of this problem to all coefficients. These conditions will be examined in more detail in the next paper.

1. Let us denote (following [1]) by L_m^k the Lie group of all invertible k -jets of maps from R^m into R^{km} with source and target in $0 \in R^m$. The composition of such two elements $\xi = j^k \circ f$ and $\eta = j^k \circ g$ is given by $\xi \circ \eta = j^k \circ (f \circ g)$.

If M is a differentiable manifold (of class C^∞) and $m \in M$, we denote by

$A(m)$ the set of all differentiable functions defined on some neighbourhood of m;

$A_0(m) \subset A(m)$ the subset of all functions which have the value 0 at the point m ;

$A_c(m) \subset A(m)$ the subset of all functions which are constant on some neighbourhood of m ;

$A_0^{\nu+1}(m) \subset A(m)$ the subset of all multiples of $\nu + 1$ elements of $A_0(m)$.

The R -tangent space $T_m^\nu(M)$ of the manifold M at the point m may be defined as a vector space whose elements are linear functionals

(1) $\qquad X : A(m) \longrightarrow R$

such that $A_0^{\nu+1}(m) \cup A_c(m) \subset Ker\, X$ $\qquad$ (see [1]).

If $h = (x^1 \ldots x^m)$ is a coordinate system in a neighbourhood of $m \in M$, then $X \in T_m^\nu(M)$ can be expressed uniquely in the form

(2) $\qquad X = \sum_{i=1}^{m} X^i \frac{\partial}{\partial x^i} + \ldots + \sum_{1 \leq i_1 \leq i_\nu \leq m} \frac{\partial^\nu}{\partial x^{i_1} \partial x^{i_\nu}}$.

For $\nu = 1$, we obtain the ordinary tangent space of the manifold.

The space $T^\nu(M) = \bigcup_{m \in M} T_m^\nu(M)$ can be endowed with a structure of a differentiable manifold such that $T^\nu(M)$ is a vector bundle over M . A section of the bundle $T^\nu(M)$ is a map $D : U \longrightarrow T^\nu(M)$ such that $D(x) \in T_x^\nu(M)$ for all $x \in U$, and (2) shows that D is a differentiable operator of order $\leq \nu$ on U .

2. Let us denote by $F^\nu = T_0^\nu(R^m)$ the ν -tangent space at $0 \in R^m$. If $(z^1 \ldots z^m)$ is a fixed c.s. on R^m , then the ν -vectors

(3) $\partial_{i_1 \cdots i_p} = (\frac{\partial^p}{\partial x^{k_1} \cdots \partial x^{k_p}})_0$ $\quad 1 \le i_1 \le \cdot \le k_p \le n$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad p \le k$

form a basis of F^k .

Every differentiable mapping φ of the manifold M into the manifold N induces a differentiable map

$$\varphi_*^k : T^k(M) \longrightarrow T^k(N)$$

given by

(4) $\varphi_*^k(X)f = X(f \circ \varphi)$, $X \in T_m^k(M)$, $f \in A_M(\varphi(m))$.

If φ is a diffeomorphism, then φ_*^k is also a diffeomorphism for every k and it holds

$$(\varphi \circ \psi)_*^k = \varphi_*^k \circ \psi_*^k \quad .$$

Now we define the action $\wp$ of the group L_m^k on F^k. Every $\alpha \in L_m^k$ is given by a map $f : R^n \longrightarrow R^n$ such that $f(0) = 0$. The mapping f determines the mapping f_*^k : $: T_0^k(R^n) \longrightarrow T_0^k(R^n)$ depending on $\alpha = \jmath^k \circ f$ only; it does not depend on f . Let us denote this induced mapping by $\tilde{\alpha}$ and define $\wp(\alpha) : F^k \longrightarrow F^k$ by

(5) $\qquad \wp(\alpha)X = (\tilde{\alpha}^{-1})X = (f^{-1})_*^k X$

for $X \in F^k$ $\quad \alpha = \jmath^k \circ f$.

If $\beta = \jmath^k \circ q$, then $\wp(\alpha \cdot \beta)X = \wp(\beta)(\wp(\alpha)X)$.

Lemma 1 [1] . L_m^k acts on F^k effectively from the left.

3. Let $H^k(M)$ be the principal fibre bundle of k-frames on M and let $\pi : H^k(M) \longrightarrow M$ be the

canonical projection.

The element $\mu \in H^{\varkappa}(M)$ is the $\varkappa$-jet at $0 \in R^m$ of a local diffeomorphism F of a neighbourhood of $o \in$ $\in R^m$ into M satisfying $F(o) = \pi(\mu)$.

The mapping F^{-1} is a c.s. defined on a neighbourhood of $\pi(\mu)$ and it determines an isomorphism

(6) $\qquad (F^{-1})^{\varkappa}_{*} : T^{\varkappa}_{\pi(\mu)}(M) \longrightarrow F^{\varkappa}$

which depends on μ only and which does not depend on F.

Let us denote, for every $X \in T^{\varkappa}_{\pi(\mu)}$, by $\Phi_X(\mu) =$ $= (F^{-1})^{\varkappa}_{*} X$ the element of $F^{\varkappa}$ obtained in the above described way, and denote by

(7) $\qquad \Phi_X : \pi^{-1}(\pi(\mu)) \longrightarrow F^{\varkappa}$

the obtained mapping.

If D is a differential operator of order $\varkappa$ on an open set $U \subset M$, then D can be treated as a field of $\varkappa$-vectors on U, and we can construct the map

(8) $\qquad \Phi_D : H^{\varkappa}(U) \longrightarrow F^{\varkappa}$

such that $\Phi_D(\mu) = \Phi_{D(\pi(\mu))}(\mu)$.

The Lie group $L^{\varkappa}_m$ acts on $H^{\varkappa}(U)$ and also on $F^{\varkappa}$. Let us examine $\Phi_D(\mu \cdot \alpha)$.

Let $\mu = j^{\varkappa}_0 F$, $\alpha = j^{\varkappa}_0 f$, then $\mu \cdot \alpha = j^{\varkappa}_0(F \circ f)$, $(F \circ f)^{-1} = f^{-1} \circ F^{-1}$ and $[(F \circ f)^{-1}]^{\varkappa}_{*} X = (f^{-1})^{\varkappa}_{*} (F^{-1})^{\varkappa}_{*} X = \varphi(\alpha) \Phi_D(\mu)$.

<u>Lemma 2</u>. The mapping Φ_D is a differentiable mapping of $H^{\varkappa}(U)$ into $F^{\varkappa}$, commuting with the action of the group $L^{\varkappa}_m$, e.g.,

$$\Phi_D(\mu \cdot \alpha) = \rho(\alpha)\,\Phi_D(\mu)\ .$$

Proof. The first assertion follows directly expressing Φ_D in local coordinates. The second assertion has been shown above.

Remark. Choosing a c.s. on R^n and a base of F^k in the sense of (3), Φ_D is determined by the system of functions $\varphi^i(\mu),\ \dots\ \varphi^{k\cdots i_k}(\mu)$

$$\Phi_D(\mu) = \sum_{i=1}^{n} \varphi^i(\mu)\,\frac{\partial}{\partial x_i}(0) + \dots + \sum_{1 \leq i_k \leq i_n \leq} \varphi^{i\cdots i_n}(\mu)\,\frac{\partial^k}{\partial x \cdots \partial x}$$

this equation corresponding to the expression of D in the coordinate system (more accurately, in the germ of c.s.) given by F^{-1} at the point $\pi(\mu)$ $(\mu = j^k \circ F)$.

With any c.s. $h = (x^1 \dots x^n)$ on the neighbourhood U we can canonically associate a section $\tilde{h}$ of the principal fibre bundle $H^k(M)$ over U in the following way: Let us denote by $t_z : R^n \longrightarrow R^n$ the map defined by

(10) $\qquad t_z(x) = (x + z)$ $\qquad\qquad$ (translation).

For $\bar{y}_0 \in U$, $h(\bar{y}_0) = y_0$, we have $\bar{y}_0 = h^{-1}(y_0) = h^{-1}(t_{y_0}(0)) = h^{-1} \circ t_{y_0}(0)$, and $\tilde{h}(\bar{y}_0) = j_0^k(h^{-1} \circ t_y)$ is well defined and it is an element of $H_y^k(M)$.

If there exists a c.s. on the neighbourhood U such that D has constant coefficients with respect to it, then Φ_D is constant on the section $\tilde{h}(U)$, and according to Lemma 2 we have

Lemma 3. If there exists a c.s. on M in the neighbourhood V such that D expressed in it has constant

coefficients, then it holds:

(11) $\Phi_D (H^{\varkappa}(V))$ is contained in one orbit of the
action of the group $L^{\varkappa}_m$ on $F^{\varkappa}$.
Hence $\Phi_D (H^{\varkappa}(V))$ fills out the whole orbit.

Condition (11) is therefore a necessary condition for
the existence of a c.s. in a neighbourhood of $m \in M$
such that D has constant coefficients in it.

4. Our next task is to produce necessary and suffici-
ent conditions. We solve this problem locally.

Let us assume that M is a neighbourhood of $o \in R^n$,
D a differential operator of order $\varkappa$ on M satisfy-
ing (11), and denote by $\mathcal{O}$ the orbit $\Phi(H^{\varkappa}(M))$. Let us
choose $\eta \in H^{\varkappa}(M)$, write $\Phi_D (\eta_o) = Z \in \mathcal{O}$ and let
$G_Z \subset L^{\varkappa}_m$ be the isotropic group of Z with respect to the action, e.g.,
$$G_Z = \{\alpha \in L^{\varkappa}_m , \quad \varphi(\alpha)z = z \} .$$
The following lemma is obvious.

<u>Lemma 3</u>. G_Z is a closed subgroup of $L^{\varkappa}_m$ and
thus it is a Lie subgroup of $L^{\varkappa}_m$.

Now, the main result is given by the following.

<u>Proposition 1</u>. Let D be a differential operator of
order $\varkappa$ on a neighbourhood M of the point $0 \in R^n$
satisfying the condition (11). Let us choose a $Z \in \mathcal{O} =$
$= \Phi_D(H^{\varkappa}(M))$. Then $\mathcal{P}_Z = \Phi_D^{-1}(Z)$ is a G_Z -structure
on M , where G_Z is the isotropic group of z . If z' is
another element of the orbit, then $\mathcal{P}_Z$ and $\mathcal{P}_Z'$ are conjuga-
te structures on M .

<u>Proof</u>. The proposition follows from the bundle struc-
ture theorem for Lie groups.

Remark. A differential operator of order κ satisfying the conditions of Prop. 1 on M determines a system of conjugate structures of order κ on M.

Let us now define the notions of equivalence and integrability for G-structures of order κ. Every diffeomorphism $g : M \to N$ determines a diffeomorphism $g^\kappa : H^\kappa(M) \to H^\kappa(M)$. For $n = j_0^\kappa F$ define $g^\kappa(n) = j_0^\kappa(g \circ F)$.

If G is a Lie subgroup of L_m^κ and $\mathcal{P}_1 \to M$, $\mathcal{P}_2 \to M$ are two G-structures of order κ on M and N, resp. then we say that $\mathcal{P}_1$ and $\mathcal{P}_2$ are equivalent if there exists a diffeomorphism $g : M \to M$ such that $g^\kappa(\mathcal{P}_1) = \mathcal{P}_2$.

By the standard G-structure of order κ on an open set $\mathcal{U} \subset \mathbb{R}^m$ we mean a G-structure φ_G given by

$$\varphi_G = \{\widetilde{h}(x) \cdot \alpha, \, x \in \mathcal{U}, \, \alpha \in G\},$$

h being a fixed global c.s. and $\widetilde{h}$ being the canonical section determined by h on $\mathcal{U}$.

The G-structure on M is called integrable if it is equivalent to a standard G-structure of order κ on $\mathbb{R}^m$. Similarly as in the case $\kappa = 1$ we may define the notions of the local equivalence and the local integrability.

Lemma 4. The G-structure $\mathcal{P}$ of order κ on M is integrable if and only if there is a c.s. h on M such that $\widetilde{h}(M)$ is a section of $\mathcal{P}$

Proposition 2. Under the assumptions of Proposition 1 the following statements are equivalent:

(i) $\mathcal{P}_z$ is an integrable G_z-structure on M for some (and for all) $z \in \mathcal{O}$.

(ii) There is a c.s. on M such that D has constant coefficients in it.

Example. It can be easily shown that D being an operator of the first order on $M \subset R^n$, e.g. D being a vector field on M, there is such a c.s. if and only if the field is nonzero everywhere on M.

R e f e r e n c e s

[1] CENKL B.: On the G-structure of higher order, Čas.pěst. mat.90(1963),26-32.

[2] CLARK R.S.: Special geometric object fields on a differentiable manifold , Tensor 13(1965),60-70.

[3] GUILLEMIN - SINGER: Diff. equations and G-structures, Proc.of US Japan Seminar in Dif.Geom.Kyoto(1965), 34-36.

Matematicko-fyzikální fakulta

Karlova universita

Sokolovská 83, Praha 8

Československo