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ON PROBLEMS CONCERNING UNIQUENESS OF THE EXTENSION OF LINEAR OPERATIONS ON LINEAR SPACES

František CHARVÁT, Praha

The aim of this paper is the formulation of the so-called Φ -unique extensibility of linear operators (i.e. linear transformations of linear space into another one) which is a generalization of the traditional uniqueness of the extensibility of linear functionals preserving the norm (see [1]). The necessary and sufficient conditions for Φ -unique extensibility and for the uniqueness of the extensibility of bounded linear operators are proved. The paper further contains a generalization of the Phelps' result (see [1]).

This note follows the paper [2], and the same conventions are used here.

<u>Definition 1</u>. Let Φ be a mapping from P into $\exp Q$ (i.e. the set of all subsets of the linear space Q). The operator will be called Φ -unique extensionable, if there is one and only one operator B such that

$$\operatorname{def} B = P ,$$

$$x \in \operatorname{def} A \Longrightarrow A(x) = B(x) ,$$

$$x \in P \Longrightarrow B(x) \in \Phi(x) .$$

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Remark 1. It is true that every Φ -unique extensionable operator is a Φ -extensionable operator (see Definition 2 in [2]).

Definition 2. Let Φ be a mapping from P into $exp\ Q$. The mapping is called a uniquely linearly covering P in respect to Q , if the following statement is satisfied:

Let A be a Φ -admissible operator (see Definition 1 in [2]), then for every $y \in P$ there is one and only one $a \in Q$ such that

$$A(x) + \alpha a \in \Phi(x + \alpha y)$$

for all $x \in def\ A$ and $\alpha \in K$.

Remark 2. It is true that every uniquely linearly covering mapping is a linearly covering mapping in respect to Q .

Theorem 1. Let Φ be a mapping from P into Q . Then the following statements are equivalent:

(i) Every Φ -admissible operator is a Φ -unique extensionable operator;

(ii) The mapping Φ is a uniquely linearly covering P in respect to Q .

Proof. Let (i) be true, but (ii) untrue. From Remark 1 and Theorem 1 in [2] it follows that Φ is linearly covering P in respect to Q . Then there is also a Φ -admissible operator A and an element $y \in P$ as well as the different elements $a_1, a_2 \in Q$ such that

$A(x) + \alpha a_1 \in \Phi(x + \alpha y)$,

$A(x) + \alpha a_2 \in \Phi(x + \alpha y)$ for all $x \in def\ A$ and $\alpha \in K$.

We define the operators B_1, B_2 as follows:

$def\, B_1 = def\, B_2 = [def\, A \cup y]$,

if $z = x + \alpha y$, $x \in def\, A$, $\alpha \in K$, then

$B_1(z) = A(x) + \alpha a_1$,

$B_2(z) = A(x) + \alpha a_2$.

B_1 and B_2 are Φ -admissible operators. From Theorem 1 in [2] it follows that there are Φ -admissible operators B_3, B_4 which are the extensions of the operators B_1, B_2 and $def\, B_3 = def\, B_4 = P$. It is true that B_3 and B_4 are different operators being the extensions of the operator A. This gives a contradiction.

Let (ii) be true, but (i) untrue. According to Remark 2 and Theorem 1 in [2] it follows that every Φ -admissible operator is a Φ -extensionable operator and that there is also a Φ -admissible operator A such that it has two different extensions, i.e. there are B_1, B_2 such that

$def\, B_1 = def\, B_2 = P$,

$x \in def\, A \Longrightarrow A(x) = B_1(x) = B_2(x)$,

$x \in P \Longrightarrow B_1(x) \in \Phi(x),\ B_2(x) \in \Phi(x)$

and there is $y \in P$ (resp. $y \in P - def\, A$)such that

$B_1(y) \neq B_2(y)$.

If we denote $a_1 = B_1(y)$, $a_2 = B_2(y)$, it follows

$A(x) + \alpha a_1 \in \Phi(x + \alpha y)$,

$A(x) + \alpha a_2 \in \Phi(x + \alpha y)$ for all $x \in def\, A$ and $\alpha \in K$.

This is a contradiction. The proof is complete.

Convention. In the following K will denote a field of real or complex numbers. Let P, Q be normed linear spaces. We denote the norm on P by the same way as in [2]

$^1\| \cdot \|$, the norm on Q $^2\| \cdot \|$.
Analogously, the symbol $S(a; \varepsilon)$ is used for the set

$\{b \in Q;\ ^2\|a - b\| \leqq \varepsilon\}$, $\varepsilon \geqq 0$.

Definition 3. Let $k \geqq 0$. Let P , Q be normed linear spaces. The linear space Q is called k-productively uniquely centred in respect to P , if the following is satisfied:

Let A be such that

$S(A(x_1), k\ ^1\|x_1 + y\|) \cap S(A(x_2), k\ ^1\|x_2 + y\|) \neq \emptyset$

for all $x_1, x_2 \in \mathrm{def}\, A$ and $y \in P$, then

$\bigcap_{x \in \mathrm{def}\, A} S(A(x), k\ ^1\|x + y\|)$ contains only one element for every $y \in P$.

Remark 3. It is true that every k-productively uniquely centred linear space Q in respect to P is k-productively centred in respect to P (see Definition 4 in [2]).

Theorem 2. Let $k \geqq 0$. Let P, Q be normed linear spaces. Then the following statements are equivalent:

(i) The mapping Φ from linear space P to $\exp Q$ defined by the following

$x \in P \Rightarrow \Phi(x) = \{a \in Q;\ ^2\|a\| \leqq k\ ^1\|x\|\}$

is uniquely linearly covering P in respect to Q ;

(ii) The linear space Q is k-productively uniquely centred in respect to P .

Proof. Let (i) be true, but (ii) untrue. From Remark 2 and Theorem 2 in [2] Q is k-productively centred in respect to Q and there is also A such that

$S(A(x_1), k\ ^1\|x_1 + y\|) \cap S(A(x_2), k\ ^1\|x_2 + y\|) \neq \emptyset$

for all $x_1, x_2 \in def\, A$ and $y \in P$ and there is at least one element $y \in P$ such that

$\bigcap_{x \in def\, A} S(A(x), k\ {}^1\| x + y \|)$ contains at least two different elements. We denote these elements $-a_1$, $-a_2$. It follows

$${}^2\| A(x) + a_1 \| \leqq k\ {}^1\| x + y \| ,$$

$${}^2\| A(x) + a_2 \| \leqq k\ {}^1\| x + y \| \text{ for all } x \in def\, A .$$

From there it follows that for all $\alpha \in K$, $\alpha \neq 0$

$${}^2\| A(x) + \alpha a_1 \| \leqq k\ {}^1\| x + \alpha y \| ,$$

$${}^2\| A(x) + \alpha a_2 \| \leqq k\ {}^1\| x + \alpha y \| ,$$

in other words

$A(x) + \alpha a_1 \in \Phi(x + \alpha y)$,

$A(x) + \alpha a_2 \in \Phi(x + \alpha y)$ for all $x \in def\, A$ and $\alpha \in K$ (for $\alpha = 0$ trivially). However, this is a contradiction.

Let (ii) be true, but (i) untrue. From Remark 3 and Theorem 2 in [2] it follows that Φ is linearly covering P in respect to Q and there is also a Φ-admissible operator A and $y \in P$ and two different $-a_1$, $-a_2$ such that

$${}^2\| A(x) + \alpha a_1 \| \leqq k\ {}^1\| x + \alpha y \| ,$$

$${}^2\| A(x) + \alpha a_2 \| \leqq k\ {}^1\| x + \alpha y \|$$

for all $x \in def\, A$ and $\alpha \in K$.

From it

$$S(A(x_1), k\ {}^1\| x_1 + y \|) \cap S(A(x_2), k\ {}^1\| x_2 + y \|) \neq \emptyset$$

for all $x_1, x_2 \in def\, A$ and $y \in P$ because it follows

$$\mathcal{k} \left({}^1\| x_1 + y \| + {}^1\| x_2 + y \| \right) \geqq \mathcal{k} \, {}^1\| x_1 - x_2 \| \geqq$$
$$\geqq {}^2\| A(x_1 - x_2) \| = {}^2\| A(x_1) - A(x_2) \|$$

and that

$$-a_1 , -a_2 \in \bigcap_{x \in \operatorname{def} A} S(A(x), \mathcal{k} \, {}^1\| x + y \|) .$$

This gives a contradiction. The proof is complete.

Definition 4. We call the linear space Q productively uniquely centred in respect to P if this linear space is $\mathcal{k}$ -productively uniquely centred in respect to P for every Q .

Theorem 3. Let P, Q be normed linear spaces. Let P be productively uniquely centred in respect to P . Then every bounded operator from P into Q has only one extension on the whole P preserving the norm.

Proof. This theorem is a result of Theorem 1.2 and Definition 4.

Remark 4. In the following we shall be concerned with a slightly different problem formulated for linear functionals in [1]:

Let P, Q be normed linear spaces. Let R be a subspace of the space P . Let Q be productively centred in respect to P . We want to formulate a necessary and sufficient condition for the uniqueness of the extension preserving the norm of every bounded operator such that $\operatorname{def} A = R$, more exactly, there is only one operator B such that

$$\operatorname{def} B = P , \; x \in R \Longrightarrow A(x) = B(x), \quad {}^3\| A \| = {}^3\| B \|$$

(in this way we denote the norm on a linear space of all bounded operators from P into Q).

It follows from Theorem 2, Remark 1 from [2] respectively, that there is an extension of this operator. The problem lies in the uniqueness of such an extension.

Convention. Let P, Q be normed linear spaces. By the symbol $\mathcal{L}$ we shall denote a normed linear space of all bounded operators from P into Q such that their domain is the whole P. Analogously, we denote by the symbol $\mathcal{L}_R$ a normed linear space of all bounded operators from P into Q such that their domain is the subspace R.

Furthermore, let $A \in \mathcal{L}$. By the symbol A_R, we denote an operator such that $A_R \in \mathcal{L}_R$, $x \in R \Rightarrow A_R(x) = A(x)$. The set $\{B \in \mathcal{L};\ x \in R \Rightarrow B(x) = 0\}$ we denote ${}_Q R^{\perp}$ and call Q - anihilator of the product R.

Definition 5. Let P be a normed linear space. Let R be a subspace of the space P. We say that R has the Haar's characteristic (see [1]), if the following is valid:

if $x \in P$, then there is at most one element $y \in R$ such that

$$^1\| x - y \| = \inf \{ ^1\| x - z \|\ z \in R \} .$$

Lemma 1. Let P be a normed linear space. Let R be a subspace of the space P. Then the following statements are equivalent:

(i) R has not the Haar's characteristic;

(ii) there are $x \in P$ and $y \in R$, $y \neq 0$ such that $^1\| x \| = {}^1\| x - y \| = {}^1\| x - z \|$ for all $z \in R$.

Proof. Let (i) be true. Thus, there are $x_0 \in P$, different $y_1, y_2 \in R$ so that

$$^1\| x_0 - y_1 \| = {}^1\| x_0 - y_2 \| = \inf \{ {}^1\| x - z \| ; z \in R \} .$$

We denote $x = x_0 - y_1$, $y = y_2 - y_1$. It follows

$$^1\| x \| = {}^1\| x - y \| , \quad y \in R , \quad y \neq 0 .$$

Let $z \in R$, then $z + y_1 \in R$ and further

$$^1\| x_0 - (z + y_1) \| \geq {}^1\| x_0 - y_1 \| ;$$

in other words,

$^1\| x \| \leq {}^1\| x - z \|$. Thus, (ii) is satisfied.

If (ii) is true, then (i) is trivially satisfied. The proof is complete.

<u>Lemma 2.</u> Let P, Q be normed linear spaces. Let Q be productively centred in respect to P . Let R be a subspace of the space P . Let $A \in \mathcal{L}$. Then

$$^3\| A_R \| = \inf \{ k ; {}^2\| A(x) \| \leq k \, {}^1\| x \| , x \in R \} =$$

$$= \inf \{ {}^3\| A - B \| , B \in {}_Q R^{\perp} \} .$$

<u>Proof</u>. If $B \in {}_Q R^{\perp}$, then

$$^3\| A_R \| = \inf \{ k ; {}^2\| (A-B)(x) \| \leq k \, {}^1\| x \| , x \in R \} \geq {}^3\| A - B \| .$$

Also, it follows that: $^3\| A_R \| \leq \inf \{ {}^3\| A - B \| , B \in {}_Q R^{\perp} \}$.

According to the assumption that Q is productively centred in respect to P , from Remark 1 in [2] it follows that there is an operator C such that

$$^3\| A_R \| = {}^3\| C \| , \; x \in R \Rightarrow A_R(x) = C(x) .$$

Since

$^3\| A_R \| = {}^3\| C \| = {}^3\| A - (A - C) \|$, and $A - C \in {}_Q R^{\perp}$,

the proof is complete.

<u>Theorem 4</u>. Let P, Q be normed linear spaces. Let

Q be productively centred in respect to P. Let R be a subspace of the space P. Then the following statements are equivalent:

(i) For every $B \in \mathcal{L}_R$ there is one and only one $C \in \mathcal{L}$ such that

$$x \in R \Longrightarrow B(x) = C(x), \quad {}^3\|B\| = {}^3\|C\| .$$

(ii) The linear space ${}_Q R^{\perp}$ has the Haar's characteristic ("in respect to the linear space $\mathcal{L}$ ").

Proof. Let (i) be true, but (ii) untrue. From Lemma 1 it follows that there is $C \in \mathcal{L}$ and $D \in {}_Q R^{\perp}$, $D \neq 0$ such that

$${}^3\|C\| = {}^3\|C - D\| = \inf\{{}^3\|C - E\| ; E \in {}_Q R^{\perp}\} .$$

From Lemma 2 it follows that

$${}^3\|C_R\| = \inf\{{}^3\|C - E\| ; E \in {}_Q R^{\perp}\} .$$

Also, the operator $C_R \in \mathcal{L}_R$ has two different extensions, i.e. C and $C - D$, on the whole P preserving the norm but this is a contradiction.

Let (ii) be true, but (i) untrue. There is an operator $B \in \mathcal{L}_R$ having at least two different extensions on the whole P preserving the norm. We denote these extensions C_1, C_2. It is true that $C_1 - C_2 \in {}_Q R^{\perp}$, and, further, from Lemma 2 it follows that

$${}^3\|C_1\| = {}^3\|C_1 - (C_1 - C_2)\| = {}^3\|B\| = \inf\{{}^3\|C_1 - D\|, D \in {}_Q R^{\perp}\} ,$$

however, it is a contradiction (see Lemma 1).

The proof is complete.

Theorem 5. Let P, Q be normed linear spaces. Let Q be productively centred in respect to P. Then the following statements are equivalent:

(i) Every bounded operator is uniquely extensionable on the whole P preserving the norm;

(ii) Q -anihilator of every subspace of the space P has the Haar's characteristic.

The proof is easy.

References

[1] PHELPS R.P.: Uniqueness of Hahn-Banach extensions and unique best approximation, TAMS 95(1960),238-255.

[2] CHARVÁT F.: On problems concerning extension of linear operations on linear spaces, Comment.Math.Univ. Carolinae 12(1971),105-115.

Praha - Vinohrady

Ambrožova 13

Československo

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