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A REMARK ON THE THEORY OF DIOPHANTINE APPROXIMATIONS

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Let β be an irrational number and $(b_0; b_1, b_2, \dots)$ its (simple) continued fraction expansion. For $t \geq 1$ let

$$\psi_\beta(t) = \min_{\substack{p,q \text{ int.} \\ 0<q\leq t}} |q\beta - p| .$$

It is well known that $0 < t\,\psi_\beta(t) < 1$ for every $t \geq 1$. Let us set

$$\lambda(\beta) = \liminf_{t\to+\infty} t\,\psi_\beta(t), \quad \mu(\beta) = \limsup_{t\to+\infty} t\,\psi_\beta(t) .$$

The aim of this paper is to prove some theorems for the numbers $\mu(\beta)$ which were announced in Preliminary communication [2].

First, we introduce some notation. For any positive integer N we denote by $\mathcal{B}(N)$ the set of all β for which $\limsup_{k\to+\infty} b_k = N$ (i.e. from certain suffix k_0 on is $b_k \leq N$ and $b_k = N$ for infinitely many k). A number $\alpha = (a_0; a_1, a_2, \dots)$ will be called equivalent to β if there exists an integer n such that $a_{k+n} = b_k$ for all sufficiently large k. We use the symbol $\alpha \sim \beta$ or $\alpha \nsim \beta$ according to whether α and β are equivalent or not. If $\alpha \sim \beta$ then obviously $\lambda(\alpha) = \lambda(\beta)$,

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$\mu(\alpha) = \mu(\beta)$. We shall use a standard notation for the period of a continued fraction; e.g.

$$(\overline{1;2}) = (1; 2, 1, 2, \dots) = \frac{1}{2}(1+\sqrt{3}).$$

Let us start with the following simple

Lemma.

$$\mu(\beta) = \frac{1}{1+\frac{1}{R_\beta}},$$

where

$$R_\beta = \limsup_{k\to+\infty} (b_k; b_{k-1}, \dots, b_1).(b_{k+1}; b_{k+2}, \dots)$$

$\left(\frac{1}{R_\beta} = 0\right.$ for $\left. R_\beta = +\infty\right)$.

It is sufficient to prove the lemma for $0 < \beta < 1$. If $\frac{p_n}{q_n}$ denotes the n-th convergent of β, then clearly

$$\mu(\beta) = \limsup_{k\to+\infty} q_{k+1}|q_k\beta - p_k|.$$

Now (see e.g. [1] chapter I, § 2)

$$q_{k+1}|q_k\beta - p_k| = (1+\Theta_{k+1}\varphi_k)^{-1},$$

where

$$\Theta_{k+1} = (0; b_{k+1}, b_{k+2}, \dots), \quad \varphi_k = \frac{q_k}{q_{k+1}} = (0; b_k, b_{k-1}, \dots, b_1).$$

Let $\mathfrak{M}(N)$ be the set of all R_β with $\beta \in \mathcal{L}(N)$, and let $\mathfrak{M} = \bigcup_{N=1}^{\infty} \mathfrak{M}(N)$. By the lemma we see immediately that

$$\frac{1}{2} \le \mu(\beta) \le 1.$$

Further $\mu(\beta) = 1$ if and only if the sequence $b_1, b_2, \dots$ is unbounded, and thus $\mu(\beta) < 1$ if and only if $\beta \in \bigcup_{N=1}^{\infty} \mathcal{L}(N)$. Now the structure of the sets $\mathfrak{M}(N)$ and $\mathfrak{M}$ will be studied.

Theorem 1. [1] Let

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1) This theorem was first proved by J. Lesca [6]; it was proved by B. Diviš independently in 1968 (see [2]). See also [7].

$c_j = 1\,,\ j = 0, 1, 2, \dots\,,\ \alpha_0 = (c_0\,;\, c_1, c_2, \dots)\,,$

$\alpha_m = (\overline{2\,;\, c_1, c_2, \dots, c_{2m-1}})\,,\quad m = 1, 2, \dots\ .$

Then

a) $R_{\alpha_0} = \frac{1}{2}(3 + \sqrt{5})\,,$

b) $R_{\alpha_j} < R_{\alpha_{j+1}}\,,\quad j = 0, 1, 2, \dots\,,$

c) $\lim\limits_{j \to +\infty} R_{\alpha_j} = 2 + \sqrt{5}\ .$

d) If $R_\beta < 2 + \sqrt{5}$ then there exists a non-negative integer j such that $\beta \sim \alpha_j$.

The proof may be found in [6].

Theorem 2. Let N be a positive integer, $\alpha = (\overline{1\,;\, N})$. If $\beta \in \mathcal{L}(N)$, then $R_\beta \geq R_\alpha = \alpha N + 1 =$ $= \frac{1}{2}(N + 2 + \sqrt{N^2 + 4N})$. 2)

Moreover, there exists a positive constant c_N depending only on N such that $R_\beta \geq R_\alpha + c_N$ whenever $\beta \in$ $\in \mathcal{L}(N)$ and $\beta \nsim \alpha$.

Proof. We denote by c (in general different) positive constants which depend only on N . Without loss of generality we may restrict ourselves to the case $N \geq 2$ and $1 \leq b_k \leq N$, $k = 1, 2, \dots$. Notice that

(1) $$\alpha^2 N = \alpha N + 1\ .$$

Evidently, it is sufficient to prove that $R_\beta \geq R_\alpha + c$ whenever $\beta \in \mathcal{L}(N)$ and $\beta \nsim \alpha$. Denote this statement by (T) . We have that (T) holds:

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2) See also P. Flor, Inequalities among some real modular functions, Duke Math.J.26(1959),679-682 (added in proof).

a) If for infinite number of positive integers k we have $b_k = N$, and $max(b_{k-1}, b_{k+1}) > 1$. (Obviously, $R_\beta > 2N > R_\alpha$.)

b) If either

$$b_k = 1,\ b_{k+1} = N,\ b_{k+2} = 1,\ b_{k+3} = a \le \frac{1}{2}N,$$

or

$$b_k = a \le \frac{1}{2}N,\ b_{k+1} = 1,\ b_{k+2} = N,\ b_{k+3} = 1.$$

for an infinite number of positive integers k.

In this case obviously we have

$$R_\beta \ge (N; 1, \dots).(1; a, \dots) \ge (N; 1, \alpha).(1; a, \alpha)$$

i.e. $R_\beta \ge (N + \frac{\alpha}{\alpha+1})(1 + \frac{\alpha}{\alpha a + 1})$.

According to (1), the difference

$$(N + \frac{\alpha}{\alpha+1})(1 + \frac{\alpha}{\alpha a+1}) - (\alpha N + 1)$$

can be written as follows

$$\frac{1}{a\alpha + 1}(N - 2a + \frac{\alpha^2 + a - 1}{\alpha + 1}).$$

The last expression is at least

$$\frac{\alpha^2}{(\alpha+1)(\alpha N + 1)} = \frac{1}{N\alpha + N} = c$$

because $a \le \frac{1}{2}N$.

c) If either

$$b_k = 1,\ b_{k+1} = N,\ b_{k+2} = 1,\ b_{k+3} = b,\ b_{k+4} = a,$$

or

$$b_k = a,\ b_{k+1} = b,\ b_{k+2} = 1,\ b_{k+3} = N,\ b_{k+4} = 1$$

with $b > \frac{1}{2}N$ and $a > 1$, for an infinite number of positive integers k.

With respect to a) it is sufficient to consider only the case $b_k \neq N$ (i.e. $N > 2$). We have

$$R_\beta \ge (2; \dots).(b; 1, N, 1, \dots) > 2(b; 1, N) = 2b + \frac{2N}{N+1}.$$

Since $R_\alpha < N + 2$, $2b \ge N + 1$

we get $R_\beta - R_\alpha > \frac{N-1}{N+1} = c$.

If $\beta \in \mathscr{L}(N)$ and $\beta \not\sim \alpha$, then, according to a) and b), it is sufficient to consider only the case when the number N occurs infinitely many times in a group

$$a, 1, N, 1, b ,$$

where $\frac{1}{2} N < \min(a, b) < N$. Hence, according to a), and c), it is sufficient to assume that the number N occurs infinitely many times in a group

$$1, a, 1, N, 1, b, 1 ,$$

where $\frac{1}{2} N < \min(a, b) < N$.

But then

$$R_\beta \geq (N; 1, b, 1, \dots) . (1; N-1, \alpha) \geq$$

$$\geq (N; 1, b, 1, \alpha) . (1; N-1, \alpha) ,$$

where $N \geq b > \frac{1}{2} N$.

Since $(1; N-1, \alpha) = \dfrac{\alpha N + 1}{\alpha(N-1)+1}$,

it is sufficient to prove the inequality

$$(N; 1, b, 1, \alpha) > \alpha(N-1) + 1$$

or, as we easily see, the inequality

$$b + \frac{\alpha}{\alpha+1} > \frac{(N-1)(\alpha-1)}{\alpha + N - \alpha N} .$$

Using (1), this inequality can be rewritten in the form

$$b > \frac{\alpha(N-2)}{\alpha+1} .$$

Since $b > \frac{1}{2} N$, it is sufficient to show that

$$\frac{1}{2} N > \frac{\alpha(N-2)}{\alpha+1}$$

or

$$N + \alpha (4 - N) > 0 .$$

The last inequality is trivial for $N \leq 4$. For $N > 4$ we get

$$\alpha < \frac{N}{N-4} = 1 + \frac{4}{N-4} ,$$

which is true.

Remark. Theorem 2 can also be formulated as follows: the minimal point of the set $\mathfrak{M}(N)$ is its isolated point. Also the following estimates of the constants c_N can be determined: $c_N \approx \frac{1}{N}$.

Theorem 3. Let α be as in Theorem 2. If $\beta \in \mathcal{L}(N)$, then

$$R_\beta \leq N R_\alpha = \frac{1}{2} N (N + 2 + \sqrt{N^2 + 4N}) . \quad ^{2)}$$

If $N > 1$ and $\varepsilon > 0$, then there exist uncountable sets $\mathfrak{N}, \mathfrak{N}_\varepsilon \subset \mathcal{L}(N)$ of mutually inequivalent numbers such that

$$\beta \in \mathfrak{N} \Longrightarrow R_\beta = N R_\alpha ,$$

$$\gamma \neq \beta , \ \gamma, \beta \in \mathfrak{N}_\varepsilon \Longrightarrow R_\beta \neq R_\gamma ,$$

$$N R_\alpha - \varepsilon < R_\beta < N R_\alpha .$$

Proof. Let $\beta \in \mathcal{L}(N)$; i.e. we may assume that $1 \leq b_i \leq N$, $i = 1, 2, \dots$. Obviously

$$(b_k ; b_{k-1}, b_{k-2}, \dots, b_1) . (b_{k+1} ; b_{k+2}, \dots) \leq$$

$$\leq (N; \overline{1, N}) . (N ; \overline{1, N}) = (N; \alpha)^2 = N(\alpha N + 1) = N R_\alpha .$$

Let $N > 1$. Since there are only countably many numbers equivalent to a given number, it is sufficient in both cases to prove existence of uncountable sets $\mathfrak{N}, \mathfrak{N}_\varepsilon \subset \mathcal{L}(N)$ with the required properties.

Let $\mathscr{A}$ be the set of all sequences on 1 and 2 . For $A = \{a_j\}_{j=1}^{\infty} \in \mathscr{A}$ we define $A_n = (a_1, a_2, \dots, a_n)$, $n = 1, 2, \dots$. We construct the elements of $\mathscr{N}$ as follows:

$\beta_A = (0; A_1, N, N, A_2, 1, N, N, 1, A_3, N, 1, N, N, 1, N, \dots)$

i.e. between A_n and A_{n+1} there is always a group of $2n$ numbers

$$\underbrace{1, N, 1, N, \dots, 1, N}_{n \text{ numbers}}, \qquad \underbrace{N, 1, N, 1, \dots, N, 1}_{n \text{ numbers}}$$

for n even, and

$$\underbrace{N, 1, N, \dots, 1, N}_{n \text{ numbers}}, \qquad \underbrace{N, 1, N, 1, \dots, N, 1, N}_{n \text{ numbers}}$$

for n odd.

For distinct elements $A \in \mathscr{A}$ we get different numbers $\beta = \beta_A \in \mathscr{L}(N)$ and, obviously, $R_\beta = N R_\alpha$.

For the proof of the second part of the theorem, let $\mathscr{A}$ be the set of all $\beta \in (0,1) \cap \mathscr{L}(N)$ such that $1 \le b_j \le N$, $j = 1, 2, \dots$, with the following property: if $b_j = N$ for some j, then $b_{j-1} = b_{j+1} = 1$ (for $j = 1$ we set $b_2 = 1$). If n is a positive integer, we denote by A_n the following group of $4n + 6$ numbers

$$1, 1, \underbrace{N, 1, N, 1, \dots, N, 1}_{2n}, N, N, \underbrace{1, N, \dots, 1, N, 1, N}_{2n}, 1, 1 \quad .$$

To given β we order a number

$$\varphi_n(\beta) = (0; b_1, A_n, b_1, 1, b_2, b_1, A_n, b_1, b_2, 1, \dots,$$

$$\dots 1, b_k, b_{k-1}, \dots, b_1, A_n, b_1, b_2, \dots, b_k, 1, \dots) = (0; c_1, c_2, \dots) .$$

Since (as can be shown by a direct computation)

$(N; \dots)(1, \dots) \le (N; \alpha) . (1; \alpha) < (N; N, \alpha)^2 \le (N; \dots)(N; \dots)$

we have

$$R_{\varphi_n(\beta)} = \limsup_{j \to +\infty} (c_{k_j}; c_{k_j-1}, \dots, c_1) \cdot (c_{k_j+1}; c_{k_j+2}, \dots),$$

where $k_1, k_2, \dots$ is the set of all positive integers k for which $c_k = c_{k+1} = N$. From this it follows that

$$R_{\varphi_n(\beta)} = (N; \underbrace{1, N, 1, N, \dots, 1, N}_{2n}, 1, 1, \beta^{-1})^2 < NR_\alpha .$$

Now the set $\mathcal{U}$ is uncountable, $\lim_{n \to \infty} R_{\varphi_n(\beta)} = NR_\alpha$ for each fixed β, and, finally, $R_{\varphi_n(\beta)}$ is a continuous and increasing function of β for each fixed n. This completes the proof of Theorem 3.

Remark. Thus, for $N > 1$, the maximal point of the set $\mathfrak{M}(N)$ is its condensation point and it is assumed for uncountably many $\beta \in \mathcal{L}(N)$.

Remark. Analogous statements for the values $\lambda(\beta)$ are proved in [4] and in some other papers of the same author. For each positive integer N we denote by $\mathfrak{M}_1(N)$ the set of all $\lambda(\beta)$ with $\beta \in \mathcal{L}(N)$. Then the maximal point of the set $\mathfrak{M}_1(N)$ (which is its isolated point) is the number $(N^2+4)^{-\frac{1}{2}}$ and the minimal point of this set (which for $N > 1$ is its point of condensation) is the number $(N^2+4N)^{-\frac{1}{2}}$.

Remark. A natural question that arises is that of studying the minimal condensation point of $\mathfrak{M}(N)$. This question will be the subject of a further paper.

Using the results of [3], one can show that there exists a number λ_0 such that $\lambda(\beta)$ assumes every value of the interval $[0, \lambda_0]$ (see [1], p.44). An analogous result is shown in

Theorem 4. a) There exists a number R^* such that $[R^*, +\infty) \subset \mathfrak{M}$,

b) for all sufficiently large N ($N \geq 5$) the set $\mathfrak{M}(N)$ contains some interval,

c) $R^* \leq \bar{R} = 12 + 8\sqrt{2} = 23.3136\ldots$.

Proof. For each positive integer n we denote by $F(n; 4)$ the set of all real numbers $\beta = (b_0; b_1, b_2, \ldots)$ for which $b_0 = n$, $b_j \leq 4$ ($j \geq 1$). Marshall Hall Jr. proved (see [3], Theorem 3.2, p.974) that for $n \geq 1$ each number $\gamma \in J_n$,

$$J_n = [n^2 + (\sqrt{2}-1)n + \tfrac{1}{4}(3 - 2\sqrt{2}),\ n^2 + 4(\sqrt{2}-1)n + 12 - 8\sqrt{2}],$$

can be written in a form $\gamma = \beta_1 \cdot \beta_2$, where $\beta_1 \in F(n; 4)$, $\beta_2 \in F(n; 4)$. Similarly, each number $\delta \in K_n$,

$$K_n = [n^2 + \sqrt{2}\,n + \tfrac{1}{4},\ n^2 + (4\sqrt{2} - 3)n + 10 - 6\sqrt{2}]$$

can be written in a form $\delta = \beta_3 \cdot \beta_4$, where $\beta_3 \in F(n; 4)$, $\beta_4 \in F(n+1; 4)$.

Evidently, $\bigcup_{n=5}^{\infty} (J_n \cup K_n) = [\frac{83}{4} + \frac{9}{2}\sqrt{2}, +\infty)$.

Thus an arbitrary $\lambda \geq \frac{83}{4} + \frac{9}{2}\sqrt{2} = 27.11\ldots$ can be written in a form $\lambda = (a_0; a_1, a_2, \ldots) . (b_0; b_1, b_2, \ldots)$, where $b_0 + 1 \geq a_0 \geq b_0 \geq 5$ and $a_j \leq 4$, $b_j \leq 4$ for $j \geq 1$. We construct a number $\varkappa = (d_0; d_1, d_2, \ldots)$ as follows:

$$\varkappa = (a_0; b_0, a_1, a_0, b_0, b_1, a_2, a_1, a_0, b_0, b_1, b_2, \ldots,$$
$$\ldots, a_n, a_{n-1}, \ldots, a_1, a_0, b_0, b_1, \ldots, b_{n-1}, b_n, \ldots).$$

We claim that $R_\varkappa = \lambda$.

Let us put $s_k = (d_{k-1}; d_{k-2}, \ldots, d_1) . (d_k; d_{k+1}, \ldots)$.

Then, by the lemma, $R_\varkappa = \limsup_{k \to +\infty} s_k$.

Now, for all positive integers n

$$d_{n^2} = b_0 , \quad d_{n^2-1} = a_0 ,$$

and thus

$$\limsup_{n\to+\infty} s_{n^2} = \limsup_{n\to+\infty} (d_{n^2-1}; d_{n^2-2},\dots, d_1).(d_{n^2}; d_{n^2+1},\dots)=$$

$$= \limsup_{n\to+\infty} (a_0; a_1,\dots, a_{n-1}, b_{n-2},\dots, a_0).(b_0; b_1, b_2,\dots, b_{n-1}, a_n,\dots)=$$

$$= \limsup_{n\to+\infty} (a_0; a_1,\dots, a_{n-1}).(b_0; b_1,\dots, b_{n-1}) =$$

$$= \limsup_{n\to+\infty} (a_0; a_1,\dots, a_{n-1}).(b_0; b_1,\dots, b_{n-1}) = \lambda .$$

Similarly,

$$\limsup_{n\to+\infty} s_{n^2-1} = \limsup_{n\to+\infty} (d_{n^2-2}; d_{n^2-3},\dots, d_1).(d_{n^2-1}; d_{n^2},\dots)=$$

$$= \limsup_{n\to+\infty} (d_{n^2-2}; d_{n^2-3},\dots, d_1). (a_0; b_0, d_{n^2+1},\dots) \leq$$

$$\leq (\overline{4;1}).(a_0; b_0) < 5\,(a_0 + \frac{1}{b_0}) \leq b_0\,(a_0 + \frac{1}{b_0}) < \lambda .$$

Analogously, we have

$$\limsup_{n\to+\infty} s_{n^2+1} = \limsup_{n\to+\infty} (d_{n^2}; d_{n^2-1},\dots, d_1).(d_{n^2+1}; d_{n^2+2},\dots)=$$

$$= \limsup_{n\to+\infty} (b_0; a_0, d_{n^2-2},\dots, d_1).(d_{n^2+1}; d_{n^2+2},\dots) \leq$$

$$\leq (b_0; a_0).(\overline{4;1}) < \lambda .$$

Finally, let k be a positive integer, $|k - n^2| \geq 2$ for $n = 1, 2, \dots$. Then

$$s_k = (d_{k-1}; d_{k-2},\dots).(d_k; d_{k+1},\dots) < 5.5 < \lambda .$$

Hence $R_{\mathfrak{x}} = \limsup_{k\to+\infty} s_k = \limsup_{n\to+\infty} s_{n^2} = \lambda$.

Thus, we have proved that for $N \geq 5$

$$J_N \subset \mathfrak{M}(N) , \quad K_N \subset \mathfrak{M}(N+1)$$

and

$$\bigcup_{n=5}^{\infty} (J_n \cup K_n) = [\,\frac{83}{4} + \frac{9}{2}\sqrt{2} , +\infty) \subset \mathfrak{M} ;$$

in particular, $R^* \leq \frac{83}{4} + \frac{9}{2}\sqrt{2} = 27.11 \ldots$.

It remains for us to prove the last part of Theorem 4, namely, that even $R^* \leq \bar{R} = 12 + 8\sqrt{2} = 23.3136 \ldots$.

Let us denote by $F(5, 1, 3; 4)$ the set of all $\beta = (b_0; b_1, b_2, \ldots)$ for which

$$b_0 = 5,\ b_1 = 1,\ b_2 = 3 \quad \text{and} \quad b_j \leq 4 \ (j \geq 3) .$$

From the proof of the above mentioned statement of Marshall Hall Jr. ([3], Theorem 3.2, p.974), it immediately follows that each number $\gamma \in L_1$, where

$$L_1 = [\, \min F(5, 1, 3; 4) \,.\, \min F(4; 4) ,$$
$$\max F(5, 1, 3; 4) \,.\, \max F(4; 4)]$$

can be written in a form $\gamma = \beta_1 \cdot \beta_2$, where $\beta_1 \in F(5, 1, 3; 4)$, $\beta_2 \in F(4; 4)$. By a direct computation, we get that

$$L_1 = [20 + 3\sqrt{2},\ 11 + 12\sqrt{2}] = [24.24.2\ldots,\ 27.97\ldots] .$$

Thus an arbitrary $\lambda \in L$ can be written in a form

$$\lambda = (a_0; a_1, a_2, \ldots).(b_0; b_1, b_2, \ldots) ,$$

where $a_0 = 5, a_1 = 1, a_2 = 3, a_j \leq 4 \ (j \geq 3), b_0 = 4, b_j \leq 4 \ (j \geq 1)$.

Now, let $\varkappa = (d_0; d_1, d_2, \ldots)$ be constructed as follows:

$$\varkappa = (a_0; b_0, a_1, a_0, b_0, b_1, \ldots, a_n, a_{n-1}, \ldots, a_1, a_0, b_0, b_1, \ldots, b_{n-1}, b_n, \ldots).$$

We claim that $R_\varkappa = \lambda$.

By the lemma, we have

$$R_\varkappa = \limsup_{k \to +\infty} s_k ,$$

where $s_k = (d_{k-1}; d_{k-2}, \ldots, d_1).(d_k; d_{k+1}, \ldots)$.

For sufficiently large integer n we have

$$d_{n^2} = b_0 = 4,\ d_{n^2-1} = a_0 = 5,\ d_{n^2-2} = a_1 = 1,\ d_{n^2-3} = a_2 = 3 .$$

Thus we have

$$\limsup_{n\to+\infty} s_{n^2} = \limsup_{n\to+\infty} (d_{n^2-1}; d_{n^2-2}, \dots, d_1).(d_{n^2}; d_{n^2+1}, \dots) =$$

$$= \lim_{n\to+\infty} (a_0; a_1, a_2, \dots).(b_0; b_1, b_2, \dots) = \lambda .$$

Further,

$$\limsup_{n\to+\infty} s_{n^2-1} = \limsup_{n\to+\infty} (1; 3, d_{n^2-4}, \dots, d_1).(5; 4, d_{n^2+1}, \dots) < 2.6 < \lambda .$$

Finally, for each positive integer k, $k \neq n^2$, $k \neq n^2-1$ $(n \geq 1)$ we have

$$s_k < (4; 1).(4; 1, 5) = 5 . \frac{29}{6} = 24.166 \dots < \lambda .$$

Hence we have

$$R_{\varkappa} = \limsup_{k\to+\infty} s_k = \limsup_{n\to+\infty} s_{n^2} = \lambda ,$$

thus proving $R^* \leq 20 + 3\sqrt{2} = 24.242 \dots$.

In the last part of the proof, let us denote by $F(5, 2; 4)$ the set of all $\beta = (b_0; b_1, b_2, \dots)$ for which

$$b_0 = 5, \ b_1 = 2, \ b_j \leq 4 \ (j \geq 2) .$$

Analogously, from the proof of the Hall's assertion mentioned above, it follows immediately that each number $\gamma \in L_2$, where

$$L_2 = [\min F(5, 2; 4) . \min F(4; 4) ,$$
$$\max F(5, 2; 4) . \max F(4; 4)]$$

can be written in a form $\gamma = \beta_1 . \beta_2$, where $\beta_1 \in F(5, 2; 4)$, $\beta_2 \in F(4; 4)$. By a direct computation, we find that $L_2 = [\frac{1}{8}(142 + 27\sqrt{2}), \frac{1}{7}(74 + 78\sqrt{2})] = [23.1819\dots, 26.3297\dots]$.

Thus, if we take an arbitrary $\lambda \in L_2$, $\lambda \geq \bar{R}$, we can write it in a form $\lambda = (a_0; a_1, a_2, \dots).(b_0; b_1, b_2, \dots)$, where $a_0 = 5$, $a_1 = 2$, $a_j \leq 4 (j \geq 2)$, $b_0 = 4$, $b_j \leq 4 \ (j \geq 1)$.

Let $\varkappa = (d_0; d_1, d_2, \dots)$ be constructed as follows:

$$\varkappa = (a_0; b_0, a_1, a_0, b_0, b_1, \dots, a_n, a_{n-1}, \dots, a_1, a_0, b_0, b_1, \dots, b_{n-1}, b_n, \dots).$$

We claim that $R_{\varkappa} = \lambda$.

By the lemma, $R_{\varkappa} = \limsup_{k \to +\infty} s_k$,

where $s_k = (d_{k-1}; d_{k-2}, \dots).(d_k; d_{k+1}, \dots)$.

By the construction of $\varkappa$, for sufficiently large positive integers n we have

$$d_{n^2} = b_0 = 4 , \; d_{n^2-1} = a_0 = 5 , \; d_{n^2-2} = a_1 = 2 .$$

Thus $\limsup_{n \to +\infty} s_{n^2} =$

$$= \limsup_{n \to +\infty} (d_{n^2-2}; d_{n^2-2}, \dots).(d_{n^2}; d_{n^2+1}, \dots) =$$

$$= \limsup_{n \to +\infty} (a_0; a_1, a_2, \dots).(b_0; b_1, b_2, \dots) = \lambda .$$

Further we have

$$\limsup_{n \to +\infty} s_{n^2-1} = \limsup_{n \to +\infty} (d_{n^2-2}; d_{n^2-3}, \dots, d_1).(d_{n^2-1}; d_{n^2}, \dots) =$$

$$= \limsup_{n \to +\infty} (2; d_{n^2-3}, \dots, d_1).(5; d_{n^2}, \dots) < 3.6 < \lambda .$$

Similarly,

$$\limsup_{n \to +\infty} s_{n^2+1} = \limsup_{n \to +\infty} (d_{n^2}; d_{n^2-1}, \dots, d_1).(d_{n^2+1}; d_{n^2+2}, \dots) =$$

$$= \limsup_{n \to +\infty} (4; 5, d_{n^2-2}, \dots, d_1).(d_{n^2+1}, \dots, d_{n^2+2n-2}, 2, 5, \dots) <$$

$$< (\overline{4;1}).(\overline{4;1}) = \overline{R} \leq \lambda ,$$

since for sufficiently large n , $d_j \leq 4$ when $n^2 + 1 \leq j \leq n^2 + 2n - 2$.

By an analogous argument,

$$\limsup_{n \to +\infty} s_{n^2-2} = \limsup_{n \to +\infty} (d_{n^2-3}; d_{n^2-4}, \dots, d_1).(d_{n^2-2}; d_{n^2-1}, \dots) =$$

$$= \limsup_{n \to +\infty} (d_{n^2-3}; d_{n^2-4}, \dots, d_1).(2; d_{n^2-1}, \dots) < 5.3 < \lambda .$$

Finally, if k is a positive integer, $|k - n^2| \geq 2$, $k \neq n^2 - 2$ $(n \geq 1)$ and $m^2 + 1 < k < m^2 + 2m - 1$ for some positive integer $m \geq 2$, say, then

$$s_k = (d_{k-1}; d_{k-2}, \dots, d_1).(d_k; d_{k+1}, \dots) =$$

$$= (d_{k-1}; \dots, d_{m^2+1}, 4, 5, \dots, d_1).(d_k; \dots, d_{m^2+2m-2}, 2, 5, \dots) <$$
$$< (\overline{4;1}).(\overline{4;1}) \leq \lambda ,$$

because $d_j \leq 4$ when $m^2 + 1 \leq j \leq m^2 + 2m - 2$.
Hence

$$R_{\varkappa} = \limsup_{k \to +\infty} s_k = \limsup_{m \to +\infty} s_{m^2} = \lambda$$

which concludes the proof of Theorem 4.

Remark. One could easily show that the sets $\mathfrak{M}(N)$ for $N \geq 5$ contain essentially bigger intervals than established in Theorem 4. Also, by a modification of Hall's proof, one could show that the set $\mathfrak{M}(4)$ already contains a certain interval.

Remark. Using the lemma, all the above theorems can be formulated in terms of $\mu(\beta)$. We have chosen the above formulation because of the simpler expressions for the values R_β .

Remark. Some interesting results concerning the solvability of the inequalities

$$0 < q < ct, \quad |q\beta - p| < \frac{1}{t}$$

with p and q integer may be derived from a more detailed consideration of the quantities R_β . These questions will be studied in a subsequent paper.

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