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ERROR BOUNDS FOR EIGENVALUES AND EIGENFUNCTIONS OF SOME ORDINARY DIFFERENTIAL OPERATORS BY THE METHOD OF LEAST SQUARES

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1. We shall consider a numerical approximation by the method of least squares for the eigenvalues and eigenfunctions of the following real boundary value problem

(1) $\mathcal{M}u(x) = \lambda \cdot u(x), \quad x \in (0,1)$

subject to the homogeneous boundary conditions

(2) $\mathcal{U}(u(x)) = 0,$

where

$$\mathcal{M}u(x) = \sum_{j=0}^{m} (-1)^{j} \cdot [p_j(x) u^{(j)}(x)]^{(j)} ,$$

(3) $p_j(x) \in C^{(j)}_{\langle 0,1 \rangle}, \; j = 1,\dots,m, \; p_m(x) > 0$ on $\langle 0,1 \rangle$

and the homogeneous boundary conditions of (2) consist of $2m$ linearly independent conditions of the form

(4) $\sum_{k=1}^{2m} \{ m_{jk} u^{(k-1)}(0) + n_{j,k} u^{(k-1)}(1) \} = 0, \; 1 \le j \le 2m .$

We assume that the eigenvalue problem (1) - (2) is self-adjoint in the sense that

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(5) $(\mathcal{M}u, v) = (u, \mathcal{M}v)$ for all $u, v \in \mathcal{D}$,

where $\mathcal{D}$ denotes the set of real-valued functions of the class $C^{(2m)}_{\langle 0,1\rangle}$ which satisfy the homogeneous boundary conditions (2) and

$(u, v) = \int_0^1 u(t)\cdot v(t)dt$ for $u(t)$, $v(t)$ in $L^2_{\langle 0,1\rangle}$.

We also assume that there exists a real constant K such that

(6) $(\mathcal{M}u, u) \geq K\cdot(u, u)$ for all $u \in \mathcal{D}$.

With the assumptions (5) and (6) the eigenvalue problem of (1) - (2) has countably many eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ which are real and have no finite limit point, and can be arranged as follows:

(7) $$\lambda_1 \leq \lambda_2 \leq \dots \lambda_k \leq \dots \ .$$

The associated normalized eigenfunctions $\{\varphi_j(x)\}_{j=1}^{\infty}$, $\varphi_j \in C^{(2m)}_{\langle 0,1\rangle}$ form a complete orthogonal system in $L^2_{\langle 0,1\rangle}$.

For each positive integer k let $K_2^k\langle 0,1\rangle$ denote the collection of all real-valued functions u defined on $\langle 0,1\rangle$ such that each $u \in C^{(k-1)}_{\langle 0,1\rangle}$ and $u^{(k-1)}(x)$ is absolutely continuous with $u^{(k)} \in L^2_{\langle 0,1\rangle}$. Now let M denote a differential operator of the form (1) with the domain $\mathcal{D}(M)$ in $L^2_{\langle 0,1\rangle}$ – a real separable Hilbert space, where

$$\mathcal{D}(M) = \{u \in K_2^{2m}\langle 0,1\rangle ;\ u \text{ satisfies } (2)\}.$$

Let $\{\Psi_i\}_{i=1}^{\infty}$, $\Psi_i \in \mathfrak{D}(M)$ be a totally complete system (cf.[1]) and μ be a real number such that

(8) $$\inf_k |\lambda_k - \mu| = |\lambda_j - \mu| > 0 .$$

By Theorem 3 of [1], we have

$$\lim_{N \to \infty} q_N = |\lambda_j - \mu| ,$$

where q_N^2 is the smallest eigenvalue of the algebraic eigenvalue problem

$$\mathcal{A}_N u - \sigma \mathcal{B}_N u = 0 ;$$

the matrices $\mathcal{A}_N = \{\alpha_{ij}\}_{i,j=1}^N$ and $\mathcal{B}_N = \{\beta_{ij}\}_{i,j=1}^N$ have their entries given by

$$\alpha_{ij} = (M_\mu \Psi_i, M_\mu \Psi_j), \ \beta_{ij} = (\Psi_i, \Psi_j), \ i, j = 1, \ldots, N,$$

$$M_\mu v = Mv - \mu \cdot v \quad \text{for} \quad v \in \mathfrak{D}(M) .$$

Let R_N and $\mathcal{R}_N$ be subspaces of $L^2_{\langle 0,1 \rangle}$ determined by the functions $\{\Psi_i\}_{i=1}^N$ and $\{M_\mu \Psi_i\}_{i=1}^N$, respectively.

By Theorem 1 of [3] there exists a constant C_1, independent of N, such that

$$q_N - |\lambda_j - \mu| \leq C_1 \cdot \delta_N^2 ,$$

$$\delta_N = \inf_{t \in \mathcal{R}_N} \| \varphi_j - t \| ,$$

where φ_j is a normalized eigenfunction of M associated with the eigenvalue λ_j. We shall call

$$\lambda_j^N = \mu + q_N \cdot \operatorname{sign} [\lambda_j - \mu]$$

an approximate eigenvalue. Thus

(9) $$|\lambda_j - \lambda_j^N| \leq C_1 \cdot \sigma_N^2 \quad .$$

Suppose the eigenvalues $\{\lambda_i\}$ of (1) - (2) satisfy the following assumption

(10) $$|\lambda_{j-1}| < |\lambda_j| < |\lambda_{j+1}| \quad .$$

Construct $\{u_N\}$ such that the following conditions be satisfied:

1) $u_N \in R_N , \quad \| u_N \| = 1 ,$

2) $q_N = \| M_\mu u_N \| ,$

3) $(u_N , u_{N+1}) \geq 0 .$

By Theorems 2 and 3 of [3] there exist constants C_2, C_3, K_1, K_2, K_3 and an integer N_1 such that for $N \geq N_1$

(11) $$\lambda_1^N = \mu + q_N \cdot sign\,[(M_\mu u_N , u_N)] \quad ,$$

$$C_2 \cdot \sigma_N^2 \leq |\lambda_j - \lambda_j^N| \leq C_1 \cdot \sigma_N^2 \quad ,$$

(12) $$\| u_N - \varphi_j \| \leq C_3 \cdot \sigma_N \quad ,$$

and

$$K_2 \cdot \varepsilon_N^2 \leq |\lambda_j - \lambda_j^N| \leq K_1 \cdot \varepsilon_N^2 \quad ,$$

$$\| u_N - \varphi_j \| \leq K_3 \cdot \varepsilon_N \quad ,$$

where $\varepsilon_N = q_N - |(M_\mu u_N , u_N)|$.

We shall call u_N an approximate eigenfunction for (1) - (2).

We now apply the method of least squares to appropri-

ately selected finite dimensional subspaces R_N of $\mathcal{D}(M)$.
In particular, we consider polynomial subspaces and subspaces of L-spline functions. We derive the asymptotic order of accuracy for the approximate eigenvalues, as well as for the approximate eigenfunctions.

2. As our first example, we consider $P_0^{(N)}$, the $(N+1-2m)$-dimensional subspace of $L^2_{<0,1>}$ consisting of all real polynomials of degree $\leq N$ which satisfy the boundary conditions of (2).

Let B be the operator with the domain $\mathcal{D}(M)$ defined by

(13) $$Bx = x^{(2m)} \quad \text{for} \quad x \in \mathcal{D}(M) .$$

The problem $Bx = 0,\ x \in \mathcal{D}(M)$ has only the trivial solution. On the basis of the functional analytical theory of differential equations there exists a continuous operator B^{-1} mapping $L^2_{<0,1>}$ into $L^2_{<0,1>}$ such that

$$B^{-1}u = \int_0^1 G(t,\tau)\, u(\tau)\, d\tau, \quad u \in L^2_{<0,1>} ,$$

where $G(t,\tau)$ is the Green's function for the problem $Bx = 0$.
We now present an elementary lemma which will be essentially used later.

Lemma 1. With the assumptions of (3), (8) and (13), let $C = M_u B^{-1}$ be a linear operator whose domain is $\mathcal{D}(C)$, $\mathcal{D}(C) = \{u \in L^2_{<0,1>};\ u$ is piecewise continuous on $\langle 0, 1 \rangle$ } and whose range is in $L^2_{\langle 0,1\rangle}$. Then C is continuous.

<u>Proof</u>. If $f \in \mathcal{D}(C)$ then there exist the points $\{x_i\}_{i=1}^{k}$, $x_i \in (0, 1)$ such that $f \in C(\bigcup_{i=0}^{k} (x_i, x_{i+1}))$, where $x_0 = 0$, $x_{k+1} = 1$.

If $x \in (x_i, x_{i+1})$, $0 \leq i \leq k$, it follows from the definition of the Green's function that

$$(B^{-1}f)^{(j)}(x) = \int_0^1 G_x^{(j)}(x,t) \cdot f(t)\,dt \text{ for } 0 \leq j \leq 2m-1$$

and $(B^{-1}f)^{(2m)}(x) = f(x)$.

Since M_μ can be written as

$$M_\mu[u] = \sum_{i=0}^{2m} a_i(x) u^{(i)}(x),\ a_i(x) \in C_{\langle 0,1\rangle},\ 0 \leq i \leq 2m,$$

we have $Cf = M_\mu B^{-1} f = v$, where

$$v(x) = a_{2m}(x) \cdot f(x) + \int_0^1 (\sum_{i=0}^{2m-1} a_i(x) G_x^{(i)}(x,t)) \cdot f(t)\,dt$$

for each $x \in (x_j, x_{j+1})$, $0 \leq j \leq k$.

It follows by direct computation that $\| Cf \| \leq Q \cdot \| f \|$, where

$$Q = a + b, \qquad a = \max_{x \in \langle 0,1\rangle} |a_{2m}(x)| ,$$

$$b = (\int_0^1 \int_0^1 | \sum_{i=0}^{2m-1} a_i(x) G_x^{(i)}(x,t)|^2 dt\,dx)^{\frac{1}{2}} .$$

Note that Q does not depend on $\{x_i\}_{i=1}^{k}$ and this completes the proof of the lemma.

Corollary. With the assumptions (3) and (8), let $R_N \subset \mathcal{D}(M) \cap \mathcal{D}(C)$. Then there exists a constant C_4, dependent on j and m but independent of N, such that

$$\delta_N \equiv \inf_{t \in R_N} \| \varphi_j - t \| \leq C_4 \cdot \inf_{t \in R_N} \| \varphi_j^{(2m)} - t^{(2m)} \|^2 .$$

(We make use of the fact that the eigenfunctions $\{\varphi_j\}$ of (1) - (2) are of the class $C^{(2m)}_{\langle 0,1 \rangle}$ and $M_\mu \varphi_j = (\lambda_j - \mu) \cdot \varphi_j$.)

We remark that if $N \geq 2m$, then the set $P = \{ t^{(2m)}, t \in P_0^{(N)} \}$ is a finite dimensional subspace of $\mathcal{D}(M) \cap \mathcal{D}(C)$ consisting of all real polynomials of the degree $\leq N - 2m$. The following result is obtained from Corollary and Jackson's Theorem of [4], p.113.

Theorem 1. (a) With the assumptions (3) and (8), let λ_j^N be the approximate eigenvalue of (1) - (2), obtained by applying the method of least squares to the subspace $P_0^{(N)}$ of $L^2_{\langle 0,1 \rangle}$, where $N \geq 2m$. If the eigenfunction φ_j of (1) - (2) is in $C^{(t)}_{\langle 0,1 \rangle}$, with $t \geq 2m$, then there exists a constant D_1 dependent on m and j but independent of N, such that

$$(14) \quad | \lambda_j - \lambda_j^N | \leq D_1 \cdot \left[\frac{1}{(N-2m)^{t-2m}} \cdot \omega\left(\varphi_j^{(t)}, \frac{1}{N-2m}\right)\right]^2$$

for all $N \geq 2m$, where ω is the modulus of continuity.

(b) With the assumptions (a) let

$$|\lambda_{j-1}| < |\lambda_j| < |\lambda_{j+1}|$$

and let u_N be the approximate eigenfunction for (1) - (2), obtained by applying the method of least squares to $P_0^{(N)}$. Then there exists a constant D_2 and an integer N_0, dependent on j and m but independent of N, such that

$$(15) \quad \| \varphi_j - u_N \| \leq D_2 \cdot \frac{1}{(N-2m)^{t-2m}} \cdot \omega\left(\varphi_j^{(t)}, \frac{1}{N-2m}\right)$$

for all $N \geq m$.

(c) If, in addition, the eigenfunction φ_j is analytic in some open set of the complex plane containing the interval $\langle 0, 1 \rangle$, then there exist constants μ_1 and μ_2, $\mu_i \in \langle 0, 1 \rangle$, $i = 1, 2$, such that

$$\overline{\lim_{N \to \infty}} |\lambda_j^N - \lambda_j|^{\frac{1}{N}} = \mu_1 ,$$

and

$$\overline{\lim_{N \to \infty}} (\| \varphi_N - u_N \|)^{\frac{1}{N}} = \mu_2 .$$

Remark 1. If there exists a constant $K_2 \geq 0$ such that $\max_{x \in \langle 0,1 \rangle} |u(x)| \leq K_2 \cdot \| M_\mu u \|$ for all $u \in \mathcal{D}(M)$, then we may obtain error estimates in the uniform norm for the approximate eigenfunctions.

Remark 2. If the hypotheses of Theorem 1 hold, then the error of the approximate eigenvalue λ_j^N has the order of magnitude $\sigma(d^{-2t+4m})$ and the error of the approximate eigenfunction u_N in the norm $\| \cdot \|_{L^2\langle 0,1 \rangle}$ has the order of magnitude $\sigma(d^{-t+2m})$, where $d = \dim P_0^{(N)} = N + 1 - 2m$.

We now assume that $\lambda_i \neq 0$ for $i = 1, 2, \ldots$ and consider S_N, the $(N+1)$-dimensional subspace of $L^2_{\langle 0,1 \rangle}$ consisting of all real functions of the form

$M^{-1}t$, where t is a real polynomial of the degree $\leqq N$. From Lemma 1 and Lemma 5 of [3], we obtain

Theorem 2. Let the assumptions (a) in Theorem 1 be satisfied and let $\lambda_i \neq 0$ for any integer i . Let λ_j^N be the approximate eigenvalue of (1) - (2) obtained by applying the method of least squares to the subspace $R_N \equiv S_N$ of $L^2_{<0,1>}$. Then there exists a consonant D_3 , dependent on j and m but independent of N , such that

(16) $$|\lambda_j^N - \lambda_j| \leq D_3 \cdot \frac{1}{N^{2t}} \cdot [\omega(u^{(t)}, \frac{1}{N})]^2$$

for all $N \geqq 1$.

If, in addition, the assumptions (b) in Theorem 1 are satisfied, then there exist a constant D_4 and an integer N_0 such that

(17) $$\| u_N - \varphi_j \| \leq D_4 \cdot [\frac{1}{N^t} \cdot \omega(u^{(t)}, \frac{1}{N})]$$

for $N \geq N_0$.

Remark 3. Theorem 2 gives us that

$$|\lambda_j^N - \lambda_j| = o(d^{-2t}) ,$$

and $\| u_N - \varphi_j \| = o(d^{-t})$, where $d = \dim S_N = N+1$.

3. As our second example, we consider subspaces of L -spline functions introduced in [5]. We now restrict for reasons of brevity to the special homogeneous boundary conditions of the following form

(18) $$u^{(k)}(0) = u^{(k)}(1) = 0, \quad 0 \leq k \leq m-1 .$$

Let L be the m -th order linear differential operator defined by

$$Lu = \sum_{k=0}^{m} a_k(x) \cdot u^{(k)}(x) , \quad x \in \langle 0, 1 \rangle$$

for all $u \in K_2^m \langle 0, 1 \rangle$. We assume that $a_k(x) \in K_2^m \langle 0,1 \rangle$, $0 \le k \le m$, and $a_m(x) \ge \omega > 0$ for all $x \in \langle 0, 1 \rangle$.

Let $\pi : 0 = x_0 < x_1 < \dots < x_N < x_{N+1} = 1$ denote a partition of the interval $\langle 0, 1 \rangle$ and let $z = (z_0, z_1, \dots, z_N, z_{N+1})$, the incidence vector, be an $(N+2)$ - vector with positive integer components each less than or equal to m , i.e., $1 \le z_j \le m$, $j = 0, \dots, N+1$. The class of all L -splines for fixed π and z with $z_0 = z_{N+1} = m$ we denote by $Sp(L, \pi, z)$, which corresponds to the boundary interpolation of Type I in [5]. Note that if $Lu = u^{(m)}$ and $z = (m, 1, \dots, 1, m)$ then $Sp(L, \pi, z)$ is the space of ordinary spline functions $Sp(\pi)$. If $z = (m, m, \dots, m)$ and $Lu = u^{(m)}$, then $Sp(L, \pi, z)$ is the Hermite space $H^{(m)}(\pi)$ of piecewise polynomial functions.

We remark that if $m > n$, then $Sp_0(L, \pi, z)$, the subset of elements of $Sp(L, \pi, z)$ which satisfy the boundary conditions of (18), is a finite-dimensional subspace of $\mathcal{D}(M) \cap \mathcal{D}(C)$.

Let $\{\pi_k\}_{k=1}^{\infty}$ be a sequence of partitions of $\langle 0, 1 \rangle$ such that $\lim_{k \to \infty} \bar{\pi}_k = 0$, $\bar{\pi}_k = \max_{i=0,\dots,N_k} | x_i - x_{i+1} |$ and let σ be a positive constant such that $\sigma \underline{\pi}_k \ge \bar{\pi}_k$ for all $k \ge 1$, $\underline{\pi}_k = \min_{i=0,\dots,N_k} | x_i - x_{i+1} |$. Let $z^{(k)}$

be an incidence vector associated with π_k .

If $\varphi_j \in K_2^{2m} \langle 0,1 \rangle$, $m > n$, then there exist a positive integer k_0 and a constant G , dependent on j and m but not on k, such that

$$\| \varphi_j^{(2n)} - s_k^{(2n)} \| \leq G \cdot (\bar{\pi}_k)^{2m-2n} , \quad k \geq k_0 ,$$

where s_k is a unique $Sp(L, \pi_k, x^{(k)})$ -interpolate of φ_j (cf.[5]). Since $s_k \in Sp_0(L, \pi_k, x^{(k)})$, the following result follows immediately from Corollary.

Theorem 3. Let $\{\pi_k\}_{k=1}^{\infty}$ be a sequence of partitions of $\langle 0, 1 \rangle$ such that $\lim_{k \to \infty} \bar{\pi}_k = 0$ and $\sigma \cdot \underline{\pi}_k \geq \bar{\pi}_k$ for all $k \geq 1$, where σ is a positive constant. Let $\{x^{(k)}\}_{k=1}^{\infty}$ be a corresponding sequence of incidence vectors associated with $\{\pi_k\}_{k=1}^{\infty}$.
With the assumptions (3) and (8), let λ_j^N be the approximate eigenvalue of (1) - (18) obtained by applying the method of least squares to the subspace

$R_N \equiv Sp_0(L, \pi_k, x_k)$ of $L^2\langle 0,1 \rangle$.

If the eigenfunction φ_j of (1) - (13) is in $K_2^t \langle 0,1 \rangle$ with $t \geq 2m > 2n$, then there exist a constant G , dependent on j and n and m but independent of k, and a positive integer k_0 such that

$$(19) \qquad | \lambda_j^N - \lambda_j | \leq G \cdot (\bar{\pi}_k)^{4m-4n}$$

for all $k \geq k_0$.

If, in addition,

$$|\lambda_{j-1}| < |\lambda_j| < |\lambda_{j+1}| ,$$

then there exist a constant G_1 dependent on j, n and m but independent of k, and a positive integer k_1 such that

(20) $$\| u_N - \varphi_j \| \leq G_1 \cdot (\bar{\pi}_k)^{2m-2n}$$

for all $k \geq k_1$.

<u>Remark 4.</u> Let $\{\pi_k\}_{k=1}^{\infty}$ be a sequence of partitions of $\langle 0, 1 \rangle$ such that $\lim_{k \to \infty} \bar{\pi}_k = 0$ and let $\{x^{(k)}\}_{k=1}^{\infty}$ be a corresponding sequence of incidence vectors associated with $\{\pi_k\}_{k=1}^{\infty}$.

Define $\mathcal{S}_k$ as the class of real-valued functions of the form

$$u = B^{-1}\Psi, \ \Psi \in Sp\,(L, \pi_k, x^{(k)}), \ k = 1, 2, \dots .$$

With the assumptions of (3) and (8), let λ_j^N be the approximate eigenvalue of (1) - (18) obtained by applying the method of least squares to the subspace $R_N \equiv \mathcal{S}_k$.

If $\varphi_j \in K_2^t \langle 0, 1 \rangle$, $t \geq 2n + 2m$, then there exist constants G_2 , G_3 and a positive integer k_0 such that

$$|\lambda_j - \lambda_j^N| \leq G_2 \cdot (\bar{\pi}_k)^{4m}$$

for any $k \geq k_0$.

If, in addition, $|\lambda_{j-1}| < |\lambda_j| < |\lambda_{j+1}|$, then there exist a constant G_4 and an integer k_1 such that

$$\| u_N - \varphi_j \| \leq G_4 \cdot (\bar{\pi}_k)^{2m}$$

for any $k \geq k_1$. This follows from Lemma 1 and Theorem 9 of [5].

In [7], Ciarlet, Schultz and Varga obtain the asymptotic order of accuracy for the approximate eigenvalues and for the approximate eigenfunctions by applying the Rayleigh-Ritz method to $P_0^{(N)}$ and to $Sp(L, \pi, z)$. Comparing the above theorems and remarks with the results of [7] we see that the asymptotic order of accuracy for the approximate eigenvalues and the approximate eigenfunctions obtained by the method of least squares are very close to those of [7]; more precisely, (16), (17), (19) and (20) correspond to (5.1), (5.4), (5.9) and (5.10) of [7], respectively.

We remark on the other hand that the principal advantage of the method of least squares is that we need not know the eigenvalue λ_i for $i < j$ and the corresponding eigenfunctions to obtain an approximation of λ_j . Moreover, one can obtain upper or lower numerical approximations of the eigenvalues and the eigenfunctions of (1) - (2) by choosing a parameter μ appropriately.

The behaviour of the constants C_i and K_i , $i = 1, 2, 3$ of (11) depending on j are studied and the results will be published later.

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