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AN EXAMPLE CONCERNING SET-FUNCTORS

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In her paper [2], V. Trnková studied set-functors preserving limits of certain diagrams, leaving open the problem of the existence of a big set-functor preserving finite limits. The aim of this note is to construct a big set-functor preserving finite limits and colimits up to a given cardinal (see Definition 4). The existence of a proper class of measurable cardinals is assumed (see Definition 2).

First we shall recall some well-known definitions:

Definition 1 . Let $\mathcal{F}$ be an ultrafilter on a set A. Let α be a cardinal. Then $\mathcal{F}$ is said to be α-complete if for every collection $\{X_\iota ; \iota \in J\}$ of sets of $\mathcal{F}$, $card\ J < \alpha$ implies $\bigcap_{\iota \in J} X_\iota \in \mathcal{F}$.

Definition 2. A cardinal α is said to be measurable if there exists an α-complete ultrafilter on α.

Convention 1. Throughout this note, the word functor denotes a covariant functor from the category of sets into itself.

Definition 3. A functor F is said to be small if

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there exists a set A such that for every set $X \neq \emptyset$

$$F(X) = \bigcup_{f: A \to X} F(f)[F(A)] .$$

A functor F is said to be big if it is not small.

Definition 4. Let $D : \mathcal{D} \longrightarrow \mathcal{S}$ be a diagram ($\mathcal{S}$ is the category of sets). Let $(X, \{\pi_d ; d \in \mathcal{D}^{obj}\})$ be its limit (or colimit resp.). Let F be a functor. We shall say that F preserves limit of D if $(F(X), \{F(\pi_d) ; d \in \mathcal{D}^{obj}\})$ is a limit (or colimit resp.) of $F \circ D$.

We shall say that F preserves limits (or colimits resp.) up to a cardinal α if it preserves limit of any diagram $D : \mathcal{D} \longrightarrow \mathcal{S}$ such that $\mathrm{card}\, \mathcal{D}^m < \alpha$. ($\mathcal{D}^m$ is the set of all morphisms of $\mathcal{D}$.)

We shall say that F preserves finite limits if it preserves limits up to $\aleph_0$.

Convention 2. Let F, G be functors. Denote $F \subset G$ if

(1) $F(X) \subset G(X)$,

(2) $x \in F(X) \Longrightarrow F(f)(x) = G(f)(x)$

holds for every X and every $f : X \longrightarrow Y$.

Definition 5. Let $\mathcal{J}$ be a directed class. Let a functor F_ι be given for every $\iota \in \mathcal{J}$. Assume

(3) $\iota < \iota' \Longrightarrow F_\iota \subset F_{\iota'}$,

(4) $\bigcup_{\iota \in \mathcal{J}} F_\iota(X)$ is a set for every set X .

Define a functor F by

$F(X) = \bigcup_{\iota \in \mathcal{J}} F_\iota(X)$ for every set X ,

$F(f)(x) = F_\alpha(f)(x)$ for every $x \in F(X)$, $f : X \to Y$,

$\alpha \in \mathcal{J}$ is arbitrary with $x \in F_\alpha(X)$.
(The correctness of the definition of $F(f)$ is guaranted by (3).)

We shall call F the union of F_ι, $\iota \in \mathcal{J}$ and we shall write

$$F = \bigcup_{\iota \in \mathcal{J}} F\iota .$$

Lemma 1. Let $F\iota$; $\iota \in \mathcal{J}$, $F = \bigcup_{\iota \in \mathcal{J}} F\iota$ be as in Definition 5. If $F\iota$; $\iota \in \mathcal{J}$ preserve **finite limits, so does** F .

Proof. I. It is well-known [1] that a functor preserving equalisers and products of any two sets preserves all finite limits.

II. F preserves equalisers. Really, if $f, g : X \rightarrow$ $\rightarrow Y$ are arbitrary, $E = \{x ; f(x) = g(x)\}$, $j : E \rightarrow X$ is the inclusion then

$$\{x \in F(X); F(f)(x) = F(g)(x)\} = \bigcup_{\iota \in \mathcal{J}} \{x ; F_\iota(f)(x) = F_\iota(g)(x)\} = \bigcup_{\iota \in \mathcal{J}} F_\iota(j)[F_\iota(E)] = F(j)F[E] .$$

III. F preserves products of any two sets: Let X_1, X_2 be sets, let $\pi_i : X_1 \times X_2 \rightarrow X_i$ $(i = 1, 2)$ be the canonical projections. We have to prove that for every $x_1 \in F(X_1)$, $x_2 \in F(X_2)$ there is exactly one $z \in F(X_1 \times X_2)$ with $F(\pi_i)(z) = x_i$, $i = 1, 2$.

The existence of z : Choose $\iota \in \mathcal{J}$ with $x_i \in F_\iota(X_i)$, $i = 1, 2$; as F_ι preserves products, there is exactly one $z \in F_\iota(X_1 \times X_2)$ with $F_\iota(\pi_i)(z) = x_i$, $i = 1, 2$. As $F_\iota \subset F$, the last equalities are equivalent to those which we had to prove.

The unicity of x : Assume that $F(\pi_i)(x') = x_i$, $i = 1, 2$ for some $x' \in F(X_1 \times X_2)$. Choose ι with $x, x' \in F_\iota(X_1 \times X_2)$. Thus we have $F_\iota(\pi_i)(x') = x_i$, $F_\iota(\pi_i)(x) = x_i$ which implies $x = x'$.

Lemma 2. Let F_ι ; $\iota \in \mathcal{J}$, $F = \bigcup_{\iota \in \mathcal{J}} F_\iota$ be as in Definition 5. If $\mathcal{J}$ is a linear ordered proper class and if for any $\iota \in \mathcal{J}$ there is $\iota' \in \mathcal{J}$ such that $\iota < \iota'$ and $F_\iota \neq F_{\iota'}$, then F is big.

Proof. Assume that F is small i.e. that there exists a set A such that for every set $X \neq \emptyset$

$$F(X) = \bigcup_{f: A \to X} F(f)[F(A)] .$$

As the ordering of $\mathcal{J}$ is linear, there is $\alpha \in \mathcal{J}$ with $F(A) = F_\alpha(A)$. Consequently,

$$F(X) = \bigcup_{f: A \to X} F_\alpha(f)[F_\alpha(A)] \subset F_\alpha(X)$$

for every $X \neq \emptyset$. Choose $\beta \in \mathcal{J}$ with $F(\emptyset) = F_\beta(\emptyset)$ and put $\varepsilon = \max\{\alpha, \beta\}$. Thus, we have $F(X) \subset \subset F_\varepsilon(X) \subset F_\iota(X) \subset F(X)$ for $\iota > \varepsilon$; hence $F_\varepsilon = F_\iota$ for every $\iota > \varepsilon$ which is in contradiction with the assumptions of the lemma.

Definition 6. Let F_ι ; $\iota \in \mathcal{O}rd$ be a system of functors such that $I \subset F_\iota$ for $\iota \in \mathcal{O}rd$ (I is the identical functor.) Define functors G_ι, $\iota \in \mathcal{O}rd$ by the transfinite induction as follows:

(5) $G_0 = F_0$, $G_\iota = F_\iota \circ \bigcup_{\beta < \iota} G_\beta$.

(Evidently, G_β, $\beta < \iota$ form an increasing sequence and thus the definition of $\bigcup_{\beta<\iota} G_\beta$ is correct.) Let us assume that

(6) for every set X there is an ordinal α such that $F_\gamma(G_\alpha(X)) = G_\alpha(X)$ for every $\gamma > \alpha$.

Then G_ι, $\iota \in \mathcal{O}rd$ satisfy the conditions (3),(4) from Definition 5 and we can define a functor $\mathcal{S}upp\, F_\iota$ by $\mathcal{S}upp\, F_\iota = \bigcup_{\iota \in \mathcal{O}rd} G_\iota$.

Remark 1. If F, G preserve finite limits, so does $F \circ G$.

Lemma 1′. Let F_ι, $\iota \in \mathcal{O}rd$, $F = \mathcal{S}upp\, F_\iota$ be as in Definition 6. If F_ι, $\iota \in \mathcal{O}rd$ preserve finite limits, so does F.

Lemma 2′. Let F, $\iota \in \mathcal{O}rd$, $F = \mathcal{S}upp\, F_\iota$ be as in Definition 6. If for any $\iota \in \mathcal{O}rd$ there is $\beta > \iota$ with $F_\beta \neq I$, then F is big.

Proofs of the last two lemmas follow from the definition of $\mathcal{S}upp\, F_\iota$ and from Lemma 1, Lemma 2, Remark 1.

Now, we recall the definition of a functor $Q_{A,\mathcal{F}}$ where A is a set and $\mathcal{F}$ a filter on A (see [2]): If X is a set, then the elements of $Q_{A,\mathcal{F}}(X)$ are equivalence-classes on the set of all $f: A \to X$ with respect to the equivalence $f \sim g \equiv \{x;\ f(x) = g(x)\} \in \mathcal{F}$. For every $f: A \to X$ define $[f]$ by $f \in [f] \in Q_{A,\mathcal{F}}(X)$. If $f: X \to Y$ is an arbitrary mapping then $Q_{A,\mathcal{F}}(f)([g]) = [f \circ g]$. For every X, $x \in X$ define $\hat{x}: A \to X$ by $\hat{x}(a) = x$ for every $a \in A$ and put $\mu^X(x) = [\hat{x}]$. Evidently, μ is a monotransformation from I to $Q_{A,\mathcal{F}}$.

Hence there is a functor $\tilde{Q}_{A,\mathcal{F}}$ and an isotransformation $\varepsilon : Q_{A,\mathcal{F}} \longrightarrow \tilde{Q}_{A,\mathcal{F}}$ such that $\varepsilon^X \circ \mu^X(x) = x$ for every set X and $x \in X$. Thus, $I \subset \tilde{Q}_{A,\mathcal{F}}$.

Remark 2. [2] $\tilde{Q}_{A,\mathcal{F}}$ preserves finite limits.

Remark 3. (a) If $\mathcal{F}$ is a filter on A and X a set, then

$$\operatorname{card} Q_{A,\mathcal{F}}(X) \leq (\operatorname{card} X)^{\operatorname{card} A} .$$

(b) If $\mathcal{F}$ is an α-complete ultrafilter and if X is a set with $\operatorname{card} X < \alpha$ then $\tilde{Q}_{A,\mathcal{F}}(X) = X$.

Proof of (a) is easy, (b) follows from the well-known fact that every function $f : A \longrightarrow X$ is (under our assumptions on X and $\mathcal{F}$) constant on a set of the filter $\mathcal{F}$.

Theorem. For every cardinal α there exists a functor preserving finite limits and colimits up to α.

Proof. Let $\{ m_\iota ; \iota \in \mathcal{O}rd \}$ be a class of measurable cardinals such that $m_0 > \alpha$ and $m_\beta < m_\gamma$ whenever $\beta < \gamma$.

For every $\iota \in \mathcal{O}rd$ choose a m_ι-complete ultrafilter $\mathcal{F}_\iota$ on m_ι. Put $F_\iota = \tilde{Q}_{m_\iota,\mathcal{F}_\iota}$ and define G_ι as in Definition 6.

Let X be a set with $\operatorname{card} X < m_\iota$ for some $\iota \in \mathcal{O}rd$. As each measurable cardinal is unaccessible, we can easily prove by the transfinite induction that $\operatorname{card} G_\beta(X) < m_{\iota+1}$ for $\beta \leq \iota$. In particular, $\operatorname{card} G_\iota(X) < m_{\iota+1}$ which implies (see Remark 3(b)) that $F_\beta(G_\iota)(X)) = G_\iota(X)$ for $\beta > \iota$.

Hence we can define $F = \mathrm{Supp}\, F_\iota$.

F is big by Lemma 2′ and it preserves finite limits by Remark 2 and Lemma 1′ .

As $\mathcal{F}_\iota$ are α-complete ultrafilters, F_ι defined above preserve coproducts up to α (see [2]). It may be easily proved that $F = \mathrm{Supp}\, F_\iota$ also does.

There was proved in [2] that a functor preserving coproducts up to α, $\alpha > \aleph_0$, preserves coequalisers and thus preserves colimits up to α .

R e f e r e n c e s

[1] J.M. MARANDA: Some remarks on limits in categories, Canad.Math.Bull.5(1962),133-136.

[2] V. TRNKOVÁ: On descriptive classification of set-functors, I.II. Part I,Comment.Math.Univ.Carolinae 12(1971),143-175; Part II to appear later in the same journal.

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