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ON LOCAL MEROTOPIC CHARACTER

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Merotopic spaces represent one type of non-classical continuity structures. They were introduced by M. Katětov in [4]. It is known that there are certain relations between merotopic spaces and other structures. In the present paper, we shall study the merotopic spaces and the topology induced by the given merotopy on the same set.

In the first part, we recall preliminary definitions and propositions; see [3],[4]. In the second part, we give a construction of an important class of merotopis spaces over the given closure space , and we define the notion of the local merotopic character. We shall study those subsets of a given merotopy which determine in a specified sense (see 2.4) a neighbourhood system of a given point. We call the least cardinality of such a subset a local merotopic character of a point. We are interested in the problem, what are the values of the local merotopic character of a fixed point for merotopies inducing the given closure. Such a set of cardinal numbers may be regarded as "the merotopic spectrum" of the given closure

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(in a fixed point). We shall prove that this "spectrum" contains always a certain interval of cardinal numbers. (Theorem 2.11.)

In the third part, we shall solve the same problem in special cases. We shall find the "merotopic spectrum" of merotopies inducing the finest non-discrete topology. Further, we shall restrict ourselves only to the "natural" merotopies; i.e., merotopies which may be considered as images of closures under an embedding of the category of closure spaces into the category of merotopic spaces. We shall study the set of possible local characters of a fixed point with respect to an embedding functor and a given closure. We shall show that under these conditions there are large "gaps" in the "spectrum".

The notation and symbols from [1] are used.

We assume the generalized continuum hypothesis (GCH) in the form $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$ for each cardinal $\aleph_\alpha$.

1.

Let E be a set. Let $\Gamma \subset \exp \exp E$ be such that

(i) If $\mathfrak{M} \in \Gamma$, $\mathfrak{M}_1 \subset \exp E$ and to each $M \in \mathfrak{M}$ there is an $M_1 \in \mathfrak{M}_1$ with $M_1 \subset M$ (we say that $\mathfrak{M}_1$ minorizes $\mathfrak{M}$), then also $\mathfrak{M}_1 \in \Gamma$;

(ii) if $\mathfrak{M}_1 \cup \mathfrak{M}_2 \in \Gamma$ then $\mathfrak{M}_1 \in \Gamma$ or $\mathfrak{M}_2 \in \Gamma$;

(iii) $((x)) \in \Gamma$ for all $x \in E$;

(iv) $(\emptyset) \in \Gamma$, $\emptyset \notin \Gamma$.

Then Γ is called a merotopic structure, or a merotopy on E ; $\langle E , \Gamma \rangle$ is termed a merotopic space. Members of Γ are said to be micromeric.

Γ -continuous (or continuous) mapping $f : \langle E_1 , \Gamma_1 \rangle \longrightarrow \langle E_2 , \Gamma_2 \rangle$ is such a mapping that $f[\mathfrak{M}_1] \in \Gamma_2$ whenever $\mathfrak{M}_1 \in \Gamma_1$. Merotopic spaces with Γ -continuous mappings form a category. We shall say that Γ_1 is finer than Γ_2 (and note $\Gamma_1 \leq \Gamma_2$), iff the identity mapping $i : \langle E , \Gamma_1 \rangle \rightarrow \langle E , \Gamma_2 \rangle$ is Γ -continuous, or, equivalently, iff $\Gamma_1 \subset \Gamma_2$.

A merotopic cover (Γ -cover) $\mathfrak{Z}$ of the space $\langle E , \Gamma \rangle$ is such a cover of the set E that for any $\mathfrak{M} \in \Gamma$ there exist a $Z \in \mathfrak{Z}$ and an $M \in \mathfrak{M}$ with $M \subset Z$. All merotopic covers form a filter under the refinement order. On the other hand, for a given non-void system Ω of covers of a set E there exists only one merotopy Γ on E such that Ω is the collection of all Γ -covers, assuming just that Ω is a filter under the refinement order.

Let Γ be a merotopy on a set E. A system θ, $\theta \subset \subset \Gamma$ will be called fundamental, if $\Gamma \subset \Gamma_1$ whenever Γ_1 is a merotopy on E with $\theta \subset \Gamma_1$.

A merotopic space $\langle E , \Gamma \rangle$ will be called a filter-merotopic space (and Γ a filter-merotopy) if there exists a fundamental system for Γ consisting of filters.

Let $\langle E , \Gamma \rangle$ be a filter-merotopic space. Then there exists a Γ -fundamental system θ consisting of filters such that Γ is exactly the collection of all $\mathfrak{M} \subset \exp E$, minorizing some $\mathfrak{M}_1 \in \theta$.

The micromeric collection $\mathfrak{M}$ is localized at a point $x \in E$ if $[\mathfrak{M}] \cup (x) \in \Gamma$. The merotopy Γ

(and also the space $\langle E, \Gamma \rangle$) will be called localized iff either $E = \emptyset$ or every micromeric $\mathfrak{m}$ is localized (at some point of E).

Let Γ be a merotopy on a set E ; for every $X \subset E$ let $u_\Gamma X$ consist of all $x \in E$ such that for some micromeric $\mathfrak{m}$, $M \in \mathfrak{m}$ implies $x \in M$ and $M \cap X \neq \emptyset$. Then u_Γ is a closure structure on E , induced by the merotopy Γ .

There exist many merotopies on a set E , inducing a given closure u for a set E . We shall notice two of them: the coarsest localized filter-merotopy Γ_u which has a fundamental system consisting of all neighbourhood systems $\sigma(x)$ of all points $x \in E$, and the finest merotopy Γ with a fundamental system θ defined in the following way: $\mathfrak{m} \in \theta$ iff there exist $x \in E$ and $A \subset E$ such that $x \in A'$, and $\mathfrak{m} = \{(x, y) \mid y \in A\}$.

A merotopic space $\langle E, \Gamma \rangle$ will be called a semi-separated merotopic space iff the induced closure u_Γ is semi-separated. This condition is fulfilled if and only if $((x, y)) \notin \Gamma$ for any two distinct points $x, y \in E$.

2.

2.1. <u>Definition</u>. Let $\langle E, \Gamma \rangle$ be a merotopic space, We shall call a merotopy $\mathrm{ess}\, \Gamma \subset \Gamma$ an essential part of the merotopy Γ , iff $\mathrm{ess}\, \Gamma$ is the coarsest localized merotopy finer than Γ .

The merotopy $\mathrm{ess}\, \Gamma$ exists for any $\langle E, \Gamma \rangle$. This follows from the definition of induced topology.

2.2. <u>Proposition</u>. $u_{ess\,\Gamma} = u_\Gamma$ for every merotopic space $\langle E, \Gamma \rangle$. (The proof is obvious.)

2.3. <u>Theorem</u>. Let $\langle E, u \rangle$ be a semi-separated closure space. Let $x \in E$, let $\sigma(x)$ be a neighbourhood system of x and let $U \in \sigma(x)$. Let $\mathcal{N}_U \subset \exp E$ be a system satisfying

(i) $N \in \mathcal{N}_U$ implies $x \in N$;

(ii) $\bigcup \mathcal{N}_U = U$.

Put $\mathfrak{X}_U = \mathcal{N}_U \cup \{(y) \mid y \neq x\}$ and let Γ_x be a merotopy on E such that $\{\mathfrak{X}_U \mid U \in \sigma(x)\}$ forms a subbase of all Γ_x-covers. Let $\Gamma = \bigcup \{\Gamma_x \mid x \in E\}$.

Then $u_\Gamma = u$. Moreover, whenever Γ_1 is a merotopy such that $u_{\Gamma_1} = u$, then putting $\mathcal{N}_U = star((x), \mathfrak{X})$ for $x \in E$ and every Γ_1-cover $\mathfrak{X}$, we obtain the merotopy $ess\,\Gamma_1$. (The symbol $star((x), \mathfrak{X})$ denotes the set $\{Z \mid Z \in \mathfrak{X}, x \in Z\}$.)

Proof of the first part is obvious and the second follows immediately from this easy proposition: Let $\langle E, u \rangle$ be a closure space, $\langle E, \Gamma \rangle$ a merotopic space and $u_\Gamma = u$. Then $\{st_{\mathfrak{X}}(x) \mid \mathfrak{X}$ is a Γ-cover$\}$ is a neighbourhood system of x. (The symbol $st_{\mathfrak{X}}(x)$ denotes the set $\bigcup \{Z \mid Z \in \mathfrak{X}, x \in Z\}$.)

2.4. <u>Definition</u>. Let $\langle E, \Gamma \rangle$ be a merotopic space, let $x \in E$. Local merotopic character of a point x is the least cardinality σx such that there exists a system $\Delta \subset \Gamma$ with $card\, \Delta = \sigma x$ for which these two conditions are satisfied:

(i) $\mathfrak{M} \in \Delta$ implies $x \in \bigcap \mathfrak{M}$;

(ii) for every choice $M_{\mathfrak{m}} \in \mathfrak{m}$ there exists a neighbourhood O of the point x in the closure space $\langle E, u_{\Gamma} \rangle$ such that $O \subset \bigcup \{ M_{\mathfrak{m}} \mid \mathfrak{m} \in \Delta \}$.

It follows from Theorem 2.3 that the system $\Delta = \{ \mathfrak{m} \mid \mathfrak{m} \in ess\, \Gamma,\ x \in \bigcap \mathfrak{m} \}$ satisfies (i) and (ii).

2.5. Theorem. Let $\langle E, \Gamma \rangle$ be a semi-separated merotopic space. Then $\sigma x = 1$ for all $x \in E$ if and only if $ess\, \Gamma = \Gamma_{u_{\Gamma}}$.

Proof. Let $x \in E$ and let $\mathcal{O}(x)$ be its neighbourhood system. Let $ess\, \Gamma = \Gamma_{u_{\Gamma}}$. Then obviously $\sigma x = 1$ since the system Δ equals to $(\mathcal{O}(x))$.
Let $\sigma x = 1$. The system Δ contains exactly one micromeric collection, say $\mathfrak{m}$. Each $M \in \mathfrak{m}$ is a neighbourhood of x in $\langle E, u_{\Gamma} \rangle$ by 2.4 (ii). If there exists a neighbourhood O of a point x such that for all $M \in \mathfrak{m}$ is $M - O$ non-void, then $x \in u_{\Gamma}(E - O)$, which is a contradiction. Thus $\mathfrak{m} = \mathcal{O}(x)$ and $ess\, \Gamma = \Gamma_{u_{\Gamma}}$.

2.6. Theorem. Let $\langle E, \Gamma_1 \rangle$ and $\langle E, \Gamma_2 \rangle$ be merotopic spaces, for which $u_{\Gamma_1} = u_{\Gamma_2}$ holds, and let Γ_1 be finer than Γ_2. Then $\sigma_1 x \geq \sigma_2 x$ for every $x \in E$ ($\sigma_i x$ is the local merotopic character of x in the space $\langle E, \Gamma_i \rangle$, $i = 1, 2$).

Proof. Clearly $\Gamma_1 \leq \Gamma_2$ implies $ess\, \Gamma_1 \leq ess\, \Gamma_2$. For $i = 1, 2$ let $\Delta_i = \{ \mathfrak{m} \mid \mathfrak{m} \in ess\, \Gamma_i,\ x \in \bigcap \mathfrak{m} \}$. Clearly $\Delta_1 \subset \Delta_2$. If $\Delta^1 \subset ess\, \Gamma_1$ is the system satisfying 2.4 (i), (ii) and if $card\, \Delta^1 = \sigma_1 x$, then (since $\Delta^1 \subset \Delta_1 \subset \Delta_2$) Δ^1 has the same properties in $\langle E, \Gamma_2 \rangle$. Thus $\sigma_2 x \leq card\, \Delta^1 = \sigma_1 x$.

2.7. Definition. Let $\langle E, u \rangle$ be a closure space, let $x \in E$. γ -character of x (notation γx) is the least cardinality of a neighbourhood of x; i.e. $\gamma x = card\ O_0$, where O_0 is a neighbourhood of x such that $card\ O_0 \leq card\ \mathcal{U}$ for every neighbourhood $\mathcal{U}$ of x.

2.8. Definition. Let $\langle E, u \rangle$ be a closure space, let $x \in E$. Consider the index set A with the following property:

(1) There exists a neighbourhood O of x and a disjoint system $\{R_\alpha \mid \alpha \in A\}$ where $R_\alpha \subset E$, $x \in R'_\alpha$ for every $\alpha \in A$, and $\bigcup \{R_\alpha \mid \alpha \in A\} \supset O$.

σ -character of x (notation σx) is the least upper bound of the set $\{card\ A \mid A$ satisfies (1)$\}$.

2.9. Let $\langle E, u \rangle$ be a semi-separated closure space, $x \in E$ and χx the local character of x. Then

$$1 \leq 6x \leq \gamma x^{\chi x},$$

$$1 \leq 6x \leq \exp \gamma x$$

for every merotopy Γ on E inducing u.

Proof. As a consequence of 2.6 it suffices to verify the proposition for the finest merotopy Γ inducing u. Let O_0 be a neighbourhood of x with $card\ O_0 = \gamma x$. Let $\sigma(x)$ be a neighbourhood base of x with cardinality χx such that $O \in \sigma(x)$ implies $O \subset O_0$.

Let us choose an $x_{\mathcal{U}} \in \mathcal{U}$ for each $\mathcal{U} \in \sigma(x)$ and form $\mathcal{M} = \{(x, x_{\mathcal{U}}) \mid \mathcal{U} \in \sigma(x)\}$. Let Δ be the system of all such $\mathcal{M}$. Clearly, the first inequality holds for Δ. The second inequality holds for $\Delta_1 \subset \Gamma$

consisting of all $\mathfrak{m}_X = \{ (x, y) \mid y \in X \}$ for all $X \subset O_o$ with $x \in X'$.

Both systems satisfy 2.4 (i),(ii).

2.10. Remark. The bounds given in 2.9 are the best possible in the sense that there are examples of spaces with $\sigma x = \gamma x^{\chi x}$ or $\sigma x = exp\ \gamma x$. On the other hand, there is a topology such that the upper bounds from 2.9 can be reached by no merotopy inducing it. To see a part of it let us consider the following two spaces:

a) $\langle P, u \rangle$ is a set $(0) \cup \{ \frac{1}{n} \mid n < \omega_0 \}$ endowed with the relativization of usual topology for real numbers.

b) $\langle Q, v \rangle$ is the set of real numbers with this topology: $A \subset Q$ is closed iff it is finite or $A = Q$.

In the case a) both upper bounds for σx are equal to $exp\ \aleph_0$ as $\chi 0 = \aleph_0$ and $card\ P = \aleph_0$ in $\langle P, u \rangle$ and $\sigma 0 = exp\ \aleph_0$ for the finest merotopy Γ on P inducing u.

In the case b) $\chi 0 = exp\ \aleph_0$ and the cardinality of each neighbourhood of 0 is $exp\ \aleph_0$, so both upper bounds for $\sigma 0$ are the same and equal to $exp\ exp\ \aleph_0$. Choosing Δ consisting of all $\mathfrak{m} = \{ (0, x) \mid x \in S$, S is countable infinite $\}$ in the finest merotopy Γ , we obtain $\sigma x \leq card\ \Delta = exp\ \aleph_0$.

An interesting question remains: Let $\langle E, u \rangle$ be a semi-separated closure space and let $\aleph_\beta$ equal to σx for the finest merotopy inducing u . Given a cardinal number $\aleph_\alpha$ with $1 \leq \aleph_\alpha \leq \aleph_\beta$, does there exist a merotopy Γ for $\langle E , u \rangle$ such that $u_\Gamma = u$ and $\sigma x = \aleph_\alpha$?
In other words, we are interested in the question what are

the possible characters. The following theorem gives a partial answer to this question.

2.11. <u>Theorem</u>. Let $\langle E, u \rangle$ be a semi-separated closure space, $x \in E$. Then for every cardinal number $\aleph_\beta$, $1 \leq \aleph_\beta \leq \sigma x$, there exists a filter-merotopy Γ inducing u with $\sigma x = \aleph_\beta$.

2.12. <u>Lemma</u>. Let $\langle E, u \rangle$ be a semi-separated closure space, $x \in E$ and let there exist a system $\mathcal{B} \subset \exp E$ with the following properties:

(i) $\mathcal{B}$ is a filter base of some proper filter $\mathcal{F}$ on E;

(ii) $B \in \mathcal{B}$ implies $x \in B'$, $x \in (E - B)'$;

(iii) for each $B_1 \in \mathcal{B}$ there exists a $B_2 \in \mathcal{B}$ with $x \in (B_1 - B_2)'$;

(iv) if $\mathcal{B}_1 \subset \mathcal{B}$, $\operatorname{card} \mathcal{B}_1 < \operatorname{card} \mathcal{B}$ then $\bigcap \mathcal{B}_1 \in \mathcal{F}$.

Then there exists a filter-merotopy Γ inducing u with $\sigma x = \operatorname{card} \mathcal{B}$.

<u>Proof</u> of 2.11: Let $\aleph_\beta$ be a cardinal number. Let $\aleph_\sigma$ be the least cardinal number such that there exists a transfinite sequence of ordinal numbers $\{ \xi_\iota \mid \iota < \omega_\sigma \}$ converging to ω_β. We shall write $\aleph_\sigma = \operatorname{cf} \aleph_\beta$.

Since $\sigma x \geq \aleph_\beta$, we can find a system $\{ S_\gamma \mid \gamma \in C \}$ which is disjoint, $x \in S'_\gamma$ for every $\gamma \in C$, $\bigcup \{ S_\gamma \mid \gamma \in C \}$ is a neighbourhood of x and $\operatorname{card} C = \aleph_\beta$.

First, suppose that $\operatorname{cf} \aleph_\beta = \aleph_\beta$. Denote $B_K = \bigcup \{ S_\gamma \mid \gamma \in C - K \}$ for every non-void $K \subset C$ with $\operatorname{card} K < \operatorname{card} C$. Then the family $\mathcal{B} = \{ B_K \mid K \subset C, \operatorname{card} K < \operatorname{card} C \}$ satisfies 2.12 (i),(ii),(iii),(iv) ((iv) follows from the condition $\mathrm{cf}\, \aleph_\beta = \aleph_\beta$). The statement of Theorem 2.11 follows from 2.12.

Secondly, let $\aleph_\sigma = \mathrm{cf}\, \aleph_\beta < \aleph_\beta$. Then there exists a sequence $\{ \aleph_\iota \mid \iota < \omega_\sigma \}$ such that $\{ \omega_\iota \mid \iota < \omega_\sigma \}$ converges to ω_β and $\mathrm{cf}\, \aleph_\iota = \aleph_\iota < \aleph_\beta$ holds for all $\aleph_\iota$, $\iota < \omega_\sigma$.

Consider a disjoint union $C = \bigcup \{ H_\iota \mid \iota < \omega_\sigma ,$ $\mathit{card}\, H_\iota = \aleph_\iota \}$. Define on each subspace $D_\iota = \bigcup \{ S_\gamma \mid \gamma \in H_\iota \} \cup (x)$ the merotopy Γ_ι with $\sigma_\iota x = \aleph_\iota$ in the same way as in the first part. Let Φ_ι be a Γ_ι -fundamental system and let $\Delta_\iota \subset \mathit{ess}\, \Gamma_\iota$ with $\mathit{card}\, \Delta_\iota = \sigma_\iota x$ satisfy 2.4 (i),(ii). Let $\mathcal{M}_1$ be a filter with the base: $\{ \bigcup \{ D_\iota \mid \iota < \omega_\sigma , \iota \notin F \} \cap O \mid O$ a neighbourhood of x , F a finite set of ordinal numbers $\}$ and let $\bigcup \{ \Phi_\iota \mid \iota < \omega_\sigma \} \cup (\mathcal{M}_1)$ be a Γ -fundamental system.

Put $\Delta = \bigcup \{ \Delta_\iota \mid \iota < \omega_\sigma \} \cup (\mathcal{M}_1)$. It is easy to prove that Δ satisfies 2.4 (i),(ii), and that $\mathit{card}\, \Delta = \aleph_\beta$. In the way of contradiction let us suppose $\aleph_\varkappa = \sigma x < \aleph_\beta$. Let us choose $\aleph_\iota > \aleph_\varkappa$. Then the restriction of Γ to D_ι has its local merotopic character at x not greater than $\aleph_\varkappa$. Since this restriction coincides with Γ_ι , this is a contradiction with $\sigma_\iota x = \aleph_\iota$.

It remains to prove 2.12.

Proof of 2.12. Let $\mathcal{O}(y)$ be the neighbourhood system of y for each $y \in E$. Let us define the Γ -fundamental system Θ in the following way:

$$\Theta = \{ \mathcal{M}_y \mid y \neq x,\ y \in E \} \cup \{ \mathcal{M}_B \mid B \in \mathcal{B} \} \cup (\mathcal{M}_0)$$

$$\mathfrak{M}_y \doteq \sigma(y) ,$$

$$\mathfrak{M}_0 = \{ B \cap U \cup (x) \mid U \in \sigma(x), B \in \mathcal{B} \} ,$$

$$\mathfrak{M}_B = \{ (E - B) \cap U \cup (x) \mid U \in \sigma(x) \}$$

and let $\Delta = (\mathfrak{M}_0) \cup \{ \mathfrak{M}_B \mid B \in \mathcal{B} \}$.

It is simple to verify that Γ is a filter-merotopy inducing u and $card\ \Delta = card\ \mathcal{B}'$.

We verify 2.4 (i),(ii) for Δ . 2.4(i) is obvious. Let $M_{\mathfrak{M}} \in \mathfrak{M}$ be chosen for each $\mathfrak{M} \in \Delta$. Then $M_{\mathfrak{M}_0}$ is of the form $M_{\mathfrak{M}_0} = B_1 \cap U_1 \cup (x)$ and there exists a $V \in \sigma(x)$ contained in $\bigcup \{ M_{\mathfrak{M}} \mid \mathfrak{M} \in \Delta \}$ because

$$\bigcup \{ M_{\mathfrak{M}} \mid \mathfrak{M} \in \Delta \} \supset M_{\mathfrak{M}_0} \cup M_{\mathfrak{M}_{B_1}} =$$

$$= B_1 \cap U_1 \cup (x) \cup (E - B_1) \cap U_2 \cup (x) \supset U_1 \cap U_2 .$$

Thus we have 2.4 (ii).

It remains to prove that $card\ \Delta$ is really the minimal cardinality of the system fulfilling 2.4 (i),(ii). Suppose that there exists $\Delta_1 \subset \Gamma$ with $card\ \Delta_1 < card\ \Delta$ having all the needed properties. We may assume $\Delta_1 \subset \Delta$, because Γ is a filter-merotopy and $\Delta \subset \Theta$. Let $\mathcal{B}_1 \subset \mathcal{B}$ be defined by $B \in \mathcal{B}_1$ iff $\mathfrak{M}_B \in \Delta_1$. Since $card\ \mathcal{B}_1 < card\ \mathcal{B}_i$ the set $L = \cap \mathcal{B}_1$ is by (iv) non-void, by (ii) x is a cluster point of L , and by (iii) there exists a $B^* \in \mathcal{B}$ with $x \in (L - B^*)'$.

Suppose that $\mathfrak{M}_0 \notin \Delta_1$. Let V be the neighborhood of x, $V \subset \bigcup \{ M_{\mathfrak{M}_B} \mid B \in \mathcal{B}_1 \}$. Since $x \in L'$,

$O \cap L - (x)$ is non-void for every $O \in \mathcal{O}(x)$. Thus

$$\emptyset \neq V \cap L - (x) \subset \bigcup \{ M_{\mathfrak{m}_B} \mid B \in \mathcal{B}_1 \} \cap L \subset$$

$$\subset \bigcup \{ E - B \mid B \in \mathcal{B}_1 \} \cap L =$$

$$= (E - \bigcap \{ B \mid B \in \mathcal{B}_1 \}) \cap L = (E - L) \cap L = \emptyset$$

which is a contradiction. We see that $\mathfrak{m}_o$ must belong to Δ_1.

Since $x \in (L - B^*)'$, it is $O \cap (L - B^*) - (x) \neq \emptyset$ for every $O \in \mathcal{O}(x)$. Let us choose $M_o \in \mathfrak{m}_o$ such that $M_o = (B^* \cap U) \cup (x)$. Let V be a neigbourhood of x with $V \subset \bigcup \{ M_{\mathfrak{m}_B} \mid B \in \mathcal{B}_1 \} \cup M_o$. We have

$$\emptyset \neq (L - B^*) \cap V - (x) \subset (L - B^*) \cap$$

$$\cap (\bigcup \{ M_{\mathfrak{m}_B} \mid B \in \mathcal{B}_1 \} \cup M_o) \subset (L - B^*) \cap (\bigcup \{ E - B \mid B \in \mathcal{B}_1 \} \cup$$

$$\cup B^*) = (L - B^*) \cap ((E - \bigcap \{ B \mid B \in \mathcal{B}_1 \}) \cup B^*) =$$

$$= (L - B^*) \cap ((E - L) \cup B^*) = (L - B^*) \cap (E - (L - B^*)) = \emptyset$$

which is a contradiction.

Thus we have that the cardinality of Δ cannot decrease. The lemma is proved.

2.13. Lemma (Kuratowski, [6]). Let F be a mapping defined for all ordinal numbers $\xi < \omega_\sigma$ such that $F(\xi)$ is a set of cardinality $\aleph_\sigma$ whenever $\xi < \omega_\sigma$. Then there exists a mapping $G(\xi)$ defined for all $\xi < \omega_\sigma$ with the following properties:

$G(\xi) \cap G(\xi') = \emptyset$ for $\xi \neq \xi'$;

$G(\xi) \subset F(\xi)$ for all $\xi < \omega_\sigma$;

$\operatorname{card} G(\xi) = \aleph_\sigma$ for all $\xi < \omega_\sigma$.

Using Kuratowski's Lemma one can prove that in every space with $\chi X \leq \gamma X$ there exists a collection

$\{R_\alpha \mid \alpha \in A\}$ having the property 2.8 (1) and with $\mathrm{card}\ A = \gamma x$. In this case we have the following

Corollary. Let $\langle E, u \rangle$ be a semi-separated space and let $x \in E$, $\chi x \leq \gamma x$. Then for every cardinal number $\aleph_\alpha$ with $1 \leq \aleph_\alpha \leq \gamma x$ there exists a merotopy Γ inducing u such that $\sigma x = \aleph_\alpha$.

3.

3.1. Now we shall study the spaces with fine non-discrete topology, i.e. the spaces with exactly one non-isolated point x, for whose neighbourhood system $\mathcal{O}(x)$ the family $\mathcal{U} = [\mathcal{O}(x)] \cap (E - (x))$ is an ultrafilter on $E - (x)$. Since $\sigma x = 1$ for fine non-discrete spaces, Theorem 2.11 says nothing new about its merotopies.

3.2. Proposition. Let $\langle E, u \rangle$ be a fine non-discrete closure space and let Γ be the finest merotopy on E inducing u. Then $\sigma x = \chi x$ for the non-isolated point $x \in E$.

Proof. Let $\Delta \subset \Gamma$ satisfy 2.4 (i),(ii). Then $\mathrm{card}\ \Delta = \mathrm{card}\ \{P_m \mid P_m - (x) = \{y \mid (x, y) \in m\}, m \in \Delta\}$ and the collection on the right hand side must be a base of a neighbourhood system.

3.3. Definition. Let $\mathcal{U}$ be an ultrafilter on a set E. We shall call $\mathcal{U}$ to be an $\aleph_\alpha$-ultrafilter, if $\bigcap\{U_\iota \mid U_\iota \in \mathcal{U}, \iota \in I\}$ belongs to $\mathcal{U}$ for every I, $\mathrm{card}\ I \leq \aleph_\alpha$.

3.4. Theorem (GCH). Let $\langle E, \Gamma \rangle$ be a semi-separated merotopic space, u_Γ the fine non-discrete closure, x

the non-isolated point of $\langle E, u_{\Gamma} \rangle$, $\sigma(x)$ its neighbourhood system and $[\sigma(x)] \cap (E - (x))$ an $\aleph_\alpha$-ultrafilter. Then $\sigma x \neq 1$ implies $\sigma x \geq \aleph_{\alpha+2}$.

Proof. We shall show that Δ cannot fulfil 2.4 (ii) for every $\Delta \subset \Gamma$ with $1 \neq \mathrm{card}\, \Delta \leq \aleph_{\alpha+1}$. Let $\Delta \subset \Gamma$ be fixed with $\mathrm{card}\, \Delta \leq \aleph_{\alpha+1}$ which has the property 2.4 (i). Let us write $\Delta = \{ \mathfrak{m}_\iota \mid \iota < \omega_{\alpha+1} \}$. Put $\mathcal{U} = [\sigma(x)] \cap (E - (x))$ and $\mathfrak{m}'_\iota = [\mathfrak{m}_\iota] - (x)$.

Put $U_1 = E$. We can find $A_1 \in \mathfrak{m}'_1$, $A_1 \notin \mathcal{U}$ (since $\sigma x \neq 1$), thus $V_1 = U_1 - A_1 - (x)$ belongs to $\mathcal{U}$, consequently there exists a $B_1 \in \mathfrak{m}'_1$, $B_1 \subset V_1$, $B_1 \notin \mathcal{U}$ (since $\mathfrak{m}_1$ minorizes $\sigma(x)$).

Put $U_1 = E - \bigcup \{ A_\varkappa \cup B_\varkappa \mid \varkappa < \iota \}$ for $\iota < \omega_{\alpha+1}$. Since $\mathcal{U}$ is an $\aleph_\alpha$-ultrafilter and all the sets $E - (A_\varkappa \cup B_\varkappa) - (x)$ belong to $\mathcal{U}$, it is $U_\iota \in \mathcal{U}$. Since $\mathfrak{m}'_\iota$ minorizes $\mathcal{U}$ and $\sigma x \neq 1$, there exists an $A_\iota \in \mathfrak{m}'_\iota$ with $A_\iota \subset U_\iota$, $A_\iota \notin \mathcal{U}$. Put $V_\iota = U_\iota - A_\iota - (x)$ and let us choose a $B_\iota \in \mathfrak{m}'_\iota$ with $B_\iota \notin \mathcal{U}$, $B_\iota \subset V_\iota$.

We can put $A = \bigcup \{ A_\iota \mid \iota < \omega_{\alpha+1} \}$, $B = \bigcup \{ B_\iota \mid \iota < \omega_{\alpha+1} \}$. If the collection Δ fulfils the condition 2.4 (ii), then both A and B belong to $\mathcal{U}$. Clearly $A \cap B = \emptyset$ and so at least one of the sets A, B is not a member of $\mathcal{U}$. This is a contradiction, thus $\sigma x \geq \aleph_{\alpha+2}$.

3.5. It was shown in 2.11 that there are many different values of σx for a given closure space. The only restriction on the merotopy was given, namely that it induces the given closure. The situation changes if we add another natural condition.

Let $\langle E_1, \Gamma_1 \rangle$, $\langle E_2, \Gamma_2 \rangle$ be merotopic spaces. Let a mapping $f: \langle E_1, \Gamma_1 \rangle \to \langle E_2, \Gamma_2 \rangle$ be continuous if and only if the mapping $f: \langle E_1, u_{\Gamma_1} \rangle \to \langle E_2, u_{\Gamma_2} \rangle$ is continuous. In other words, we shall study a local merotopic character with respect to embeddings F of the category of semi-separated closure spaces into the category of merotopic spaces which preserve the underlying set functor. A trivial example of such embedding is the functor $\langle E, u \rangle \to \langle E, \Gamma_u \rangle$. In this case $\sigma x = 1$ for all $x \in E$ and every $\langle E, u \rangle$.

Other examples: $\langle E, \Gamma_1 \rangle$ is an image of $\langle E, u \rangle$, iff Γ_1 is the finest merotopy inducing u; $\langle E, \Gamma_2 \rangle$ is an image of $\langle E, u \rangle$ iff the collections $\mathfrak{M}_X = \{M_a \cup (x) \mid a \in A, M_a = \{x_{a'} \mid a' > a\}\}$ form a Γ_2-fundamental system for each $x \in E$ and for each net $X = \{x_a \mid a \in A, A$ is directed$\}$ converging to $x \in E$; $\langle E, \Gamma_3 \rangle$ is an image of $\langle E, u \rangle$ iff the Γ_3-fundamental system Φ_3 consists of all $\mathfrak{M}_{\mathcal{U},x} = \{U \cup (x) \mid U \in \mathcal{U}\}$, where $\mathcal{U}$ is an ultrafilter converging to $x \in E$.

3.6. Proposition. Let F be an embedding of the category of semi-separated closure spaces into the category of merotopic spaces. Let $\sigma x \neq 1$ in $F\langle E, u \rangle$ for a space $\langle E, u \rangle$ and an $x \in E$. Then for every cardinal $\aleph_\alpha$ there exists a space $\langle E_0, \Gamma_0 \rangle$ with a point x_0 such that $\sigma_0 x_0 \geq \aleph_\alpha$.

This follows from the fact that the continuity of projections implies the existence of such x_0 in $F\langle E, u \rangle^{\aleph_\alpha}$.

3.7. Theorem. Given a cardinal number $\aleph_\alpha$, there exists an embedding F of the category of semi-separated closure spaces into the category of merotopic spaces such that $\sigma x \neq 1$ implies $\sigma x > \aleph_\alpha$.

Proof. We shall construct a merotopy Γ for every closure space $\langle E, u \rangle$ such that $F\langle E, u\rangle = \langle E, \Gamma \rangle$ will be the desired embedding.

Let $\langle E, u \rangle$ be a semi-separated closure space and $x \in E$. Let A_x be a set $A_x = \{\mathcal{U} \mid \mathcal{U}$ is an ultrafilter on $E - (x)$, x is a cluster point of $\mathcal{U}\}$. The fundamental system for Γ consists of all $((x))$ and of all collections $\mathcal{M}_{A,x} \subset \exp E$ of the form $\mathcal{M}_{A,x} = \bigcap \{[\mathcal{U}] \cup (x) \mid \mathcal{U} \in A\}$ where $A \subset A_x$, $card\, A \leq \aleph_\alpha$. It is clear that F defined by $F\langle E, u \rangle = \langle E, \Gamma \rangle$ is an embedding, for a continuous image of an ultrafilter converging to x is an ultrafilter converging to the image of x . Evidently, $u_\Gamma = u$.

Suppose that there exists a point $x \in E$ with $\sigma x \leq \aleph_\alpha$. Then there exists $\Delta = \{\mathcal{M} \mid \mathcal{M} \in \operatorname{esb} \Gamma\}$ with $card\, \Delta = \sigma x \leq \aleph_\alpha$. Further, let $M_{\mathcal{M}} \in \mathcal{M}$. Then there exists a neighbourhood O of a point x such that $O \subset \bigcup \{M_{\mathcal{M}} \mid \mathcal{M} \in \Delta\}$. Since we may assume that all $\mathcal{M} \in \Delta$ are filters of the form mentioned above, we have $\bigcap \{\mathcal{M} \mid \mathcal{M} \in \Delta\} = \mathcal{O}(x)$ where $\mathcal{O}(x)$ is a neighbourhood system of x . But $\mathcal{M} = \bigcap\{[\mathcal{U}] \cup (x) \mid \mathcal{U} \in A_{\mathcal{M}}\}$ and $card\, A_{\mathcal{M}} \leq \aleph_\alpha$, thus the inequality $card\, A \leq \aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$ holds for $A = \bigcup\{A_{\mathcal{M}} \mid \mathcal{M} \in \Delta\}$ and consequently $\bigcap\{[\mathcal{U}] \cup (x) \mid \mathcal{U} \in A\}$ belongs to Γ . As $\bigcap\{[\mathcal{U}] \cup (x) \mid \mathcal{U} \in A\} = \bigcap\{\mathcal{M} \mid \mathcal{M} \in \Delta\} = \mathcal{O}(x)$ it is $\mathcal{O}(x) \in \Gamma$ and $\sigma x = 1$.

3.8. Theorem. Let F be an embedding of the category of semi-separated closure spaces into the category of merotopic spaces. Consider $[0,1]$ with its usual topology. Then $\sigma x \neq 1$ implies $\sigma x > \aleph_0$ for every $x \in F[0,1]$. Assuming (CH), then σx in $F[0,1]$ can reach only two values, 1 and $\exp \aleph_0$.

<u>Proof</u>. Let F be an embedding and let $\langle [0,1], \Gamma \rangle = F[0,1]$. Let I_n denote the interval $[1/n+1, 1/n]$. W.l.o.g. we may assume that $x = 0$.

We say that a set $L \subset [0,1]$ has a property (V), if there exists a continuous mapping $f: L \to [0,1]$ which maps L onto $[0,1]$.

We say that a set $L \subset [0,1]$ has a property (F), if there exists an infinite sequence $\{k_n\}$ of natural numbers such that the set $I_{k_n} \cap L$ has a property (V) for all $n < \omega_0$.

Denote by $\mathbb{P}$ the following proposition:

"For each non-void subset $L \subset [0,1]$ with the property (F), for each $\mathcal{M} \in \operatorname{sec} \Gamma$ and for each V neighbourhood of x there exist a $\mathcal{U}$ neighbourhood of x, $\mathcal{U} \subset V$ and a set $M \in \mathcal{M}$ with $M \subset \mathcal{U}$ such that $O \cap (L - M)$ has a property (F) for every neighbourhood $\mathcal{O}$ of x, $\mathcal{O} \subset \mathcal{U}$".

Either $\mathbb{P}$ or $\neg \mathbb{P}$ must hold.

I. We shall prove that $\mathbb{P} \Rightarrow \sigma x > \aleph_0$. Suppose that $\sigma x \leq \aleph_0$. Then there exists a system $\Delta \subset \operatorname{sec} \Gamma$, $\operatorname{card} \Delta \leq \aleph_0$, satisfying the conditions of 2.4. We shall write $\Delta = \{\mathcal{M}_i\}$. Let W_n denote the neighbourhood $[0, 1/n]$.

For $L_1 = [0, 1]$, $\mathfrak{M}_1 \in \Delta$ and $V_1 = W_1$ there exist $\mathcal{U}_1 \subset V_1$, $M_1 \in \mathfrak{M}_1$, $M_1 \subset \mathcal{U}_1$ such that the set $O \cap (L_1 - M_1)$ has a property (F) for each neighbourhood $O \subset \mathcal{U}_1$; so there exists a natural number k_1 such that $W_{k_1} \subset \mathcal{U}_1$. Since $W_{k_1} \cap (L_1 - M_1)$ has the property (F), $W_{k_1} \cap (L_1 - M_1)$ is uncountable. Choose an $x_1 \in \in W_{k_1} \cap (L_1 - M_1)$ and let $j_1 > k_1$ be a natural number with $x_1 \notin W_{j_1}$.

Let $x_1, x_2, \dots, x_{\ell-1}$ be defined. As $W_{j_{\ell-1}} \cap \cap (L_{\ell-1} - M_{\ell-1})$ has a property (F), we can set $L_\ell = W_{j_{\ell-1}} \cap \cap (L_{\ell-1} - M_{\ell-1})$. Put $V_\ell = W_{j_{\ell-1}}$ and let $\mathfrak{M}_\ell \in \Delta$. Then there exist a neighbourhood $\mathcal{U}_\ell$ of x, $\mathcal{U}_\ell \subset V_\ell$ and a set $M_\ell \in \in \mathfrak{M}_\ell$ with $M_\ell \subset \mathcal{U}_\ell$ such that the set $O \cap (L_\ell - M_\ell)$ has the property (F) for each neighbourhood O, $O \subset \mathcal{U}_\ell$. Let k_ℓ be such a natural number that $k_\ell > j_{\ell+1}$ and $W_{k_\ell} \subset \mathcal{U}_\ell$. Since the set $W_{k_\ell} \cap (L_\ell - M_\ell)$ is uncountable, there exists a point $x_\ell \in W_{k_\ell} \cap (L_\ell - M_\ell)$. Let us choose $j_\ell > k_\ell$ with $x_\ell \notin W_{j_\ell}$.

If σx is finite, say $m = \sigma x$, then $(M_1 \cup M_2 \cup \dots \cup M_m) \cap W_{k_m} \cap (L_m - M_m)$ is void, and the point x is a cluster point of the set $W_{k_m} \cap (L_m - M_m)$ (because this set has the property (F)). If $\sigma x = \aleph_0$ then the sets $\bigcup \{ M_i \mid i < \omega_0 \}$ and $\{ x_i \mid i < \omega_0 \}$ are disjoint, and the sequence $\{ x_i \}$ converges to x.

In both cases we have found $M_m \in \mathfrak{M}$, which cannot cover any neighbourhood of x. From this contradiction it follows that $\sigma x > \aleph_0$. Assuming (CH) we have $\sigma x = = \exp \aleph_0$.

II. Now we show that $\neg P \Rightarrow \sigma x = 1$. Let $\neg P$ hold. Then there exist a set $L \subset [0, 1]$ with the property (F), a micromeric collection $\mathfrak{M} \in \cancel{\text{est}}\ \Gamma$ and a neighbourhood V of x such that we can find an O_M neighbourhood of x, $O_M \subset \mathcal{U}$, for which the set $O_M \cap (L - M)$ has the property $\neg(F)$ for each neighbourhood $\mathcal{U}$ of x, $\mathcal{U} \subset V$ and for each $M \in \mathfrak{M}$, $M \subset \mathcal{U}$. Let $\{k_n\}$ be a sequence of natural numbers and let $\{f_n\}$ be a sequence of continuous mappings defined on $L \cap I_{k_n}$ such that $f_n[L \cap I_{k_n}] = [0, 1]$. Let us denote $I_{k_n} = [a_{k_n}, b_{k_n}]$. We may assume that $\lim\{f_n y \mid y \to a_{k_n}+, y \in L \cap I_{k_n}\} = 0$, $\lim\{f_n y \mid y \to b_{k_n}-, y \in L \cap I_{k_n}\} = 1$. Put $J_1 = [a_{k_1}, 1]$, $J_n = [a_{k_n}, a_{k_{n-1}}]$ for $n = 2, 3, 4, \dots$ and let h_n be a linear increasing function of $[0, 1]$ onto J_n. Let P be a Peano's mapping, i.e. the continuous mapping defined on $[0, 1]$ which maps $[0, 1]$ onto $[0, 1] \times [0, 1]$. Let π_i $(i = 1, 2)$ be the projections defined by $\pi_i \langle x_1, x_2 \rangle = x_i$ $(i = 1, 2)$. Finally, let φ_n be the composition $\varphi_n = h_n \circ \pi_1 \circ P \circ f_n$ and let $\varphi : (0) \cup (L \cap \bigcup\{I_{k_n}\}) \to [0, 1]$ be defined by $\varphi \mid L \cap I_{k_n} = \varphi_n$ for $n < \omega_0$, $\varphi 0 = 0$. Clearly φ is a continuous mapping of the set $(0) \cup (L \cap \bigcup\{I_{k_n} \mid n < \omega_0\})$ onto $[0, 1]$ which maps the neighbourhood base $\{(0) \cup \bigcup\{I_{k_n} \cap L \mid n > i\} \mid i < \omega_0\}$ of x onto the neighbourhood base $\{(0) \cup \bigcup\{J_n \mid n > i\} \mid i < \omega_0\}$ of x. (The first base is the base in the subspace $(0) \cup (L \cap \bigcup\{I_{k_n} \mid n < \omega_0\}$.)

Since the set $O_M \cap (L - M)$ has not the property (F), there exists a natural number n_M such that $I_{k_n} \subset O_M$ for all $n > n_M$ and further no continuous mapping defined on $(L - M) \cap I_{k_n}$ is onto $[0, 1]$. As a consequence, choosing $g_n = \pi_2 \circ P \circ f_n$ there exists a point $y_n \in [0, 1]$ such that the set $g_n^{-1}[y_n] \cap (L - M) \cap I_{k_n}$ is void. Since $g_n[L \cap I_{k_n}]$ contains y_n, it follows $g_n^{-1}[y_n] \subset \subset M \cap L \cap I_{k_n}$. But then $\pi_1 \circ P \circ f_n [M \cap I_{k_n} \cap L] \supset \supset \pi_1 \circ P \circ f_n [g_n^{-1}[y_n]] = [0, 1]$.

Thus we have proved that $\xi_n [M \cap I_{k_n} \cap L] = J_n$ for each $M \in \mathfrak{M}$.

The space $(0) \cup (L \cap \bigcup \{ I_{k_n} \mid n < \omega_0 \})$ is a subspace of $[0, 1]$, therefore $F((0) \cup (L \cap \bigcup \{ I_{k_n} \mid n < \omega_0 \}))$ is a subspace of $F[0, 1]$.
Let us denote by Γ_L the merotopy of $F((0) \cup \cup (L \cap \bigcup \{ I_{k_n} \mid n < \omega_0 \}))$. As $\mathfrak{M}$ belongs to Γ, the collection $[\mathfrak{M}] \cap ((0) \cup (L \cap \bigcup \{ I_{k_n} \mid n < \omega_0 \}))$ belongs to Γ_L. The mapping ξ is continuous, hence Γ-continuous and thus we have $\xi [[\mathfrak{M}] \cap ((0) \cup (L \cap \bigcup \{ I_{k_n} \mid n < \omega_0 \}))] \in \Gamma$. But this system is a neighbourhood base of x; it follows that $\mathcal{O}(x) \in \Gamma$. This completes the proof.

3.9. __Theorem__. Let $\langle E, u \rangle$ be a space which can be embedded into $[0, 1]^{\aleph_0}$ and suppose that the Cantor discontinuum can be embedded into every open subset of $\langle E, u \rangle$. (For example, all uncountable separable complete metrizable spaces with no isolated point have this property.) Let F be an embedding of the category of semi-separated closure spaces into the category of merotopic spaces. Then $\sigma x \neq 1$ implies $\sigma x > \aleph_0$ in $F\langle E, u \rangle$. Assuming (CH), then σx in $F\langle E, u \rangle$ can reach only two values, 1 and $\exp \aleph_0$.

Proof. Let us denote by $\mathcal{C}$ the Cantor discontinuum. It suffices to prove that local merotopic characters in $\mathcal{C}$ and in $[0,1]^{\aleph_0}$ are equal. $\mathcal{C}$ and $\mathcal{C}^{\aleph_0}$ being homeomorphic, their local merotopic characters are equal. $\mathcal{C}^{\aleph_0}$ can be mapped onto $[0,1]^{\aleph_0}$ in such a way that the image of every neighbourhood in $\mathcal{C}^{\aleph_0}$ is a neighbourhood in $[0,1]^{\aleph_0}$. Moreover, $\mathcal{C}^{\aleph_0}$ is a subspace of $[0,1]^{\aleph_0}$. Thus $F\mathcal{C}$ and $F[0,1]^{\aleph_0}$ have the same local merotopic characters. The rest of the statement of the theorem follows by 3.8.

3.10. Remark. The space $E = [0,1]$ has two properties which are crucial for the proof of 3.8:

a) Every point has a countable neighbourhood base ;

b) Let $A, B \subset E$, $x \in A'$ and let the set A be continuously mapped by a function f onto a neighbourhood O of a point fx. Let $V - g[B] \neq \emptyset$ for every continuous function g and for each neighbourhood V of gx. Then the set $A - B$ can be continuously mapped by a function h onto a neighbourhood U of hx. Moreover, we can find the function h independently on the choice of B. (The mapping ξ defined in the proof of 3.8 has this property.)

It is obvious that assuming a) and b) we can prove the theorem analogous to 3.8 by a mere modification of the given proof. I do not know what class of closure spaces has these properties and I have no example of a space possessing a) but not b).

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