Commentationes Mathematicae Universitatis Carolinae

12,2 (1971)

THE LATTICE OF BI-NUMERATIONS OF ARITHMETIC, II

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This paper is a direct continuation of our [6]. The knowledge of [6] is presupposed. Similarly as in [6], in the whole paper $\mathcal{A} = \langle A, K \rangle$ denotes a fixed axiomatic theory with the following properties:

(1) A is a primitive recursive set,

(2) $\mathcal{A}$ is consistent,

(3) $\mathcal{P} \subseteq \mathcal{A}$ ($\mathcal{P}$ is the Peano's arithmetic).

Numbering of definitions and theorems in this paper begins with 3.1; references like 2.24 or 1.18 refer to definitions and theorems from [6].

III. Reducibility; a non-describability theorem

We shall now study the problem of reducibility of elements of $[\mathcal{B}in]$. We recall the definition:

3.1. Definition. An element z of a lattice $\underline{M} = \langle M, \leq, \cap, \cup \rangle$ is irreducible if, for each x, $y \in M$, $x \cup y = z$ implies $x = z$ or $y = z$.

3.2. Theorem. Let $\mathcal{A}$ be reflexive, let $\gamma, \beta \in \mathcal{B}in$ and suppose $\gamma <_{\mathcal{A}} \beta$. Then there is a

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AMS, Primary 02D99 Ref.Ž. 2.664
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$\delta \in \mathcal{B}in$ such that

$$(*)\quad \begin{cases} \delta <_{\mathcal{A}} \beta \\ [\gamma] \cup [\delta] = [\beta] \end{cases} .$$

The main idea of the proof: Let $\alpha' \in \mathcal{B}in$ such that $\alpha' <_{\mathcal{A}} \gamma$. Put

$$\delta'(x) = \alpha'(x) \vee Fm_K^{(\mathcal{M})}(x) \wedge$$

$$\wedge \bigvee_{y<x} [\,Pr\,f_\beta(\bar{\varphi}_\beta, y) \wedge \bigwedge_{z<y} \sim Pr\,f_\gamma(\bar{\varphi}_\gamma, z)\,] .$$

Evidently, $\delta' \leqq_{\mathcal{A}} \beta$ and $[\gamma] \cup [\delta'] = [\beta]$. But it is not clear whether $\delta' \not\equiv_{\mathcal{A}} \beta$. So we modify the definition of δ' and find a δ satisfying $(*)$ in the form

$$\alpha(x) \vee Fm_K^{(\mathcal{M})}(x) \wedge \bigvee_{y<x} [\,Pr\,f_\beta(\bar{\eta}_1, y) \wedge \bigwedge_{z<y} \sim Pr\,f_\gamma(\bar{\eta}_2, z)\,] .$$

The following lemma gives a necessary and sufficient condition for the existence of a $\delta \in \mathcal{B}in$ with required properties $(*)$.

3.3. Lemma. Let $\beta, \gamma \in \mathcal{B}in$ and let $\gamma <_{\mathcal{A}} \beta$. There exists a $\delta \in \mathcal{B}in$ satisfying $(*)$ if and only if there exist a formula $\alpha \in \mathcal{B}in$ and a formula $\psi(y)$ which is a PR-formula in $\mathcal{P}$ with exactly one free variable y such that

(1) $\vdash_{\mathcal{A}} (\sim Con_\beta \wedge Con_\gamma) \rightarrow \bigvee_y \psi(y)$,

(2) $\nvdash_{\mathcal{A}} (\sim Con_\beta \wedge Con_\alpha) \rightarrow \bigvee_y \psi(y)$.

Proof of Lemma 3.3. Let $\delta \in \mathcal{B}in$ satisfy the conditions $(*)$. It suffices to put $[\alpha] = [\gamma] \cap [\delta]$

and $\psi(y) = \mathrm{Prf}_{\sigma}(\overline{0 \approx 1}, y)$.

Conversely, let $\psi(y)$ and $\alpha \in \mathrm{Bin}$ satisfy the conditions (1) and (2). Put

$$\sigma(x) = \alpha(x) \vee \mathrm{Fm}_K^{(\mathcal{M})}(x) \wedge \bigvee_{y_1, y_2 < x} (\psi(y_1) \wedge \mathrm{Prf}_{\beta}(\overline{0 \approx 1}, y_2)).$$

By (1) and the definition of σ, we have $\vdash_{\mathcal{A}} \mathrm{Con}_{\beta} \leftrightarrow$ $\leftrightarrow (\mathrm{Con}_{\gamma} \wedge \mathrm{Con}_{\sigma})$, i.e. $[\gamma] \cup [\sigma] = [\beta]$. By (2) and the definition of σ, we have $\not\vdash_{\mathcal{A}} \mathrm{Con}_{\sigma} \rightarrow \mathrm{Con}_{\beta}$, i.e. $\sigma <_{\mathcal{A}} \beta$.

<u>Proof</u> of Theorem 3.2. By 2.11, we can assume $\vdash_{\mathcal{A}} \bigwedge_x (\gamma(x) \rightarrow \beta(x))$. Using the diagonal construction 1.9 and Lemma 1.1 determine η such that

(1) $\vdash_{\mathcal{P}} \eta \leftrightarrow \bigwedge_y (\mathrm{Prf}_{\gamma}(\overline{\eta}, y) \rightarrow \bigvee_{x<y} \mathrm{Prf}_{\beta}(\overline{\sim \eta}, x))$.

We shall prove

(2) $\not\vdash_{\mathcal{A}} \eta$.

Let $\vdash_{\mathcal{A}} \eta$ and let d be a proof of η in $\mathcal{A}$. Then $\vdash_{\mathcal{A}} \bigvee_{x<\overline{d}} \mathrm{Prf}_{\beta}(\overline{\sim \eta}, x)$, and therefore, by Lemma 3.1 [1], $\vdash_{\mathcal{A}} \sim \eta$, because β bi-numerates A. It is a contradiction and so we obtain $\not\vdash_{\mathcal{A}} \eta$.

Put

(3) $\psi(y) = \mathrm{Prf}_{\beta}(\overline{\sim \eta}, y) \wedge \bigwedge_{x<y} \sim \mathrm{Prf}_{\gamma}(\overline{\eta}, x)$.

Evidently, $\psi(y)$ is a PR-formula in $\mathcal{P}$ and $\mathrm{Fv}(\psi) =$ $= \{y\}$. We shall prove

(4) $\vdash_{\mathcal{A}} \sim \eta \rightarrow (\sim \mathrm{Con}_{\gamma} \wedge \sim \bigvee_y \psi(y))$.

In $\mathcal{A}$, suppose $\sim \eta$. Then $\bigvee_y [\mathrm{Prf}_{\gamma}(\overline{\eta}, y) \wedge$ $\wedge \bigwedge_{x<y} \sim \mathrm{Prf}_{\beta}(\overline{\sim \eta}, x)]$ and consequently

$\sim (\bigvee_y [Prf_\beta (\overline{\sim \eta}, y) \wedge \bigwedge_{x<y} \sim Prf_\gamma (\overline{\eta}, x)]) .$

The last formula is $\sim \bigvee_y \psi(y)$. From the assumption $\sim \eta$ we have $Pr_\gamma(\overline{\eta})$. On the other hand, by 1.7, $\sim \eta$ implies $Pr_\gamma(\overline{\sim \eta})$, because $\sim \eta$ is an RE-formula in $\mathcal{P}$. Consequently, we obtain $\sim Con_\gamma$.

We shall now prove

(5) $\vdash_A (\sim Con_\beta \wedge \sim \bigvee_y \psi(y)) \rightarrow \sim \eta$.

In A, suppose $\sim Con_\beta$ and $\sim \bigvee_y \psi(y)$. Then

$\bigwedge_y (Prf_\beta(\overline{\sim\eta}, y) \rightarrow \bigvee_{x<y} (\overline{\eta}, x)),\ \bigvee_y Prf_\beta(\overline{\sim\eta}, y),$

$\bigvee_y (Prf_\gamma(\overline{\eta}, y) \wedge \bigwedge_{x<y} \sim Prf_\beta(\overline{\sim\eta}, x))$

and consequently $\sim \eta$.

(4) and (5) imply

(6) $\vdash_A (\sim Con_\beta \wedge Con_\gamma) \rightarrow \bigvee_y \psi(y)$.

Put $E = A \cup \{\sim \eta\}$. The theory $\mathcal{E} = \langle E, K \rangle$ is consistent by (2). By (4), we have

(7) $\vdash_{\mathcal{E}} \sim Con_\beta$.

Let $\varepsilon(x)$ be a PR-formula in $\mathcal{P}$ defined as follows: $\varepsilon(x) = \gamma(x) \vee x \approx \overline{\sim \eta}$. Evidently, $\varepsilon(x)$ bi-numerates E. Using the diagonal construction 1.9, determine φ such that

$$\vdash_{\mathfrak{A}} \varphi \leftrightarrow \bigwedge_x (Prf_\varepsilon(\overline{\varphi}, x) \rightarrow \sim Con_{\beta \restriction x}).$$

Put $\alpha(x) = \beta(x) \wedge \bigwedge_{y<x} \sim Prf_\varepsilon(\overline{\varphi}, y)$. Evidently, $\alpha \in Bin$. Analogously as in the proof of 7.4 [1], one can prove

(8) $\nvdash_{\mathcal{E}} \varphi$

(9) $\vdash_{\mathcal{A}} \sim \varphi \rightarrow \underset{\cdot}{Con}_{\alpha}$;

(7), (8) and (9) give

(10) $\not\vdash_{\mathcal{A}} (\sim \underset{\cdot}{Con}_{\beta} \wedge \underset{\cdot}{Con}_{\alpha}) \rightarrow \eta$.

(10) and (4) give

(11) $\not\vdash_{\mathcal{A}} (\sim \underset{\cdot}{Con}_{\beta} \wedge \underset{\cdot}{Con}_{\alpha}) \rightarrow \bigvee_{y} \psi(y)$.

(11) and (6) show that the conditions of Lemma 3.3 are satisfiable.

3.4. Corollary. If $\mathcal{A}$ is reflexive, then every element of $[\mathcal{B}in]$ is reducible.

Theorem 3.2 enables us to formulate a partial result on the "non-describability" of elements of $[\mathcal{B}in]$. First we define some notions and prove a lemma.

3.5. Definition. Let $\varphi \in Fm_{K_1}$. φ is said to be a Δ_0 -formula, $\varphi \in \Delta_0$, if it belongs to the least class containing all atomic formulas in K_1 , closed under $\wedge$ and $\sim$ and which contains with every formula φ_1 also $\bigvee_{w} (u \leq w \leq v \wedge \varphi_1)$, where u, v, w are distinct variables.

3.6. Definition. Let $\varphi \in Fm_{K_1}$. φ is said to be a Σ_1 -formula, $\varphi \in \Sigma_1$, if either $\varphi \in \Delta_0$ or φ has the form $\bigvee_{u_0} \dots \bigvee_{u_K} \varphi_1$, where $\varphi_1 \in \Delta_0$ and $u_0, \dots, u_K$ are distinct variables.

Remark. These definitions are analogous to the Lévy's definitions of Δ_0 -formulas and Σ_1 -formulas of the set theory [4].

3.7. Lemma. Let $\underline{M} = \langle M, \leq, \cap, \cup \rangle$ be a lattice, let $\varphi \in \Delta_0$ and $Fv(\varphi) = \{u_0, \dots, u_{k-1}\}$. Suppose $a, b \in M$ and $a \leq b$. Furthermore, let $a_0, \dots, a_{k-1}$ be elements of M such that $a \leq a_i \leq b$ for $i = 0, \dots, k-1$. Then $\underline{M} \models \varphi[a_0, \dots, a_{k-1}]$ if and only if $\langle a; b \rangle \models \varphi[a_0, \dots, a_{k-1}]$.

Proof by induction on formulas.

(a) If φ is atomic then the assertion is obvious.

(b) Let φ have the form $\psi_1 \wedge \psi_2$. For the sake of brevity of notation, suppose $Fv(\psi_1) = Fv(\psi_2) = Fv(\varphi)$. Then

$\underline{M} \models (\psi_1 \wedge \psi_2)[a_0, \dots, a_{k-1}]$ iff

$(\underline{M} \models \psi_1[a_0, \dots, a_{k-1}]$ and $\underline{M} \models \psi_2[a_0, \dots, a_{k-1}])$ iff $(\langle a; b \rangle \models \psi_1[a_0, \dots, a_{k-1}]$ and $\langle a; b \rangle \models \psi_2[a_0, \dots, a_{k-1}]$ iff $\langle a; b \rangle \models (\psi_1 \wedge \psi_2)[a_0, \dots, a_{k-1}]$.

(c) If φ has the form $\sim \psi$ the induction step is trivial.

(d) Let φ be $\bigvee_{v_s} (v_p \leq v_s \leq v_r \wedge \psi)$. We can suppose $s \geq p, r$. Suppose $\underline{M} \models \varphi[a_0, \dots, a_{k-1}]$. Then there is an $e \in M$ such that $a \leq a_p \leq e \leq a_r \leq b$ and $\underline{M} \models \psi[a_0, \dots, a_{k-1}, e]$. By the induction hypothesis, $\langle a; b \rangle \models \psi[a_0, \dots, a_{k-1}, e]$ and consequently $\langle a; b \rangle \models \bigvee_{v_s} (v_p \leq v_s \leq v_r \wedge \psi)[a_0, \dots, a_{k-1}]$. The converse implication is proved analogously.

3.8. Definition. Let $\underline{M} = \langle M, \leq, \cap, \cup \rangle$ be a lattice and let $\langle a_0, \dots, a_{k-1} \rangle \in M^k$. The k-tuple $\langle a_0, \dots, a_{k-1} \rangle$ is said to be Σ_1-definable in M if there is a Σ_1-formula φ such that $\langle a_0, \dots, a_{k-1} \rangle$ is the unique k-tuple satisfying φ in M.

3.9. <u>Theorem</u> on Σ_1-non-definability. Let A be reflexive. Then no k-tuple of elements of $[\underline{Bin}]$ is Σ_1-definable in $[\underline{Bin}]$. Moreover, if $\varphi \in \Sigma_1$, $Fv(\varphi) = \{u_0, \dots, u_{k-1}\}$, $[\alpha_0], \dots, [\alpha_{k-1}] \in [\underline{Bin}]$ and if $[\underline{Bin}] \models \varphi[[\alpha_0], \dots, [\alpha_{k-1}]]$, then there are $[\alpha'_0], \dots, [\alpha'_{k-1}] \in [\underline{Bin}]$ such that $[\alpha'_i] \neq [\alpha'_j]$ for all $i, j = 0, \dots, k-1$ and $[\underline{Bin}] \models \varphi[[\alpha'_0], \dots, [\alpha'_{k-1}]]$.

<u>Proof</u>. Let φ be a Σ_1-formula and let $[\underline{Bin}] \models \varphi[[\alpha_0], \dots, [\alpha_{k-1}]]$. We can suppose that φ has the form $\bigvee_{v_k} \dots \bigvee_{v_{n-1}} \psi(v_0, \dots, v_{n-1})$, where $\psi \in \Delta_0$. It follows that there are $[\alpha_k], \dots, [\alpha_{n-1}] \in [\underline{Bin}]$ such that $[\underline{Bin}] \models \psi[[\alpha_0], \dots, [\alpha_{n-1}]]$. Put $[\beta] = [\alpha_0] \cup \dots \cup [\alpha_{n-1}]$ and let $[\gamma] <_A [\beta]$, $[\gamma] \leq_A [\alpha_0] \cap \dots \cap [\alpha_{n-1}]$ (cf. 2.6). By Theorem 3.2, there is a $[\sigma] <_A [\beta]$ such that $[\gamma] \cup [\sigma] = [\beta]$. Put $[\varepsilon] = [\gamma] \cap [\sigma]$. By 1.19 there exists an isomorphism f of $\langle [\gamma] ; [\beta] \rangle$ and $\langle [\varepsilon] ; [\sigma] \rangle$. By Theorem 3.7 we have $\langle [\gamma] ; [\beta] \rangle \models \psi[[\alpha_0], \dots, [\alpha_{n-1}]]$, and putting $[\alpha'_i] = f([\alpha_i])$ $(i = 0, \dots, n-1)$ we obtain $\langle [\varepsilon], [\sigma] \rangle \models \psi[[\alpha'_0], \dots, [\alpha'_{n-1}]]$ by Theorem 1.20. Using again Theorem 3.7 we have $[\underline{Bin}] \models \psi[[\alpha'_0], \dots, [\alpha'_{n-1}]]$, which implies $[\underline{Bin}] \models \varphi[[\alpha'_0], \dots, [\alpha'_{k-1}]]$. Since the intervals

$\langle [\gamma] ; [\beta] \rangle$ and $\langle [\varepsilon] ; [\sigma] \rangle$ are disjoint we have $[\alpha_i] \neq [\alpha'_j]$ for $i, j = 0, \dots, k-1$.

3.10. Remark. It can be easily seen from the proof that we can obtain an infinite sequence of distinct k-tuples of elements of $[\underline{Bin}]$ satisfying φ.

IV. Relative complements in the lattice of bi-numerations of arithmetic

In this section we are going to study the problem of existence of relative complements in the lattice $[\underline{Bin}]$. Roughly speaking, we show that in every non-trivial interval there are many elements having relative complement (w.r.t. this interval) and many elements having no relative complement (w.r.t. this interval).

We recall the definition.

4.1. Definition. Let $\underline{M} = \langle M, \leq, \cap, \cup \rangle$ be a lattice and let $a, b, c, d \in M$. Suppose $a \leq \leq b$. Then d is said to be a relative complement to c with respect to a, b if $c \cap d = a$ and $c \cup d = b$.

4.2. Definition. Let $\underline{M} = \langle M, \leq, \cap, \cup \rangle$ be a lattice, $a, b, c \in M$ and suppose $a \leq b$. Then c is said to be complementible w.r.t. a, b if there exists a $d \in M$ which is a relative complement w.r.t. a, b.

The following lemma can be easily proved from the axioms of the lattice theory.

4.3. Lemma. Let $\underline{M} = \langle M, \leq, \cap, \cup \rangle$ be a lattice, $a, b, c, d, d' \in M$ and suppose $a \leq b$. Then

(i) c is a relative complement to d w.r.t. a, b if and only if d is a relative complement to c w.r.t. a, b ;

(ii) if c is complementible w.r.t. a, b , then $a \leq c \leq b$;

(iii) if $\underline{M}$ is distributive and d , d' are relative complements to c w.r.t. a, b , then $d =$ $= d'$.

4,4. Lemma. Let $\underline{M} = \langle M , \leq , \cap , \cup \rangle$ be a distributive lattice, $a, a_1 , b, b_1 , c \in M$ and suppose $a \leq a_1 < c < b_1 \leq b$. Then

(i) if c is complementible w.r.t. a , b , then c is complementible w.r.t. a_1 , b_1 ;

(ii) if c is complementible w.r.t. a_1 , b_1 and both a_1 and b_1 are complementible w.r.t. a , b , then c is complementible w.r.t. a, b ;

(iii) if a_1 and b_1 be complementible w.r.t. a , b , then both $a_1 \cup b_1$ and $a_1 \cap b_1$ are complementible w.r.t. a, b .

Proof. (i) Let d be the relative complement to c w.r.t. a, b . Put $d' = (d \cap b_1) \cup a_1$. By elementary calculation, $d' \cap c = a_1$ and $d' \cup c = b_1$.

(ii) Let d' be the relative complement to c w.r.t. a_1 , b_1 , let d_1 be the relative complement to a_1 w.r.t. a, b and let d_2 be the relative complement to b_1 w.r.t. a, b . Put $d = (d_2 \cup d') \cap d_1$. By elementary calculation, $d \cup c = b$ and $d \cap c = a$.

(iii) Let c_1, d_1 be the relative complements to a_1, b_1 respectively w.r.t. a, b. It can be easily shown that $c_1 \cap d_1$ is the relative complement to $a_1 \cup b_1$ w.r.t. a, b and that $c_1 \cup d_1$ is the relative complement to $a_1 \cap b_1$ w.r.t. a, b.

4.5. Lemma. Let α, β, γ, $\delta \in Bin$ and suppose $\alpha \leq_{\mathcal{R}} \gamma$, $\delta \leq_{\mathcal{R}} \beta$. Then

(i) $[\gamma] \cup [\delta] = [\beta]$ if and only if

$\vdash_{\mathcal{R}} \sim Con_\beta \wedge Con_\gamma \rightarrow \sim Con_\delta$;

(ii) $[\gamma] \cap [\delta] = [\alpha]$ if and only if

$\vdash_{\mathcal{R}} \sim Con_\gamma \wedge Con_\alpha \rightarrow Con_\delta$;

(iii) $[\delta]$ is a relative complement to $[\gamma]$ w.r.t. $[\alpha]$, $[\beta]$ if and only if $\vdash_{\mathcal{R}} (\sim Con_\beta \wedge Con_\alpha) \rightarrow$
$\rightarrow (Con_\gamma \leftrightarrow \sim Con_\delta)$.

The lemma follows from Corollaries 2.20 and 2.22.

4.6. Lemma. Let α, β, $\gamma \in Bin$ and suppose $\alpha \leq_{\mathcal{R}} \gamma \leq_{\mathcal{R}} \beta$. Then $[\gamma]$ is complementible w.r.t. $[\alpha]$, $[\beta]$ if and only if there exists a formula $\varphi(y)$ which is a PR-formula in $\mathcal{P}$ with exactly one free variable y and such that

(1) $\vdash_{\mathcal{R}} (\sim Con_\beta \wedge Con_\alpha) \rightarrow (Con_\gamma \leftrightarrow \bigvee_y \varphi(y))$.

Proof. (i) Let $[\delta]$ be the relative complement to $[\gamma]$ w.r.t. $[\alpha]$, $[\beta]$. Put $\varphi(y) = Prf_\delta(\overline{0 \approx 1}, y)$. Evidently, $\varphi(y)$ is a PR-formula in $\mathcal{P}$ and $Fv(\varphi) = \{y\}$. (1) follows from Lemma 4.5 (iii).

(ii) Let $\varphi(y)$ be a PR-formula in $\mathcal{P}$,

$Fv(\varphi) = \{y\}$ and suppose (1). Put

$$\sigma(x) = \alpha(x) \vee Fm_K^{(\mathcal{M})}(x) \wedge$$

$$\wedge \bigvee_{y_1, y_2 < x} (\varphi(y_1) \wedge Prf_\beta(\overline{0 \approx 1}, y_2)) .$$

Evidently, $\sigma \in Bin$, $\alpha \leq_{\mathcal{A}} \sigma \leq_{\mathcal{A}} \beta$ and

$$\vdash_{\mathcal{A}} (\sim Con_\beta \wedge Con_\alpha) \longrightarrow (\sim Con_\sigma \leftrightarrow \bigvee_y \varphi(y)) .$$

Therefore, by Lemma 4.5 (iii), $[\sigma]$ is the relative complement to $[\gamma]$ w.r.t. $[\alpha]$, $[\beta]$.

4.7. <u>Theorem</u>. Let α, β, $\gamma \in Bin$ and suppose $\alpha \leq_{\mathcal{A}} \gamma \leq_{\mathcal{A}} \beta$. Then

(i) if $[\gamma]$ is aomplementible w.r.t. $[\alpha]$, $[\beta]$ then there exists an $n \in \omega$ such that

(1) $\vdash_{\mathcal{A}} (\sim Con_\beta \wedge Con_\gamma) \rightarrow Pr_{[\mathcal{A} \restriction n]}(\overline{Con_\alpha \rightarrow Con_\gamma})$;

(ii) if $\mathcal{A}$ is reflexive and (1) holds then $[\gamma]$ is complementible w.r.t. $[\alpha]$, $[\beta]$; in fact, if we put

$$\sigma(x) = \alpha(x) \vee Fm_K^{(\mathcal{M})}(x) \wedge$$

$$\wedge \bigvee_{y_1, y_2 < x} (Prf_{[\mathcal{A} \restriction n]}(\overline{Con_\alpha \rightarrow Con_\gamma}, y) \wedge Prf_\beta(\overline{0 \approx 1}, y_2)) ,$$

then $[\sigma]$ is the relative complement to $[\gamma]$ w.r.t. $[\alpha]$, $[\beta]$.

<u>Proof.</u> (i) Let $[\gamma]$ be complementible w.r.t. $[\alpha]$, $[\beta]$. By Lemma 4.6, there exists a formula $\varphi(y)$ with exactly one free variable y such that $\bigvee_y \varphi(y)$ is an RE-formula in $\mathcal{P}$ and

(2) $\vdash_{\mathcal{A}} (\sim Con_\beta \wedge Con_\alpha) \rightarrow (Con_\gamma \leftrightarrow \bigvee_y \varphi(y))$.

Therefore, there exists an $n_1 \in \omega$ such that

(3) $\vdash_{\mathcal{P}} Pr_{[\mathcal{A}\restriction n_1]}(\overline{(\sim Con_\beta \wedge Con_\alpha) \rightarrow (Con_\gamma \leftrightarrow \bigvee_y \varphi(y))})$.

Let ψ be an RE-formula such that

(4) $$\vdash_{\mathcal{P}} \psi \leftrightarrow \bigvee_y \varphi(y) .$$

Evidently, we can suppose $\psi \in St_{K_0}$. Therefore, there exists an $n_2 \in \omega$ such that

(5) $$\vdash_{\mathcal{P}} Pr_{[\mathcal{A}\restriction n_2]}(\overline{\psi \leftrightarrow \bigvee_y \varphi(y)}) .$$

By Lemma 3.9 [1] and Corollary 5.5 [1], we have

(6) $$\vdash_{\mathcal{P}} \psi \longrightarrow Pr_{[Q]}(\overline{\psi}) .$$

Hence, by (4), (5), (6) there exists an $n_3 \in \omega$ such that

(7) $$\vdash_{\mathcal{P}} \bigvee_y \varphi(y) \longrightarrow Pr_{[\mathcal{A}\restriction n_3]}(\overline{\bigvee_y \varphi(y)}) .$$

$\sim Con_\beta$ is an RE-formula in $\mathcal{P}$. We can prove that there exists $n_4 \in \omega$ such that

(8) $$\vdash_{\mathcal{P}} \sim Con_\beta \longrightarrow Pr_{[\mathcal{A}\restriction n_4]}(\overline{\sim Con_\beta})$$

analogously as (7).

Taking $n = max(n_1, n_3, n_4)$ we have:

$\vdash_{\mathcal{A}} (\sim Con_\beta \wedge Con_\gamma) \rightarrow \bigvee_y \varphi(y)$ (by (2) and the assumption $\alpha \leq_{\mathcal{A}} \gamma$),

$\vdash_{\mathcal{A}} (\sim Con_\beta \wedge Con_\gamma) \rightarrow Pr_{[\mathcal{A}\restriction n]}(\overline{\bigvee_y \varphi(y)})$ (by (7)),

$\vdash_{\mathcal{A}} (\sim Con_\beta \wedge Con_\gamma) \rightarrow Pr_{[\mathcal{A}\restriction n]}(\overline{\sim Con_\beta \wedge Con_\alpha \rightarrow Con_\gamma})$ (by (2)),

$\vdash_{\mathcal{A}} (\sim Con_\beta \wedge Con_\gamma) \rightarrow Pr_{[\mathcal{A}\restriction n]}(\overline{Con_\alpha \rightarrow Con_\gamma})$ (by (8)).

(ii) Let $\mathcal{A}$ be reflexive and let δ be as indicated. Suppose that (1) holds. Evidently, $\delta \in Bin$ and $\vdash_{\mathcal{A}} (\sim Con_{\beta} \wedge Con_{\gamma}) \rightarrow \sim Con_{\delta}$. It follows from Lemma 4.5 that it suffices to show that

(9) $\vdash_{\mathcal{A}} (\sim Con_{\gamma} \wedge Con_{\alpha}) \rightarrow \sim Pr_{[\mathcal{A}\restriction m]}(\overline{Con_{\alpha} \rightarrow Con_{\gamma}})$.

If $\alpha =_{\mathcal{A}} \gamma$, then (9) is evident. Suppose $\alpha <_{\mathcal{A}} \gamma$. Then $\mathcal{A} + \{\sim Con_{\gamma} \wedge Con_{\alpha}\}$ is consistent and, by 5.8 (ii) [1], reflexive. Therefore $\vdash_{\mathcal{A}} \sim Con_{\gamma} \wedge$ $\wedge Con_{\alpha} \rightarrow Con_{[(\mathcal{A} + \{\sim Con_{\gamma} \wedge Con_{\alpha}\})\restriction m]}$ for each $m \in$ $\in \omega$. In particular, putting $n' = max(n, \sim Con_{\gamma} \wedge Con_{\alpha})$, we have $\vdash_{\mathcal{A}} (\sim Con_{\gamma} \wedge Con_{\alpha}) \rightarrow Con_{[(\mathcal{A} + \{\sim Con_{\gamma} \wedge Con_{\alpha}\})\restriction m']}$, i.e.

(10) $\vdash_{\mathcal{A}} (\sim Con_{\gamma} \wedge Con_{\alpha}) \rightarrow \sim Pr_{[\mathcal{A}\restriction m']}(\overline{Con_{\alpha} \rightarrow Con_{\gamma}})$.

Evidently,

(11) $\vdash_{\mathcal{P}} \sim Pr_{[\mathcal{A}\restriction m']}(\overline{Con_{\alpha} \rightarrow Con_{\gamma}}) \rightarrow \sim Pr_{[\mathcal{A}\restriction m]}(\overline{Con_{\alpha} \rightarrow Con_{\gamma}})$.

(10) and (11) show that (9) holds.

4.8. Corollary. Let α , β , γ , $\delta \in Bin$ and suppose $\alpha \leq_{\mathcal{A}} \beta$.

(i) If $[\delta]$ is the relative complement to $[\gamma]$ w.r.t. $[\alpha]$, $[\beta]$, then there exists an $n \in \omega$ such that

(1) $\gamma =_{\mathcal{A}} \alpha(x) \vee Fm_K^{(\mathcal{M})}(x) \wedge \bigvee_{y_1, y_2 < x} (Prf_{[\mathcal{A}\restriction m]}(\overline{Con_{\alpha} \rightarrow Con_{\delta}}, y_1) \wedge Prf_{\beta}(\overline{0 \approx 1}, y_2))$,

(2) $\delta =_{\mathcal{A}} \alpha(x) \vee Fm_K^{(\mathcal{M})}(x) \wedge \bigvee_{y_1, y_2 < x} (Prf_{[\mathcal{A}\restriction m]}(\overline{Con_{\alpha} \rightarrow Con_{\gamma}}, y_1) \wedge Prf_{\beta}(\overline{0 \approx 1}, y_2))$

and, moreover,

(3) $\vdash_{\mathcal{A}} (\sim Con_{\beta} \wedge Con_{\alpha}) \rightarrow (Pr_{[\mathcal{A} \upharpoonright m]}(\overline{Con_{\alpha} \rightarrow Con_{\gamma}}) \vee$

$\vee\, Pr_{[\mathcal{A} \upharpoonright m]}(\overline{Con_{\alpha} \rightarrow Con_{\sigma}}))$;

(ii) if $\mathcal{A}$ is reflexive and (1), (2), (3) hold, then $[\sigma]$ is the relative complement to $[\gamma]$ w.r.t. $[\alpha]$, $[\beta]$.

4.9. Theorem. Let α, β, $\xi \in Bin$ and let $\alpha <_{\mathcal{A}} \beta$. Put $\mathcal{E} = \mathcal{A} + \{\sim Con_{\beta} \wedge Con_{\alpha}\}$ and $\varepsilon(x) = \xi(x) \vee x \approx \sim \overline{Con_{\beta} \wedge Con_{\alpha}}$. Let γ be defined as follows:

$$\gamma(x) = \alpha(x) \vee Fm_K^{(\mathcal{M})}(x) \wedge \bigvee_{y_1, y_2 < x} (\sim R_{\varepsilon}(y_1) \wedge Prf_{\beta}(\overline{0 \approx 1}, y_2)).$$

Then $[\gamma]$ is complementible w.r.t. $[\alpha]$, $[\beta]$ if and only if

(1) $\vdash_{\mathcal{E}} \sim Con_{\varepsilon}$, i.e. if and only if

(1)′ $\vdash_{\mathcal{A}} (\sim Con_{\beta} \wedge Con_{\alpha}) \rightarrow Pr_{\xi}(\overline{\sim Con_{\alpha}})$.

Proof. Note that $\gamma \in Bin$, $\alpha <_{\mathcal{A}} \gamma <_{\mathcal{A}} \beta$ (cf. Theorem 2.12) and

(2) $\vdash_{\mathcal{A}} (\sim Con_{\beta} \wedge Con_{\alpha}) \leftrightarrow (Con_{\gamma} \leftrightarrow \rho_{\varepsilon})$.

(i) Let $[\gamma]$ be complementible w.r.t. $[\alpha]$, $[\beta]$. By Theorem 4.7, there exists an $m \in \omega$ such that

(3) $\vdash_{\mathcal{A}} (\sim Con_{\beta} \wedge Con_{\gamma}) \rightarrow Pr_{[\mathcal{A} \upharpoonright m]}(\overline{Con_{\alpha} \rightarrow Con_{\gamma}})$.

Hence

(4) $\vdash_{\mathcal{A}} (\sim Con_{\beta} \wedge Con_{\gamma}) \rightarrow Pr_{[\mathcal{A} \upharpoonright m]}(\overline{(\sim Con_{\beta} \wedge Con_{\alpha}) \rightarrow Con_{\gamma}})$.

(2) gives

(5) $\vdash_{\mathcal{P}} Pr_{\varepsilon}(\overline{Con_{\gamma} \leftrightarrow \varphi_{\varepsilon}})$.

(4) and (5) show that $\vdash_{\mathcal{A}} (\sim Con_{\beta} \wedge Con_{\gamma}) \rightarrow$ $\rightarrow Pr_{\varepsilon}(\overline{\varphi}_{\varepsilon})$ and therefore

(6) $\vdash_{\mathcal{A}} (\sim Con_{\beta} \wedge Con_{\gamma}) \rightarrow \sim Con_{\varepsilon}$.

By (2), $\vdash_{\mathcal{A}} (\sim Con_{\gamma} \wedge Con_{\alpha}) \rightarrow \sim \varphi_{\varepsilon}$. Hence

(7) $\vdash_{\mathcal{A}} (\sim Con_{\gamma} \wedge Con_{\alpha}) \rightarrow \sim Con_{\varepsilon}$.

(6) and (7) give $\vdash_{\mathcal{A}} (\sim Con_{\beta} \wedge Con_{\alpha}) \rightarrow \sim Con_{\varepsilon}$.

(ii) Let $\vdash_{\mathcal{C}} \sim Con_{\varepsilon}$. Put

$$\delta(x) = \alpha(x) \vee Fm_K^{(\mathcal{M})}(x) \wedge \bigvee_{y_1, y_2 < x} [(Prf_{\varepsilon}(\overline{\sim \varphi_{\varepsilon}}, y_1) \wedge \wedge \bigwedge_{z < y_1} \sim Prf_{\varepsilon}(\overline{\varphi}_{\varepsilon}, z)) \wedge Prf_{\beta}(\overline{0 \approx 1}, y_2)] .$$

Evidently, $\delta \in Bin$ and $\alpha \leq_{\mathcal{A}} \delta \leq_{\mathcal{A}} \beta$. We have

$$\vdash_{\mathcal{P}} \sim Con_{\varepsilon} \rightarrow [\varphi_{\varepsilon} \leftrightarrow \bigvee_{y} (Prf_{\varepsilon}(\overline{\sim \varphi_{\varepsilon}}, y) \wedge \wedge \bigwedge_{z < y} \sim Prf_{\varepsilon}(\overline{\varphi}_{\varepsilon}, z))]$$

and it follows that $\vdash_{\mathcal{A}} (\sim Con_{\beta} \wedge Con_{\alpha}) \rightarrow (Con_{\gamma} \leftrightarrow$ $\leftrightarrow \sim Con_{\delta})$. Hence, by Lemma 4.5, $[\gamma]$ is complementible w.r.t. $[\alpha]$, $[\beta]$.

4.10. <u>Corollary</u>. Let α , β , γ_1 , $\gamma_2 \in Bin$ and let $\alpha \leq_{\mathcal{A}} \gamma_1 <_{\mathcal{A}} \gamma_2 \leq_{\mathcal{A}} \beta$. Suppose that both $[\gamma_1]$ and $[\gamma_2]$ are complementible w.r.t. $[\alpha]$, $[\beta]$. Then there exists a $\gamma \in Bin$ such that

(i) $\gamma_1 <_{\mathcal{A}} \gamma <_{\mathcal{A}} \gamma_2$ and

(ii) $[\gamma]$ is complementible w.r.t. $[\alpha]$, $[\beta]$.

<u>Proof</u>. It suffices to take γ from Theorem 4.9, where we replace α by γ_1 , β by γ_2 and $\S$ by γ_2 .
The assertion follows from Lemma 4.4.

4.11. Corollary. Let $\alpha, \beta \in \mathcal{B}in$, $\alpha <_{\mathcal{A}} \beta$. Denote by $\mathcal{C}omp(\alpha, \beta)$ the set of all $[\gamma]$ such that

(i) $\alpha \leq_{\mathcal{A}} \gamma \leq_{\mathcal{A}} \beta$.

(ii) $[\gamma]$ is complementible w.r.t. $[\alpha], [\beta]$.

Then the structure $\langle \mathcal{C}omp(\alpha, \beta), \leq_{\mathcal{A}}, \cap, \cup \rangle$ is an atomless (denumerable) Boolean algebra. (Note that it is known that all such algebras are isomorphic.)

We shall now be interested in non-complementible elements.

4.12. Theorem. Let $\mathcal{A}$ be reflexive, $\alpha, \beta \in \mathcal{B}in$ and suppose $\alpha <_{\mathcal{A}} \beta$. Then there exists a $\gamma \in \mathcal{B}in$ such that

(i) $\alpha <_{\mathcal{A}} \gamma <_{\mathcal{A}} \beta$,

(ii) $[\gamma]$ is non-complementible w.r.t. $[\alpha], [\beta]$.

Proof. Let $E = A \cup \{\sim \mathcal{C}on_{\beta} \wedge \mathcal{C}on_{\alpha}\}$, put $\varepsilon_1(x) = \alpha(x) \vee x \approx \overline{\sim \mathcal{C}on_{\beta} \wedge \mathcal{C}on_{\alpha}}$ and let $\mathcal{E} = \langle E, K \rangle$. Evidently, $\mathcal{E}$ is consistent and reflexive (cf. Theorem 5.8 [1]) and $\varepsilon_1(x)$ is a PR-formula in $\mathcal{P}$ bi-numerating E . Using the diagonal construction 5.1 [1], determine a φ such that

$$\vdash_{\mathcal{E}} \varphi \leftrightarrow \bigwedge_{x} (\mathcal{P}rf_{\varepsilon_1}(\overline{\varphi}, x) \rightarrow \sim \mathcal{C}on_{\varepsilon_1 \restriction x}) .$$

Suppose $\vdash_{\mathcal{E}} \varphi$. Then for some n , we would have $\vdash_{\mathcal{E}} \sim \mathcal{C}on_{\varepsilon_1 \restriction \overline{n}}$, which would make $\mathcal{E}$ inconsistent. Hence

(1) $\not\vdash_{\mathcal{C}} \varphi$.

Define ξ , ε , γ as follows:

$$\xi(x) = \alpha(x) \wedge \bigwedge_{y<x} \sim Pr\, f_{\varepsilon_1}(\overline{\varphi}, y) ,$$

$$\varepsilon(x) = \xi(x) \vee x \approx \overline{\sim Con_\beta \wedge Con_\alpha} ,$$

$$\gamma(x) = \alpha(x) \vee Fm_K^{(\mathcal{M})}(x) \wedge \bigvee_{y_1, y_2 < x} \sim R_\varepsilon(y_1) \wedge Pr\, f_\beta(\overline{0 \approx 1}, y_2) .$$

Evidently, ξ , $\gamma \in Bin$ and $\alpha <_{\mathcal{A}} \gamma <_{\mathcal{A}} \beta$.

We shall show

(2) $\not\vdash_{\mathcal{C}} \sim Con_{\xi \vee x \approx \overline{\sim Con_\beta \wedge Con_\alpha}}$, i.e. $\not\vdash_{\mathcal{C}} \sim Con_\varepsilon$.

Evidently,

(3) $\vdash_{\mathcal{P}} \sim \varphi \rightarrow \bigvee_{z} [Pr\, f_{\varepsilon_1}(\overline{\varphi}, z) \wedge Con_{\varepsilon_1 \restriction z} \wedge$

$\wedge \bigwedge_{y<z} \sim Pr\, f_{\varepsilon_1}(\overline{\varphi}, z)]$,

since $\vdash_{\mathcal{P}} Con_{\varepsilon_1 \restriction z} \wedge y < z \rightarrow Con_{\varepsilon_1 \restriction y}$.

By (1), $\vdash_{\mathcal{P}} Pr\, f_{\varepsilon_1}(\overline{\varphi}, z) \rightarrow z > \overline{n}$ for every $n \in \omega$, and therefore

(4) $\vdash_{\mathcal{P}} \sim \varphi \rightarrow \bigvee_{z} [Con_{\alpha \restriction z \vee x \approx \overline{\sim Con_\beta \wedge Con_\alpha}} \wedge$

$\wedge \bigwedge_{x} (\xi(x) \leftrightarrow \alpha(x) \wedge x \leq z)]$,

which immediately gives

(5) $\vdash_{\mathcal{P}} \sim \varphi \rightarrow Con_{\xi \vee x \approx \overline{\sim Con_\beta \wedge Con_\alpha}}$.

(2) follows from (1) and (5). Non-complementibility $[\gamma]$ w.r.t. $[\alpha]$, $[\beta]$ follows from (2) and Theorem 4.9.

4.13. Corollary. Let $\mathcal{A}$ be reflexive, α, $\beta \in Bin$ and suppose $\alpha <_{\mathcal{A}} \beta$; in this corollary "non-complementible" means "non-complementible w.r.t. $[\alpha]$, $[\beta]$ ".

(i) Non-complementible elements are dense in $\langle [\alpha]; [\beta] \rangle$; i.e., for every $\sigma, \tau \in Bin$ such that $\alpha \leq_{\mathcal{A}} \sigma <_{\mathcal{A}} \tau \leq_{\mathcal{A}} \beta$ there is a non-complementible $[\gamma]$ such that $\sigma <_{\mathcal{A}} \gamma <_{\mathcal{A}} \tau$.

(ii) Non-complementible elements are not closed w.r.t. the operations $\cup$, $\cap$; in fact, for every $\gamma \in Bin$ such that $\alpha <_{\mathcal{A}} \gamma \leq_{\mathcal{A}} \beta$ there are $\sigma, \tau >_{\mathcal{A}} \alpha$ such that $[\sigma] \cup [\tau] = [\gamma]$ and $[\sigma], [\tau]$ are non-complementible. Similarly, for every $\delta \in Bin$ such that $\alpha \leq_{\mathcal{A}} \delta <_{\mathcal{A}} \beta$ there are $\sigma, \tau <_{\mathcal{A}} \beta$ such that $[\sigma] \cap [\tau] = [\delta]$ and $[\sigma], [\tau]$ are non-complementible.

(Consequently, the interval $\langle [\alpha]; [\beta] \rangle$ is generated by its non-complementible elements.)

Proof. (i) follows from Theorem 4.12 and Lemma 4.4 (i).

(ii) Let $\alpha <_{\mathcal{A}} \gamma \leq_{\mathcal{A}} \beta$. By Corollary 4.10 there are $\sigma_1, \tau_1 \in Bin$ such that $\alpha <_{\mathcal{A}} \sigma_1$, $\tau_1 <_{\mathcal{A}} \gamma$ and $[\sigma_1] \cup [\tau_1] = [\gamma]$. It follows from the part (i) of this corollary that we can define non-complementible σ, τ such that $\sigma_1 <_{\mathcal{A}} \sigma <_{\mathcal{A}} \gamma$ and $\tau_1 <_{\mathcal{A}} \tau \leq_{\mathcal{A}} \gamma$. Evidently, $[\sigma] \cup [\tau] = [\gamma]$. The second part of the assertion can be proved analogously.

The following theorem shows that the dual theorem to Theorem 3.2 does not hold.

4.14. Theorem. Let $\mathcal{A}$ be ω -consistent and let $\alpha \in Bin$. Then there exists a $\gamma \in Bin$ such that

(i) $\alpha <_{\mathcal{A}} \gamma$,

(ii) $[\gamma]$ is non-complementible w.r.t. $[\alpha], [\beta]$ for any $\beta >_{\mathcal{A}} \gamma$; in other words

(iii)' there is no $\delta >_{\mathcal{A}} \alpha$ for which $[\gamma] \cap \cap [\delta] = [\alpha]$.

<u>Proof</u>. Note that the proof will only be a deeper analysis (formalization) of the proof of 7.5 [1].

Let $\mathcal{D} = \mathcal{A} + \{\sim \dot{Pr}_\alpha (\overline{\sim \dot{Con}_\alpha})\}$. To show that $\mathcal{D}$ is consistent, we shall show that $\not\vdash_{\mathcal{A}} \dot{Pr}_\alpha (\overline{\sim \dot{Con}_\alpha})$.

Let $\vdash_{\mathcal{A}} \dot{Pr}_\alpha (\overline{\sim \dot{Con}_\alpha})$, i.e. $\vdash_{\mathcal{A}} \bigvee_y \dot{Pr} f_\alpha (\overline{\sim \dot{Con}_\alpha}, y)$. It follows from ω -consistency of $\mathcal{A}$ that there exists an $n \in \omega$ such that $\not\vdash_{\mathcal{A}} \sim \dot{Pr} f_\alpha (\overline{\sim \dot{Con}_\alpha}, \bar{n})$. The formula $\dot{Pr} f_\alpha (\overline{\sim \dot{Con}_\alpha}, \bar{n})$ is a PR-formula in $\mathcal{P}$, and therefore decidable. Consequently, there exists an $n \in \omega$ such that $\vdash_{\mathcal{A}} \dot{Pr} f_\alpha (\overline{\sim \dot{Con}_\alpha}, \bar{n})$. Hence $\vdash_{\mathcal{A}} \sim \dot{Con}_\alpha$, since $\dot{Pr} f_\alpha$ bi-numerates $Pr f_{\mathcal{A}}$. On the other hand, $\not\vdash_{\mathcal{A}} \sim \dot{Con}_\alpha$, since $\mathcal{A}$ is ω -consistent. Hence, $\not\vdash_{\mathcal{A}} \dot{Pr}_\alpha (\overline{\sim \dot{Con}_\alpha})$.

Put $\xi(x) = \alpha(x) \vee x \approx \overline{\dot{Con}_\alpha}$. Evidently,

(1) $\vdash_{\mathcal{D}} \dot{Con}_\xi$, i.e. $\vdash_{\mathcal{D}} \dot{Con}_{\alpha \vee x \approx \overline{\dot{Con}_\alpha}}$.

Using the diagonal construction 5.1 [1], we can construct a $\nu_\xi \in Fm_{K_0}$ such that $\vdash_{\mathcal{Q}} \nu_\xi \leftrightarrow \sim \bigvee_y \dot{Pr} f_\xi (\bar{\nu}_\xi)$. It follows from 5.6 [1] that

(2) $\vdash_{\mathcal{A}} \nu_\xi \to \dot{Con}_\xi$.

Hence, by (1), we have

(3) $$\vdash_{\mathcal{D}} \nu_{\xi} , \quad \text{i.e.} \quad \vdash_{\mathcal{D}} \sim Pr_{\xi} (\overline{\nu}_{\xi}) .$$

Put

$$\gamma(x) = \alpha(x) \vee Fm_{K}^{(\mathcal{M})}(x) \wedge \bigvee_{y < x} Prf_{\xi}(\overline{\nu}_{\xi}, y) .$$

Evidently, $\gamma \in Bin$ and

(4) $$\vdash_{\mathcal{P}} Con_{\gamma} \rightarrow \nu_{\xi} .$$

Hence there exists an $n_{o} \in \omega$ such that for every $n \geq n_{o}$

(5) $$\vdash_{\mathcal{P}} Pr_{[\mathcal{A} \restriction n]} (\overline{Con_{\gamma} \rightarrow \nu_{\xi}}) .$$

Since $\vdash_{\mathcal{P}} Pr_{[\mathcal{A} \restriction n]} (\overline{Con_{\alpha} \rightarrow \nu_{\xi}}) \rightarrow Pr_{\xi} (\overline{\nu}_{\xi})$, we have,

by (1),

(6) $$\vdash_{\mathcal{D}} \sim Pr_{[\mathcal{A} \restriction n]} (\overline{Con_{\alpha} \rightarrow \nu_{\xi}}) \quad \text{for every } n \in \omega .$$

(5) and (6) give

(7) $$\vdash_{\mathcal{D}} \sim Pr_{[\mathcal{A} \restriction n]} (\overline{Con_{\alpha} \rightarrow Con_{\gamma}}) \quad \text{for every } n \geq n_{o}$$

and therefore for every $n \in \omega$.

Let $\beta >_{\mathcal{A}} \gamma$ and let $[\gamma]$ be complementible w.r.t. $[\alpha]$, $[\beta]$. By Theorem 4.7, there exists an $n \in$ $\in \omega$ such that

(8) $$\vdash_{\mathcal{A}} (\sim Con_{\beta} \wedge Con_{\gamma}) \rightarrow Pr_{[\mathcal{A} \restriction n]} (\overline{Con_{\alpha} \rightarrow Con_{\gamma}}) .$$

Hence, by (7) and (8), we have

(9) $$\vdash_{\mathcal{A}} (\sim Con_{\beta} \wedge Con_{\gamma}) \rightarrow Pr_{\alpha} (\overline{\sim Con_{\alpha}}) .$$

On the other hand, $\vdash_{\mathcal{A}} Pr_{\alpha} (\overline{\sim Con_{\alpha}}) \rightarrow \sim Con_{\xi}$

and therefore, by (2) and (4),

$$(10) \qquad \vdash_{\mathcal{A}} \dot{Pr}_{\alpha}\left(\overline{\sim \dot{Con}_{\alpha}}\right) \rightarrow \sim \dot{Con}_{\gamma} \; .$$

But (9) and (10) show that $\vdash_{\mathcal{A}} \sim \dot{Con}_{\beta} \rightarrow \sim \dot{Con}_{\gamma}$, which is a contradiction with the assumption $\gamma <_{\mathcal{A}} \beta$.

4.15. <u>Theorem</u>. Let $\mathcal{A}$ be reflexive, α , β , γ , σ , $\tau \in Bin$ and $\alpha \leq_{\mathcal{A}} \tau <_{\mathcal{A}} \gamma <_{\mathcal{A}} \sigma \leq_{\mathcal{A}} \beta$. Suppose that $[\gamma]$ is not complementible w.r.t. $[\alpha]$, $[\beta]$. Then there exist γ_1 , $\gamma_2 \in Bin$ such that

(i) $\tau \leq_{\mathcal{A}} \gamma_1 <_{\mathcal{A}} \gamma <_{\mathcal{A}} \gamma_2 \leq_{\mathcal{A}} \sigma$,

(ii) if $\gamma_1 \leq_{\mathcal{A}} \gamma' \leq_{\mathcal{A}} \gamma_2$, then $[\gamma']$ is not complementible w.r.t. $[\alpha]$, $[\beta]$.

<u>Proof</u>. Let

$$E_1 = A \cup \{\sim \dot{Con}_{\beta} \wedge \dot{Con}_{\gamma}\}, \; E_2 = A \cup \{\sim \dot{Con}_{\gamma} \wedge \dot{Con}_{\tau}\} , \; \varepsilon_1(x) = \alpha(x) \vee x \approx \overline{\sim \dot{Con}_{\beta} \wedge \dot{Con}_{\gamma}} ,$$

$$\varepsilon_2(x) = \alpha(x) \vee x \approx \overline{\sim \dot{Con}_{\gamma} \wedge \dot{Con}_{\tau}} , \; \mathcal{E}_1 = \langle E_1 , K \rangle$$

and $\mathcal{E}_2 = \langle E_2 , K \rangle$. Evidently, ε_i bi-numerates E_i ($i = 1, 2$) and $\mathcal{E}_i$ ($i = 1, 2$) is consistent. Using the diagonal construction 5.1 [1], determine φ such that

$$\vdash_{\mathcal{Q}} \varphi \leftrightarrow \bigwedge_{y} [\, \dot{Pr}\, f_{\varepsilon_1}(\overline{\varphi}, y) \vee \dot{Pr}\, f_{\varepsilon_2}(\overline{\varphi}, y)) \rightarrow$$
$$\rightarrow \sim \dot{Con}_{\alpha \restriction y \vee x \approx \overline{\dot{Con}_{\alpha} \wedge \sim \dot{Con}_{\gamma}}} \,] \; .$$

Suppose $\vdash_{\mathcal{E}_1} \varphi$. Then for some n , we would have $\vdash_{\mathcal{E}_1} \sim \dot{Con}_{\alpha \restriction \overline{n} \vee x \approx \overline{\dot{Con}_{\alpha} \wedge \sim \dot{Con}_{\gamma}}}$, i.e.
$\vdash_{\mathcal{A}} (\sim \dot{Con}_{\beta} \wedge \dot{Con}_{\gamma}) \rightarrow \dot{Pr}_{[\mathcal{A} \restriction n]}\left(\overline{\dot{Con}_{\alpha} \rightarrow \dot{Con}_{\gamma}}\right)$.

But $[\gamma]$ is not complementible w.r.t. $[\alpha]$, $[\beta]$ and therefore, by Theorem 4.7,

$$\not\vdash_{\mathcal{A}} (\sim \mathit{Con}_{\beta} \wedge \mathit{Con}_{\gamma}) \rightarrow \mathit{Pr}_{[\mathcal{A} \restriction n]} (\overline{\mathit{Con}_{\alpha} \rightarrow \mathit{Con}_{\gamma}}) .$$

Hence we have proved

(1) $$\not\vdash_{\mathcal{T}_1} \varphi .$$

Suppose $\vdash_{\mathcal{T}_2} \varphi$. Then for some n , we would have

$$\vdash_{\mathcal{T}_2} \sim \mathit{Con}_{\alpha \restriction \overline{n} \vee x \approx \overline{\mathit{Con}_{\alpha} \wedge \sim \mathit{Con}_{\gamma}}} .$$

Let $n' = \max(n, \mathit{Con}_{\alpha} \wedge \sim \mathit{Con}_{\gamma})$. Then $\vdash_{\mathcal{T}_2} \sim \mathit{Con}_{[\mathcal{T}_2 \restriction n']}$. On the other hand, from reflexivity of $\mathcal{A}$, we have $\vdash_{\mathcal{T}_2} \mathit{Con}_{[\mathcal{T}_2 \restriction n']}$. Hence we have proved

(2) $$\not\vdash_{\mathcal{T}_2} \varphi .$$

Put $\xi'(x) = \alpha(x) \wedge \bigwedge_{y<x} (\sim \mathit{Pr}\, f_{\varepsilon_1}(\overline{\varphi}, y) \wedge \sim \mathit{Pr}\, f_{\varepsilon_2}(\overline{\varphi}, y))$.

Evidently, $\xi' \in \mathit{Bin}$. Analogously as in the proof of Theorem 4.12, we can show

(3) $$\vdash_{\mathcal{P}} \sim \varphi \longrightarrow \mathit{Con}_{\xi' \vee x \approx \overline{\mathit{Con}_{\alpha} \wedge \sim \mathit{Con}_{\gamma}}} ,$$

(4) $$\vdash_{\mathcal{P}} \sim \varphi \longrightarrow \bigvee_{z} \bigwedge_{x} (\xi'(x) \leftrightarrow \alpha(x) \wedge x \leq z) .$$

Let $\mu_{1,\alpha}$ be defined w.r.t. the theories

$\mathcal{A} + \{\sim \mathit{Con}_{\delta} \wedge \mathit{Con}_{\gamma}\}$, $\mathcal{A} + \{\sim \mathit{Con}_{\beta} \wedge \mathit{Con}_{\gamma} \wedge \sim \varphi\}$

and $\mathcal{A} + \{\sim \mathit{Con}_{\gamma} \wedge \mathit{Con}_{z} \wedge \sim \varphi\}$

(cf. Definition 1.16). Further let $\mu_{2,\alpha}$ be defined w.r.t. the theories $\mathcal{A} + \{\sim \mathit{Con}_{\beta} \wedge \mathit{Con}_{\gamma} \wedge \sim \varphi \wedge \mu_{1,\alpha}\}$ and

$\mathcal{A} + \{\sim \mathit{Con}_{\gamma} \wedge \mathit{Con}_{z} \wedge \sim \varphi \wedge \mu_{1,\alpha}\}$.

Put

(5) $\xi(x) = \xi'(x) \vee$

$\vee \bigvee_{y<x} [\sim M_{1,\alpha}(y) \wedge x \approx (\overline{Con_\alpha \rightarrow Con_\gamma} \mathbin{\dot\wedge} \dot{nr}_y \not\approx \dot{nr}_y)] \vee$

$\vee \bigvee_{y<x} [\sim M_{2,\alpha}(y) \wedge x \approx (\overline{Con_\alpha \wedge \sim Con_\gamma} \mathbin{\dot\wedge} \dot{nr}_y \not\approx \dot{nr}_y)]$,

(6) $\gamma_1(x) = \tau(x) \vee$

$\vee Fm_K^{(M)}(x) \wedge \bigvee_{y_1, y_2 < x} (Pr\, f_\xi(\overline{Con_\alpha \wedge \sim Con_\gamma}, y_1) \wedge$

$\wedge Pr\, f_\gamma(\overline{0 \not\approx 1}, y_2))$,

(7) $\gamma_2(x) = \gamma(x) \vee$

$\vee Fm_K^{(M)}(x) \wedge \bigvee_{y_1 y_2 < x} (Pr\, f_\xi(\overline{Con_\alpha \rightarrow Con_\gamma}, y_1) \wedge$

$\wedge Pr\, f_\sigma(\overline{0 \approx 1}, y_2))$.

Evidently, $\xi, \gamma_1, \gamma_2 \in Bin$.

(i) The inequalities $\tau \leq_R \gamma_1 \leq_R \gamma \leq_R \gamma_2 \leq_R \sigma$ are evident. We have (cf. Theorem 1.18)

(8) $\not\vdash_R (\sim Con_\sigma \wedge Con_\gamma) \rightarrow \mu_{1,\alpha}$.

It is clear that

(9) $\vdash_P \sim \mu_{1,\alpha} \rightarrow Pr_\xi(\overline{Con_\alpha \rightarrow Con_\gamma})$,

(10) $\vdash_P \sim Con_\sigma \wedge Pr_\xi(\overline{Con_\alpha \rightarrow Con_\gamma}) \rightarrow \sim Con_{\gamma_2}$.

and therefore

(11) $\vdash_P (\sim Con_\sigma \wedge Con_{\gamma_2}) \rightarrow \mu_{1,\alpha}$.

(8) and (11) immediately give

(12) $\not\vdash_R Con_\gamma \rightarrow Con_{\gamma_2}$,

i.e. we have proved $\gamma \prec_{\mathcal{A}} \gamma_2$.

We have (cf. Theorem 1.18)

(13) $\not\vdash_{\mathcal{A}} (\sim Con_{\gamma} \wedge Con_{\tau} \wedge \sim \varphi \wedge \mu_{1,\alpha}) \rightarrow \sim \mu_{2,\alpha}$.

Evidently, we have

(14) $\vdash_{\mathcal{P}} (\mu_{1,\alpha} \wedge \mu_{2,\alpha}) \rightarrow \bigwedge_{x} (\xi'(x) \leftrightarrow \xi(x))$

and therefore, by 4.4, we have

(15) $\vdash_{\mathcal{P}} (\sim \varphi \wedge \mu_{1,\alpha} \wedge \mu_{2,\alpha}) \rightarrow \bigvee_{z} \bigwedge_{x} (\xi(x) \leftrightarrow \alpha(x) \wedge x \leq z)$.

We know that

(16) $\vdash_{\mathcal{A}} Con_{\tau} \rightarrow \sim Pr_{\alpha}(\overline{Con_{\alpha}})$,

since $\vdash_{\mathcal{A}} Con_{\tau} \rightarrow Con_{\alpha}$ and $\vdash_{\mathcal{A}} Con_{\alpha} \rightarrow \sim Pr_{\alpha}(\overline{Con_{\alpha}})$ (cf. Theorem 5.6 [1]). (15) and (16) give

(17) $\vdash_{\mathcal{A}} (Con_{\tau} \wedge \mu_{1,\alpha} \wedge \mu_{2,\alpha} \wedge \sim \varphi) \rightarrow \sim Pr_{\xi}(\overline{Con_{\alpha}})$

and therefore

(18) $\vdash_{\mathcal{A}} (Con_{\tau} \wedge \mu_{1,\alpha} \wedge \mu_{2,\alpha} \wedge \sim \varphi) \rightarrow Con_{\gamma_1}$

since $\vdash_{\mathcal{P}} \sim Pr_{\xi}(\overline{Con_{\alpha}}) \rightarrow \sim Pr_{\xi}(\overline{Con_{\alpha} \wedge \sim Con_{\gamma}})$ and $\vdash_{\mathcal{P}} (Con_{\tau} \wedge \sim Pr_{\xi}(\overline{Con_{\alpha} \wedge \sim Con_{\gamma}})) \rightarrow Con_{\gamma_1}$. (13) and (18) imply

(19) $\not\vdash_{\mathcal{A}} Con_{\gamma_1} \rightarrow Con_{\gamma}$,

i.e. we have proved $\gamma_1 \prec_{\mathcal{A}} \gamma$.

(ii) Let $\gamma_1 \preccurlyeq_{\mathcal{A}} \gamma' \preccurlyeq_{\mathcal{A}} \gamma_2$ and let $[\gamma']$ be complementible w.r.t. $[\alpha]$, $[\beta]$. Then there exists an $m \in \omega$ such that $\vdash_{\mathcal{A}} (\sim Con_{\beta} \wedge Con_{\gamma'}) \rightarrow$ $\rightarrow Pr_{[\mathcal{A} \restriction m]}(\overline{Con_{\alpha} \rightarrow Con_{\gamma'}})$ (cf. Theorem 4.7) and therefore there exists an $m \in \omega$ such that

(20) $\vdash_{\mathcal{A}} (\sim Con_\beta \wedge Con_{\gamma_2}) \rightarrow Pr_{[\mathcal{A} \restriction m]}(\overline{Con_\alpha \rightarrow Con_{\gamma_1}})$.

We shall show that it is impossible.

We have (cf. Theorem 1.18)

(21) $\not\vdash_{\mathcal{A}} (\sim Con_\beta \wedge Con_\gamma \wedge \sim \varphi \wedge \mu_{1,\alpha}) \rightarrow \mu_{2,\alpha}$.

It is clear that

(22) $\vdash_{\mathcal{P}} \sim \mu_{2,\alpha} \rightarrow Pr_\xi(\overline{Con_\alpha \wedge \sim Con_\gamma})$ and in particular

(23) $\vdash_{\mathcal{P}} \sim \mu_{2,\alpha} \rightarrow Pr_\xi(\overline{\sim Con_\gamma})$.

On the other hand, we have from (22)

(24) $\vdash_{\mathcal{P}} \sim \mu_{2,\alpha} \rightarrow Pr_\xi(\overline{Pr_\xi(\overline{Con_\alpha \wedge \sim Con_\gamma})})$,

since $Pr_\xi(\overline{Con_\alpha \wedge \sim Con_\gamma})$ is an RE-formula in $\mathcal{P}$ (cf. 1.7).

(6), (23) and (24) show that

(25) $\vdash_{\mathcal{P}} \sim \mu_{2,\alpha} \rightarrow Pr_\xi(\overline{\sim Con_{\gamma_2}})$.

By (3) and (5),

(26) $\vdash_{\mathcal{P}} (\sim \varphi \wedge \mu_{1,\alpha}) \rightarrow \sim Pr_\xi(\overline{Con_\alpha \rightarrow Con_\gamma})$

and therefore

(27) $\vdash_{\mathcal{P}} (\sim \varphi \wedge \mu_{1,\alpha}) \rightarrow \sim Pr_\xi(\overline{\sim Con_\alpha})$.

On the other hand, by (26) and (7)

(28) $\vdash_{\mathcal{P}} (Con_\gamma \wedge \sim \varphi \wedge \mu_{1,\alpha}) \rightarrow Con_{\gamma_2}$.

Using (21), (25) and (28) we can easily show

(29) $\not\vdash_{\mathcal{A}} (\sim Con_\beta \wedge Con_{\gamma_2} \wedge Pr_\xi(\overline{\sim Con_{\gamma_1}}) \rightarrow Pr_\xi(\overline{\sim Con_\alpha})$.

On the other hand, using (20), we have

(30) $\vdash_{\mathcal{A}} \sim \dot{C}\dot{o}n_{\beta} \wedge \dot{C}\dot{o}n_{\gamma_2} \wedge \dot{P}r_{\xi}(\overline{\sim \dot{C}\dot{o}n_{\gamma_1}}) \rightarrow \dot{P}r_{\xi}(\overline{\sim \dot{C}\dot{o}n_{\alpha}})$,

since $\vdash_{\mathcal{P}} \dot{P}r_{[A \restriction m]}(\overline{\dot{C}\dot{o}n_{\alpha} \rightarrow \dot{C}\dot{o}n_{\gamma_1}}) \rightarrow \dot{P}r_{\xi}(\overline{\dot{C}\dot{o}n_{\alpha} \rightarrow \dot{C}\dot{o}n_{\gamma_1}})$

and $\vdash_{\mathcal{P}} (\dot{P}r_{\xi}(\overline{\dot{C}\dot{o}n_{\alpha} \rightarrow \dot{C}\dot{o}n_{\gamma_1}}) \wedge \dot{P}r_{\xi}(\overline{\sim \dot{C}\dot{o}n_{\gamma_1}})) \rightarrow \dot{P}r_{\xi}(\overline{\sim \dot{C}\dot{o}n_{\alpha}})$.

This completes the proof.

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(Oblatum 27.10.1970)

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