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SOME RESULTS ON GEOMETRICAL APPROACH TO LINEAR DIFFERENTIAL EQUATIONS OF THE n-TH ORDER

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(Preliminary communication)

Let $\underline{y}(t) = (y_1(t), \dots, y_n(t)) \in E_n \ (n \geq 1)$ for $t \in I$, $|\underline{y}(t)| = \sqrt{\sum_{i=1}^{n} y_i^2(t)}$; let $S_{n-1} = \{\underline{c} \in E_n;\ |\underline{c}| = 1\}$ be the unit sphere in E_n. Denote by $\pi(\underline{y}(t)) = \underline{y}(t)/|\underline{y}(t)|$. For $\underline{y} \in C^k(I)$, $k \geq 1$, $j \leq k$, put $d^j\underline{y}(t)/dt^j = (d^j y_1(t)/dt^j, \dots, d^j y_n(t)/dt^j)$.

Let $x : I \to J$, $x \in C^1(I)$, $dx(t)/dt \neq 0$ for all $t \in I$. Then define $T_x \underline{y} = \underline{z}$, where $z_i(x(t)) = y_i(t)$ for all $t \in I$, $i = 1, \dots, n$. Denote by $[\underline{u}_1, \dots, \underline{u}_n]$ the determinant whose i-th column is $\underline{u}_i$. Let $W_n(\underline{y}(t)) = [\underline{y}(t), d\underline{y}(t)/dt, \dots, d^{n-1}\underline{y}(t)/dt^{n-1}]$ for $\underline{y} \in C^{n-1}(I)$. If $\underline{y} \in C^1(I)$, $\underline{v} = \pi(\underline{y})$, $|d\underline{v}(t)/dt| \neq 0$ for all $t \in I$, $t_0 \in I$, $s = (t \mapsto \int_{t_0}^{t} |d\underline{v}(\sigma)/d\sigma| \cdot d\sigma)$, $s : I \to J$, $T_s \underline{v}(t) = \underline{u}(s)$, then $|d\underline{u}(s)/ds| = 1$. Denote the T_s by τ_{s,t_0}. Obviously $\underline{u}(s) = \tau_{s,t_0}\, \pi(\underline{y}(t)) \in S_{n-1}$ and $d\underline{u}(s)/ds \in S_{n-1}$ for all $s \in J$, $0 \in J$ and

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$\underline{u}(0) = \underline{y}(t_0)/|\underline{y}(t_0)|$. Also $\pi(\underline{y}(t)) = \pi(f(t)\cdot\underline{y}(t))$ for every $f > 0$ and $\tau_{s,t_0}\underline{v}(t) = \tau_{s,\sigma(t_0)} T_\sigma \underline{v}(t)$.

If $f \in C^{n-1}(I)$, $\underline{y} \in C^{n-1}(I)$, then $W_n(f(t)\cdot\underline{y}(t)) = f^n(t)\cdot W_n(\underline{y}(t))$, for $f \neq 0$, $W_n(\underline{y}(t)) \neq 0$ iff $W_n(f(t))\cdot\underline{y}(t)) \neq 0$ on I. For $x \in C^{n-1}(I)$, $dx(t)/dt \neq 0$ on I we have

$$W_n(\underline{y}(t)) = \left(\frac{dx(t)}{dt}\right)^{\frac{n(n-1)}{2}} \cdot W_n(T_x\underline{y}(t))$$

and again $W_n(\underline{y}(t)) \neq 0$ on I iff $W_n(T_x\underline{y}(t)) \neq 0$ on J.

Suppose $\underline{y} \in C^n(I)$, $W_n(\underline{y}(t)) \neq 0$ on I. Then $\underline{u}(s) = \tau_{s,t_0}\pi(\underline{y}(t))$, $s \in J$, satisfies $(' = d/ds,\ \underline{u} \equiv \underline{u}_1)$:

$$
\begin{aligned}
\underline{u}_1'(s) &= \underline{u}_2(s)\\
\underline{u}_2'(s) &= -\underline{u}_1(s) + \alpha_2(s)\underline{u}_3(s)\\
\underline{u}_3'(s) &= -\alpha_2(s)\underline{u}_2(s) + \alpha_3(s)\underline{u}_4(s) \qquad (1)\\
&\dots\\
\underline{u}_{n-1}'(s) &= -\alpha_{n-2}(s)\underline{u}_{n-2}(s) + \alpha_{n-1}(s)\underline{u}_n(s)\\
\underline{u}_n'(s) &= -\alpha_{n-1}(s)\underline{u}_{n-1}(s),
\end{aligned}
$$

where $|\underline{u}_i(s)| = 1$ for $i = 1,\dots,n$, $\underline{u}_i\cdot\underline{u}_j = 0$ for $i \neq j$, $0 < \alpha_i(s) \in C^{n-i}(J)$ (generalized Frenet formula). Constant vectors $\underline{u}_i(0)$, $i = 1,\dots,n$, can be determined from $d^{i-1}\underline{y}(0)/dt^{i-1}$ or

$\underline{u}^{(i-1)}(0)$.

Conversely, there exists the unique solution $\underline{u}_1, \ldots, \underline{u}_n$ of (1) which satisfies the initial conditions determined by $\underline{u}$ and its $(n-1)$ derivatives at 0, and $\underline{u}_1(s) = \underline{u}(s)$ for all $s \in J$. Moreover $W_n(\underline{y}(t)) = |\underline{y}(t)|^n . W_n(\pi(\underline{y}(t))) =$

$$= |\underline{y}(t)|^n . \left| d \frac{\underline{y}(t)}{|\underline{y}(t)|} / dt \right|^{\frac{n(n-1)}{2}} . W_n(\underline{u}(s)),$$

$W_n(\underline{u}(s)) = \alpha_2^{n-2}(s) . \alpha_3^{n-3}(s) . \ldots . \alpha_{n-1}(s) . [\underline{u}_1, \ldots, \underline{u}_n]$. Hence for arbitrary $0 < \alpha_i \in C^{n-i}(J)$, $i = 2, \ldots, n-1$, arbitrary conditions on $\underline{u}_1, \ldots, \underline{u}_n$ at 0 such that $[\underline{u}_1, \ldots, \underline{u}_n]_{s=0} \neq 0$, $t : J \to I$, $dt(s)/ds > 0$ on J , $t \in C^{n-1}(J)$, $f \in C^{n-1}(I)$, $f > 0$ on I , we have $W_n(f(t) . \underline{u}(s(t))) \neq 0$ on I .

Let C be a non-singular $n \times n$ matrix, $C\underline{y}(t)$ the centroaffine transform of $\underline{y}(t)$, $t \in I$. Suppose $\underline{y} \in C^n(I)$ and $W_n(\underline{y}(t)) \neq 0$ on I . If $\underline{u}(s) = \tau_{s,t_0} \pi(\underline{y}(t))$, then $\underline{y}(t) = |\underline{y}(t)| . \underline{u}(s(t))$ and $C\underline{y}(t) = |\underline{y}(t)| . C\underline{u}(s(t))$. Since $C\underline{u}, C\underline{u}_2, \ldots, C\underline{u}_n$ (for arbitrary non-singular C) is the general form of solutions of (1), all centroaffine transforms of $\underline{y}(t)$ are of the form $|\underline{y}(t)| . \underline{v}(s(t))$, where $\underline{v}$ is the first vector of any solution $\underline{v}_1, \ldots, \underline{v}_n$ of (1) such that $[\underline{v}_1, \ldots, \underline{v}_n] \neq 0$.

Let $\underline{y} \in C^n(I)$, $W_n(\underline{y}(t)) \neq 0$ on I and (1) be the corresponding system on J . If $\underline{z} \in C^n(I')$, and $\underline{z}(x) \neq f(t) . C\underline{y}(t)$ on I for any non-singular matrix C, $f \in C^n(I)$, $f > 0$ on I, $x : I \to I'$, $x \in C^n(I)$, $dx(t)/dt > 0$ on I, then $\tau_{s,x_0}\,\pi(z(x))$ does not satisfy (1) on J for any $x_0 \in I'$.

Let n be fixed. By Y denote the set of all triples $(\underline{y}, t_0, I)$, where $I \subset R$, $\underline{y} \in C^n(I)$, $t_0 \in I$, $W_n(\underline{y}(t)) \neq 0$ on I. For $(\underline{y}, t_0, I) \in Y$ define the mapping $M = ((\underline{y}, t_0, I) \mapsto (\alpha_2, \dots, \alpha_{n-1}; J))$, where α_i are the corresponding functions in (1) defined on J. Let $E(Y)$ be such a decomposition of Y that $(\underline{z}, x_0, I')$ and $(\underline{y}, t_0, I)$ belong to the same class of $E(Y)$ iff $\underline{z}(x(t)) = f(t) \cdot C\underline{y}(t)$ on I for a non-singular C, $f \in C^n(I)$, $f > 0$ on I, $x : I \to I'$, $x \in C^n(I)$, $dx(t)/dt > 0$ on I and $x(t_0) = x_0$. Denote by $\simeq$ the corresponding equivalence.

<u>Theorem 1.</u> If $(\underline{y}, t_0, I) \not\simeq (\underline{z}, x_0, I')$, then $M(\underline{y}, t_0, I) \neq M(\underline{z}, x_0, I')$.

Now, consider a differential equation

$$(2)\quad L_n(y) \equiv y^{(n)} + a_1(t)y^{(n-1)} + \dots + a_n(t)y = 0 \quad \text{on } I.$$

Let $t_0 \in I$, $\underline{y}(t) = (y_1(t), \dots, y_n(t))$ be n linearly independent solutions of (2) on I, ($\underline{y} \in C^n(I)$, $W_n(\underline{y}(t)) \neq 0$ on I). Since $C\underline{y}$ ($\det C \neq 0$) is the general form of n linearly independent solutions of (2), we may assign a fixed class $\Phi(L_n, t_0, I) \ni C\underline{y}(t)$ of the decomposition $E(Y)$ to L_n on I, $t_0 \in I$.

A differential equation $L_n(y)$ on I $(t_0 \in I)$ is said to be transformable into $L_n^*(z)$ on I' $(x_0 \in I')$ if there exist functions x and f such that $x : I \to I'$, $x(t_0) = x_0$, $x \in C^n(I)$, $dx(t)/dt > 0$ on I, $f \in C^n(I)$, $f > 0$ on I, and for every solution y of $L_n(y)$ on I, the function $z = (x \mapsto f(t).y(t)$, $x = x(t))$, is a solution of $L_n^*(z)$ on I'.
If $W_n(\underline{y}(t)) \neq 0$, then $W_n(f(t) . \underline{y}(t)) \neq 0$ and $\underline{z}(x) = (z_1(x), \dots, z_n(x))$, $z_i(x) = f(t) . y_i(t)$,

are n linearly independent solutions of $L_n^*(z)$ on I'. Hence $\Phi(L_n, t_0, I) = \Phi(L_n^*, x_0, I')$.
Conversely, if the last relation is satisfied, then $L_n(y)$ on I for $t_0 \in I$ can be transformed into $L_n^*(z)$ on I' for $x_0 \in I'$.

A solution y of (2) on $I = (a, b)$, $b \leq \infty$, is oscillatory (for $t \to b$), if it has infinitely many zeros on $[t_1, b)$, $t_1 \in I$.

$L_n(y)$ is a non-oscillatory equation on $I = (a, b)$ (for $t \to b$), if no non-trivial solution of it is oscillatory (for $t \to b$).

$L_n(y)$ is disconjugate on I, if no non-trivial solution has more than $(n-1)$ zeros (including multiplicity).

Let $d \in I$, ν a positive integer, y be a solution of $L_n(y)$ such that $y(d_i) = 0$ for $d = d_0 \leq d_1 \leq d_2 \leq \dots \leq d_{\nu+n-1}$. Then $\eta(d) = \inf_y \{d_{\nu+n-1}\}$ is called the ν - th conjugate point of d (see [1]).

For $\underline{c} \neq \underline{0}$, let $H(\underline{c}) \equiv \sum_{i=1}^{n} c_i \xi_i = 0$ be the hyperplane in E_n. Hyperplanes $H(\underline{c}_j)$, $j = 1, \dots, k \ (\leq n)$ will be called independent iff the rank of the matrix $(\underline{c}_1, \dots, \underline{c}_k)$ is k.

<u>Theorem 2</u>. Let $\underline{u}(s) \in \Phi(L_n, t_0, I)$, $s \in J = (a', b')$, $I = (a, b)$. There exists a correspondence between the solutions of $L_n(y)$ and all hyperplanes such that to linearly independent solutions y_1 and y_2 there correspond independent hyperplanes H_{y_1} and H_{y_2}. Moreover, there exists a 1-1 mapping $s: I \to J$ such that if t_1 is a k-multiple zero of a solution y of $L_n(y)$, then $\underline{u}$ and H_y have the contact of the $(k-1)$-th order at $\underline{u}(s(t_1))$.

<u>Note.</u> The mapping s and the correspondence between solutions of L_n and hyperplanes in E_n can be constructed in the following way: Let $\underline{y}$ be formed by n linearly independent solutions of L_n. Since $\underline{u}(s) \in \Phi(L_n, t_0, I)$, $s \in J$, we have

(3) $\quad A\underline{y}(t) = |A\underline{y}(t)| \cdot u(s(t)), \quad t \in I,$

for a (fixed) non-singular matrix A. Then the mapping s is given in (3), and to every solution $\underline{c}\underline{y}(t) = \underline{c}^* A\underline{y}(t)$ ($\underline{c} = (c_1, \dots, c_n) \neq \underline{0}$ and hence $\underline{c}^* \neq \underline{0}$) we assign the hyperplane $H(\underline{c}^*)$, and conversely.

<u>Corollary 1</u>. $L_n(y)$ is non-oscillatory iff no hyperplane intersects $\underline{u}(s)$ infinitely many times for $s \in [0, b')$.

<u>Corollary 2</u>. $L_n(y)$ has k linearly indepen-

dent oscillatory solutions and every other linearly independent on them is non-oscillatory iff there exist just k independent hyperplanes, every of which intersects $\underline{u}(s)$ infinitely many times for $s \in [0, b')$.

Corollary 3. $L_n(y)$ is disconjugate on I iff no hyperplane intersects $\underline{u}$ at more than $n-1$ points on J, including multiplicity.

Corollary 4. $L_n(y)$ has a non-vanishing solution on I iff there exists a hyperplane which does not intersect $\underline{u}(s)$ on J.

The oscillatory properties of solutions of $L_n(y)$ are simply recognizable from the behaviour of curves $\underline{u}$ on S_{n-1} and some known results are easy to derive, e.g.,

(Sansone 1948,[3]): There exists an equation $L_3(y)$ on $[a, \infty)$, every solution of which is oscillatory. For construction of such $L_3(y)$ only a curve $\underline{u}$, $[\underline{u}, \underline{u}', \underline{u}''] \neq 0$, on S_2 is sufficient to be considered, which is intersected infinitely many times by every plane $c_1 \xi_1 + c_2 \xi_2 + c_3 \xi_3 = 0$.

Also a construction of $L_3(y)$ having a non-trivial oscillatory solution and every linearly independent on it being non-oscillatory is rather easy.

A constructive characterization of all conjugate points for general $L_3(y)$, as required in [1],p.450, is given by the behaviour of curves on S_2. Hence Theorems 2.9, 2.10, Lemmas 2.15, 2.16 in [1], Theorems 4.1, 4.2, 4.7, 4.8 in [5] and others are obvious.

The known examples suggest the affirmative answer ([2],[4]) to the unsolved problem ([1],p.450): If $L_3(y)$ is oscillatory on $[a, \infty)$, then, is its adjoint equation also oscillatory? However, using the above considerations it can be shown that

Theorem 3. There exists an oscillatory equation $L_3(y)$ such that its adjoint equation is non-oscillatory.

The described geometrical approach makes it possible to see the whole situation and not only to consider the separate examples as motivation for possible form of theorems. And oscillatory properties of solutions can be studied for all equivalent differential equations without respect to any change of dependent or independent variables.

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