Commentationes Mathematicae Universitatis Carolinae

12,2 (1971)

MEDIAN GRAPHS

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In this paper a special kind of undirected graphs will be discussed. There exists the connection of those graphs with certain abstract algebras introduced in [4].

Let $G = (V, E)$ be a finite connected undirected graph without loops and multiple edges. Let us denote the distance in G by d. We shall say that a vertex t is a median of vertices u, v and w if it holds:

$$d(u,v) = d(u,t) + d(v,t),$$

$$d(v,w) = d(v,t) + d(w,t),$$

$$d(u,w) = d(u,t) + d(w,t).$$

<u>Proposition 1</u>. Let $\{p, q\} \in E$ and $v \in V$. Then the vertices p, q and v have at most one median. If they have a median, then it is either p or q.

<u>Proposition 2</u>. Let $\{p, q\} \in E$ and $v \in V$. Then the vertices have a median if and only if

$$|d(p,v) - d(q,v)| = 1.$$

We shall say that G is a median graph if every three its vertices have just one median. In the following we shall assume that G is a median graph. We shall denote by $M(u, v, w)$ the median of the vertices u, v and w.

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Proposition 3. Let $u, v, w \in V$. Then

(1) $M(u, u, v) = u$,

(2) $M(v, u, w) = M(u, v, w) = M(u, w, v)$.

It follows from Section 7.1 in [2] (see Problem 1 and Theorem 7.1.1)

Proposition 4. G has no circuit of an odd length.

Lemma 1. Let $p, q \in V$, $p \neq q$. A necessary and sufficient condition that $\{p, q\}$ be an edge is that $M(p, q, v)$ be either p or q for any vertex v.

Proof. The necessity follows from Proposition 1.

The sufficiency. If $\{p, q\}$ is not an edge, then there exists a vertex v, $p \neq v \neq q$ such that $d(p,q) = d(p,v) + d(q,v)$. Without loss of generality let us assume that $M(p, q, v) = p$. Then $d(q, v) = d(q,p) + d(p,v) = 2d(p,v) + d(q,v)$; thus $d(p, v) = 0$, which is a contradiction. The lemma is proved.

Let $\{p, q\} \in E$; we shall denote:

$$V_{p,q} = \{u \in V \mid d(p,u) < d(q,u)\},$$

$E_{p,q} = \{\{u,v\} \in E \mid$ either $u \in V_{p,q}$, $v \in V_{q,p}$ or $u \in V_{q,p}$, $v \in V_{p,q}\}$, $A_{p,q} = \{u \in V_{p,q} \mid$ there exists $v \in V_{q,p}$ such that $\{u, v\} \in E_{p,q}\}$.

Proposition 5. Let $\{p,q\} \in E$ and $\{u,v\} \in E_{p,q}$, $u \in V_{p,q}$. Then

$$d(p,u) = d(q,v) = d(p,v) - 1 = d(q,u) - 1.$$

Lemma 2. Let $\{p,q\} \in E$ and $\{u_0, u_1\}, \ldots, \{u_{n-1}, u_n\}$,

$n > 1$, be an arc in G such that $d(u_0, u_n) = n$ and $u_0, u_n \in V_{p,q}$. Then $u_1, \ldots, u_{n-1} \in V_{p,q}$.

<u>Proof</u>. Let us assume that $u_1 \in V_{q,p}$; then $\{u_0, u_1\} \in E_{p,q}$. There exists k, $1 \leq k < n$ such that $u_1, \ldots, u_k \in V_{q,p}$, $u_{k+1} \in V_{p,q}$ and $\{u_k, u_{k+1}\} \in E_{p,q}$. As $d(u_1, u_k) = k - 1$, then from Proposition 5 it follows that $d(u_0, u_{k+1}) = k - 1$, which is a contradiction. Thus $u_1 \in V_{p,q}$ (Proposition 4); by the induction we also get that $u_2, \ldots, u_{n-1} \in V_{p,q}$.

<u>Proposition 6</u>. Let $\{p, q\} \in E$, $u, v \in V_{p,q}$ and $w \in V$. Then

$$M(u, v, w) \in V_{p,q} .$$

<u>Theorem 1</u>. The set $\{E_{p,q} \mid \{p, q\} \in E\}$ is a disjoint partition of E.

<u>Proof</u>. Let $\{p, q\}, \{u, v\}, \{x, y\} \in E$. It is obvious that $\{p, q\} \in E_{p,q}$ and if $\{u, v\} \in E_{x,y}$ then $\{x, y\} \in E_{u,v}$. We shall assume that $\{u, v\}$, $\{x, y\} \in E_{p,q}$, $\{u, v\} \notin E_{x,y}$ and that for every $\{u', v'\} \in E_{p,q}$ such that $\min\{d(u', p), d(v', p)\} < \min\{d(u, p), d(v, p)\}$ it holds that $\{u', v'\} \in E_{x,y}$.

Without loss of generality let us assume that

$$0 \leq d(u, x) < \min\{d(u, y), d(v, x), d(v, y)\}$$

and that

$$d(u, p) = d(v, q) = d(u, q) - 1 = d(v, p) - 1 .$$

There exists a vertex $\bar{u}$ such that $\{u, \bar{u}\} \in E$ and

$d(\bar{u}, p) = d(u, p) - 1$. Thus $d(\bar{u}, q) = d(u, p)$. Denote $\bar{v} = M(\bar{u}, v, q)$. Because $\bar{u} \neq \bar{v} \neq v$, then $\{\bar{u}, \bar{v}\} \in E$ and $d(\bar{u}, p) = d(\bar{v}, q) = d(\bar{u}, q) - 1 = d(\bar{v}, p) - 1$. Thus $\{\bar{u}, \bar{v}\} \in E_{p,q}$ and $\{\bar{u}, \bar{v}\} \in E_{x,y}$.

If $d(\bar{u}, x) = d(\bar{v}, y) = d(\bar{u}, y) - 1 = d(\bar{v}, x) - 1$, then $d(\bar{v}, y) = d(u, y) \geq 2$ and $u = M(\bar{u}, v, y) = \bar{v}$, which is a contradiction. If $d(\bar{u}, y) = d(\bar{v}, x) = d(\bar{u}, x) - 1 = d(\bar{v}, y) - 1$ then $d(\bar{v}, x) = d(u, x) \geq 2$ and also $u = M(\bar{u}, v, x) = \bar{v}$, which is a contradiction, too.

Remark 1. Figure 1 gives an example of graph which is not a median graph but for which the precedent theorem also holds.

From Theorem 1 it follows

Proposition 7. G includes no subgraph which is isomorphic with the graph in Figure 2.

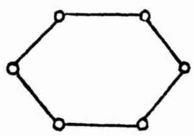

Figure 1.

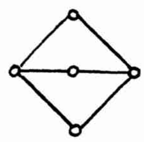

Figure 2.

Lemma 3. Let $\{p_0, q_0\}$ be an edge, $\{p_0, p_1\}, \dots, \{p_{n-1}, p_n\}$ be an arc in G such that $d(p_0, p_n) = n \geq 1$ and $p_n \in A_{p_0, q_0}$. Then $p_1, p_2, \dots, p_{n-1} \in A_{p_0, q_0}$.

Proof. The case where $n = 1$ is obvious. Let $n > 1$ and let for every arc of length $n - 1$ the lemma be proved. If there exists m, $1 \leq m < n$, such that $p_m \in A_{p_0, q_0}$, then the lemma is proved. Now, we shall assume that for every m, $1 \leq m < n$, it holds that $p_m \notin A_{p_0, q_0}$. This means that $p_1 \notin A_{p_0, q_0}$. From Lemma 2 it follows that $p_1 \in V_{p_0, q_0}$. Let q be a vertex such that $\{p_n, q\}$ be an edge and $q \in V_{q_0, p_0}$. Then $d(p_1, q) = d(q_0, q) = n \geq 2$. Obviously the vertices q_0, p_1, q have no median, which is a contradiction.

Theorem 2. Let $\{p_0, q_0\}$ be an edge and $\{p_0, p_1\}, \ldots, \{p_{n-1}, p_n\}$ be an arc in G such that $d(p_0, p_n) = n \geq 1$ and $p_n \in A_{p_0, q_0}$. Then there exists just one arc $\{q_0, q_1\}, \ldots, \{q_{n-1}, q_n\}$ such that $\{p_0, q_0\}, \ldots, \{p_n, q_n\} \in E_{p_0, q_0}$.

Proof. From Lemma 3 it follows that $p_1 \in A_{p_0, q_0}$. There exists $q_1 \in V_{q_0, p_0}$ such that $\{p_1, q_1\} \in E_{p_0, q_0}$. Thus $q_1 \in A_{q_0, p_0}$ and $\{q_0, q_1\} \in E$. The uniqueness of the vertex q_1 follows from Proposition 7. By Theorem 1 we have $E_{p_1, q_1} = E_{p_0, q_0}$. This means that $p_n \in A_{p_1, q_1}$. The continuation of the proof is easy.

Proposition 8. If some vertex of G lies on a circuit then it lies on a circuit of length 4.

Lemma 4. Let $\{p, q\}$ be an edge, $x, y \in V_{p, q}$. Then $M(p, x, y) = M(q, x, y)$.

Proof. From Proposition 6 it follows that $M(q,x,y) \in V_{p,q}$. If $d(q,M(q,x,y)) = n > 0$ and if $\{u_0, u_1\}, \ldots, \{u_{n-1}, u_n\}$ is any arc connecting q and $M(q,x,y)$, then $u_1 = p$. From this fact we easily get that $M(p,x,y) = M(q,x,y)$.

Lemma 5. Let $\{p,q\}$ be an edge, $x \in V_{p,q}$, $y \in V_{q,p}$. Then $M(p,x,y) \in A_{p,q}$.

Proof. Obviously $M(p,x,y) \in V_{p,q}$. Let $d(p,y) = n$ and $\{v_0, v_1\}, \ldots, \{v_{n-1}, v_n\}$ be any arc connecting p and y. Then there exists i and j such that $0 \le i \le j < n$ and $v_i = M(p,x,y)$, $v_j \in A_{p,q}$, $v_{j+1} \in A_{q,p}$. This means that $d(p, v_j) = j$; from Lemma 3 it follows that $v_i \in A_{p,q}$.

Lemma 6. Let $\{p,q\}$ be an edge, $x \in V_{p,q}$, $y \in V_{q,p}$. Then

$$\{M(p,x,y), M(q,x,y)\} \in E_{p,q}.$$

Proof. Denote $M(p,x,y)$ by u. There exists $v \in V$ such that $\{u,v\} \in E_{p,q}$. Obviously $d(x,v) = d(x,u) + 1$, $d(y,v) = d(y,u) - 1$ and $d(q,v) = d(p,u)$. Thus $v = M(q,x,y)$.

Theorem 3. Let $u, v, w, x, y \in V$. Then

(3) $M(M(u,v,w),x,y) = M(M(u,x,y), v, M(w,x,y))$.

Proof. Let v, w, x, y be fixed. The case where

$u = w$ is obvious. Now, let us assume that for some vertex $\bar{u}$ such that $\{u, \bar{u}\} \in E$, the theorem is proved. Denote $M(u, v, w)$ by p, $M(\bar{u}, v, w)$ by $\bar{p}$, $M(u, x, y)$ by r, $M(\bar{u}, x, y)$ by $\bar{r}$ and $M(w, x, y)$ by t. This means that $M(\bar{p}, x, y) = M(\bar{r}, v, t)$. We shall prove that $M(p, x, y) = M(r, v, t)$. Without loss of generality let us assume that $v \in V_{u,\bar{u}}$.

I) Let $w \in V_{u,\bar{u}}$. Then from Lemma 4 it follows that $p = \bar{p}$. If either $x, y \in V_{u,\bar{u}}$ or $x, y \in V_{\bar{u},u}$, then $r = \bar{r}$ and (3) holds. Now, without loss of generality let us assume that $x \in V_{u,\bar{u}}$ and $y \in V_{\bar{u},u}$. Then from Lemma 6 it follows that $\{r, \bar{r}\} \in E_{u,\bar{u}}$. Because $t \in V_{u,\bar{u}}$ and $v \in V_{u,\bar{u}}$ then $M(r, v, t) = M(\bar{r}, v, t)$ and (3) holds.

II) Let $w \in V_{\bar{u},u}$. Then $\{p, \bar{p}\} \in E_{u,\bar{u}}$. If either $x, y \in V_{u,\bar{u}}$ or $x, y \in V_{\bar{u},u}$, then $r = \bar{r}$ and $M(p, x, y) = M(\bar{p}, x, y)$; thus (3) holds. Now, without loss of generality let us assume that $x \in V_{u,\bar{u}}$ and $y \in V_{\bar{u},u}$. Then $t \in V_{\bar{u},u}$ and $\{r, \bar{r}\} \in E_{u,\bar{u}}$. From Theorem 1 it follows that $\{M(p, x, y),$ $M(\bar{p}, x, y)\} \in E_{u,\bar{u}}$ and $\{M(r, v, t), M(\bar{r}, v, t\} \in E_{u,\bar{u}}$. As $M(\bar{p}, x, y) = M(\bar{r}, v, t)$, then (3) holds.

In [4] so called simple graphic algebras were introduced. They are the abstract algebras with one ternary operation fulfilling (1), (2) and (3). By a little adaptation of results in [4] (i.e. by the substitution of graphs with a loop at every vertex by graphs without loops), we easily get that there exists a one-to-one correspondence betweem the notion of median graph and the notion of finite simple graphic algebra. The way of reconstruction of the median graph from a finite simple graphic algebra is given by Lemma 1 in the present paper.

From this result it follows that the (undirected) graph of any finite distributive lattice is a median graph; cf. the notion of median operation on distributive lattices in [1]. Similarly, every (finite) tree is a median graph; cf. the intersection vertex operation on the trees in [3].

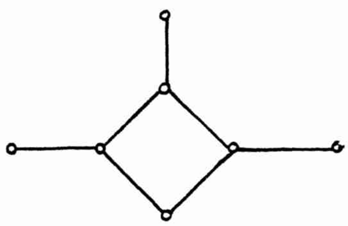

Figure 3.

An example of median graph which is neither the graph of any lattice nor a tree is given in Figure 3.

R e f e r e n c e s

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(Oblatum 16.7.1970)