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THE LATTICE OF BI-NUMERATIONS OF ARITHMETIC. I.

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Introduction.

A sufficiently strong theory $\mathcal{T}$ can be described in itself. This fact was first exploited by K. Gödel for proofs of his incompleteness theorems (the method of arithmetization of metamathematics). The notion "description" is explicated by the exact metamathematical notion bi-numeration (or strong representation). Suppose that a formula $\tau(x)$ bi-numerates in $\mathcal{T}$ the set T of axioms of $\mathcal{T}$. A formal statement $\dot{Con}_\tau$ expressing in a natural way the consistency of $\mathcal{T}$ can be constructed simply by copying the metamathematical definitions involved. Starting from different bi-numerations of T we obtain different sentences $\dot{Con}_\tau$. The sentences $\dot{Con}_{\tau_1}$, $\dot{Con}_{\tau_2}$ corresponding to two bi-numerations τ_1, τ_2 may differ not only as expressions; they may have different strengths concerning the provability or unprovability of implications $\dot{Con}_{\tau_2} \rightarrow \dot{Con}_{\tau_1}$ and $\dot{Con}_{\tau_1} \rightarrow \dot{Con}_{\tau_2}$ in $\mathcal{T}$. The Gödel's second incompleteness theorem is usually formulated as follows: if $\mathcal{T}$ is a sufficiently strong consistent theory then $\dot{Con}_T$ is not provable in $\mathcal{T}$ ($\dot{Con}_T$ means

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Con_τ for a particular τ). Feferman [1] generalized this theorem in the following way: if $\mathcal{T}$ is a sufficiently strong consistent theory and $\tau(x)$ is an RE-formula which bi-numerates the axioms of $\mathcal{T}$ then Con_τ is not provable in $\mathcal{T}$. On the other hand, Feferman shows in [1] that some limitation on $\tau(x)$ is necessary for sufficiently strong reflexive theories; for example, he constructs a bi-numeration $\pi^*(x)$ of the set of axioms of Peano's arithmetics $\mathcal{P}$, for which Con_{π^*} is provable in $\mathcal{P}$.

Let us consider for a moment the Peano's arithmetic $\mathcal{P}$ with the set of axioms P from the intuitive set-theoretical point of view. (The Peano's arithmetic can be said to be the subject of our main interest.) For every bi-numeration $\pi(x)$ of the axioms P, the formula Con_π is true in the natural model of arithmetic (i.e. in the model of natural numbers). On the other hand, for each RE-bi-numeration $\pi(x)$ of P, the formula Con_π is independent from $\mathcal{P}$. One could ask if it is possible to choose a particular bi-numeration so that the formula Con_π should most adequately express the consistency of Peano's arithmetic; then one could add the last formula to P. It would correspond to the aim of formulating axioms that describe the structure of natural numbers in a most faithful way.

In this paper, we restrict ourselves to the study of PR-bi-numerations and corresponding consistency statements. This restriction seems to be natural, because (1) every primitive recursive set (in particular, the set of axioms of Peano's arithmetic) is bi-numerable by PR-formula, (2) every PR-formula is an RE-formula and hence the PR-bi-numerations satisfy the Gödel's second incompleteness theorem, (3) PR-formulas are syntactically simplest and, say, most natural descriptions of primitive recursive sets. One of PR-bi-numerations of P seems intuitively to be the most natural one. It results by formal copying the usual definition of P as a list of finitely many formulas plus the induction schema. On the other hand, one can consider the structure $\langle \mathcal{B}in_{\mathcal{P}}, \leq_{\mathcal{P}} \rangle$ where $\mathcal{B}in_{\mathcal{P}}$ is the set of all PR-bi-numerations of P and $\alpha \leq_{\mathcal{P}} \beta$ means $\vdash_{\mathcal{P}} Con_{\beta} \rightarrow Con_{\alpha}$. (We define $\leq_{\alpha}$ following Feferman). We hypothesize that no PR-bi-numeration is preferred from the point of view of this structure.

This hypothesis will be not fully confirmed in this paper. Nevertheless, we shall present several interesting properties of this structure, confirming more or less our hypothesis. In the present first part, after collecting some preliminary results, we show that, for every theory $\mathcal{A}$ which has in some sense similar properties as Peano's arithmetic, the ordering $\leq_{\mathcal{A}}$ is dense and is not linear (in fact, in every non-trivial interval there are many mutually incomparable elements). Further, we show that $\langle \mathcal{B}in_{\mathcal{A}}, \leq_{\mathcal{A}} \rangle$ is a distributive lattice. In the second part [6] which will be a direct continuation of the first part, we shall study the problem of reducibility and the existence of relative complements. We also obtain a partial "non-describability" result, formulated in terms of a hierarchy for formulas of the lattice theory which is similar to Lévy's hierarchy for set theory [4].

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I. Preliminaries

(a) Concerning the arithmetization of metamathematics.

This paper is very closely related to the work of Feferman Arithmetization of metamathematics in general setting [1]. We take as known the theory of primitive and general recursive functions and relations (see e.g. [3]). The reader of the present paper is supposed to be familiar with §§ 2 - 5 and with a part of § 7 of [1]. The mentioned part of § 7 will be reproduced in Sect.II.of this paper. We shall consequently use all definitions, theorems and conventions from [1].

In this Section some supplements to [1] needed later on will be given.

Let $Fv(\varphi) = \{u_0, \dots, u_k\}$ and let $t_0, \dots, t_k$ be terms. If there is no danger of misunderstanding, we shall write $\varphi(t_0, \dots, t_k)$ instead of $Sb\binom{u_0, \dots, u_k}{t_0, \dots, t_k}\varphi$.

We shall add the following point (iv) to Lemma 3.5 [1]:

1.1. <u>Lemma</u>. (iv) Let φ be a formula of $\mathcal{P}'$, let $t_0, \dots, t_m$ be terms of $\mathcal{P}$ and let $u_0, \dots, u_m$ be variables. Then

$$\vdash_{\mathcal{P}} (Sb(\begin{smallmatrix} u_0,\dots,u_n \\ t_0,\dots,t_n \end{smallmatrix})\varphi)^{(\mathcal{P}')} \leftrightarrow Sb(\begin{smallmatrix} u_0,\dots,u_n \\ t_0,\dots,t_n \end{smallmatrix})\varphi^{(\mathcal{P}')}.$$

1.2. Definition. Let $\varphi \in Fm_{K_0}$. φ is said to be a PR -formula in $\mathcal{P}$ (RE -formula in $\mathcal{P}$) if there is a PR -formula (RE -formula) ψ such that $\vdash_{\mathcal{P}} \varphi \leftrightarrow \psi$.

We shall use Lemma 3.7 [1] in the following formulation:

1.3. Lemma. (i) If φ is a PR -formula in $\mathcal{P}$, then $\sim\varphi$ is a PR -formula in $\mathcal{P}$.

(ii) If φ and ψ are PR -formulas in $\mathcal{P}$, then $\varphi \wedge \psi$ and $\varphi \vee \psi$ are PR -formulas in $\mathcal{P}$.

(iii) If φ is a PR -formula in $\mathcal{P}$, u, w variables and $u \neq w$, then $\bigwedge_{u}(u < w \to \varphi)$ and $\bigvee_{u}(u < w \wedge \varphi)$ are PR -formulas in $\mathcal{P}$.

(iv) If φ is a PR -formula in $\mathcal{P}$, $Fv(\varphi) = \{u_0,\dots,u_{k-1}\}$ and $t_0,\dots,t_{k-1}$ are terms of $\mathcal{M}$, then $(Sb(\begin{smallmatrix} u_0,\dots,u_{k-1} \\ t_0,\dots,t_{k-1} \end{smallmatrix})\varphi)^{(\mathcal{M})}$ is a PR -formula in $\mathcal{P}$.

1.4. Definition. Let $\varphi \in Fm_{K_0}$ and let $Fv(\varphi) = \{v_{k_0},\dots,v_{k_m}\}$. Then

$$\tilde{\varphi} = Sb(\begin{smallmatrix} v_{k_0},\dots,v_{k_m} \\ nm_{v_{k_0}},\dots,nm_{v_{k_m}} \end{smallmatrix})\bar{\varphi}.$$ For $\varphi \in St_K$ we set $\tilde{\varphi} = \bar{\varphi}$.

1.5. Lemma. Let $\varphi \in Fm_{K_0}$ and let $Fv(\varphi) = \{u_0,\dots,u_{k-1}\}$. Then $\vdash_{\mathcal{M}} Pr_{\mathcal{L}_{K_0}}(\bigwedge_{u_0}\dots\bigwedge_{u_{k-1}}\varphi \to \tilde{\varphi})$.

The lemma follows from the assertion in [1], p. 58, the first line from above (let us remark that $\vdash_{\mathcal{M}} \bigwedge_{v_m} Tm_{K_0}(nm_{v_m})$).

Theorem 5.4 [1] can be now reformulated as follows:

1.6. Theorem. Let $\varphi \in BPF$. Then $\vdash_{\mathcal{M}} \varphi \to Pr_{[Q]}(\tilde{\varphi})$.

1.7. Corollary. (i) Let $\varphi \in Fm_{K_0}$ and suppose that there is $\psi \in BPF$ such that $\vdash_Q \varphi \leftrightarrow \psi$. Then $\vdash_{\mathcal{M}} \varphi \rightarrow \dot{P}r_{[Q]}(\tilde{\varphi})$.

(ii) Let φ be an RE-formula in $\mathcal{P}$, $Fv(\varphi) = \{u_0, \ldots, u_{k-1}\}$. Let $\mathcal{A} = \langle A, K \rangle$ be an axiomatic theory, $\mathcal{P} \subseteq \mathcal{A}$, $\alpha \in Fm_{K_0}$ and let α bi-numerate A in $\mathcal{P}$. Then

$$\vdash_{\mathcal{M}} \varphi \rightarrow \dot{P}r_{\alpha}(\tilde{\varphi}).$$

Proof. (i) We can suppose $Fv(\varphi) = Fv(\psi) = \{u_0, \ldots, u_{k-1}\}$. By 1.6, $\vdash_{\mathcal{M}} \psi \rightarrow \dot{P}r_{[Q]}(\tilde{\psi})$. From the assumption $\vdash_Q \varphi \leftrightarrow \psi$ we have $\vdash_{\mathcal{M}} \dot{P}r_{[Q]}(\overline{\bigwedge_{u_0} \ldots \bigwedge_{u_{k-1}} (\varphi \leftrightarrow \psi)})$, and therefore $\vdash_{\mathcal{M}} \dot{P}r_{[Q]}(\widetilde{\varphi \leftrightarrow \psi})$. Let us remark that $\vdash_{\mathcal{M}} \widetilde{\varphi \leftrightarrow \psi} = \tilde{\varphi} \dot{\leftrightarrow} \tilde{\psi}$. We obtain $\vdash_{\mathcal{M}} \dot{P}r_{[Q]}(\tilde{\varphi} \dot{\leftrightarrow} \tilde{\psi})$, $\vdash_{\mathcal{M}} \dot{P}r_{[Q]}(\tilde{\varphi}) \leftrightarrow \dot{P}r_{[Q]}(\tilde{\psi})$ and therefore $\vdash_{\mathcal{M}} \varphi \rightarrow \dot{P}r_{[Q]}(\tilde{\varphi})$.

(ii) From 3.9 [1] it follows that there is $\psi \in BPF$ such that $Fv(\psi) = \{u_0, \ldots, u_{k-1}\}$ and $\vdash_{\mathcal{P}} \bigwedge_{u_0} \ldots \bigwedge_{u_{k-1}} (\varphi \leftrightarrow \psi)$. By 4.4 [1], Prf_{α} bi-numerates $Prf_{\mathcal{A}}$ in $\mathcal{P}$ and therefore

$$\vdash_{\mathcal{P}} \dot{P}r_{\alpha}(\overline{\bigwedge_{u_0} \ldots \bigwedge_{u_{k-1}} (\varphi \leftrightarrow \psi)}).$$

This implies $\vdash_{\mathcal{P}} \dot{P}r_{\alpha}(\widetilde{\varphi \leftrightarrow \psi})$ by 1.5. Now we obtain

$$\vdash_{\mathcal{M}} \varphi \rightarrow \dot{P}r_{\alpha}(\tilde{\varphi}).$$

analogously as in (i).

1.8. Theorem. Let φ be a PR-formula in $\mathcal{P}$, and suppose that $\mathcal{A} = \langle A, K \rangle$ is an axiom system, $\mathcal{P} \subseteq \mathcal{A}$, $\alpha \in Fm_{K_0}$ and α bi-numerates A in $\mathcal{P}$.

Then

$$\vdash_{\mathcal{M}} (\dot{C}on_{\alpha} \wedge \dot{P}r_{\alpha}(\tilde{\varphi})) \rightarrow \varphi.$$

Proof. In $\mathcal{M}$, suppose $\dot{C}on_{\alpha}$, $\dot{P}r_{\alpha}(\tilde{\varphi})$ and $\dot{\sim}\varphi$.

By Lemma 1.3, $\sim\varphi$ is a PR-formula in $\mathcal{P}$. We obtain $Pr_\alpha(\widetilde{\sim\varphi})$ by Corollary 1.7. Let us remark that $\vdash_{\mathcal{M}} \sim\tilde{\varphi} \approx \widetilde{\sim\varphi}$. We obtain $Pr_\alpha(\sim\tilde{\varphi})$ and further $\sim Con_\alpha$, which is a contradiction in $\mathcal{M}$.

(b) Independent formulas

Feferman considers the formula ν_α (see Definition 5.2 in [1]). He proves, under certain assumptions, $\nvdash_{\mathcal{A}} \nu_\alpha$ (cf. Theorem 5.3 [1]) and $\vdash_{\mathcal{A}} Con_\alpha \longleftrightarrow \nu_\alpha$ (cf. Theorem 5.6 [1]). In this paper, we shall also use the formula ρ_α defined following Rosser and the formula μ_α defined following Mostowski. In this Section we present some results of Rosser and Mostowski in a version modified for the purpose of this paper. In particular, we stress the fact that our Theorem 1.18 is proved in [5] in a far more general formulation.

1.9. Lemma. (5.1 [1]). Let $\psi \in Fm_{K_0}$ and let $Fv(\psi) \subseteq \{x\}$. Then there is a $\varphi \in Fm_{K_0}$ such that $\vdash_{\mathcal{Q}} \varphi \longleftrightarrow \psi(\overline{\varphi})$.

1.10. Definition. Let $\alpha \in Fm_{K_0}$ and let $Fv(\alpha) = \{x\}$. Using Lemma 1.9 and Lemma 1.1 we define a formula $\rho_\alpha \in Fm_{K_0}$ such that $\vdash_{\mathcal{P}} \rho_\alpha \leftrightarrow \bigwedge_y [Prf_\alpha(\overline{\rho}_\alpha, y) \rightarrow \bigvee_{z<y} Prf_\alpha(\sim\overline{\rho}_\alpha, z)]^{(\mathcal{M})}$.

1.11. Remark. We have the following obvious fact

$$\vdash_{\mathcal{P}} \rho_\alpha \leftrightarrow \bigwedge_y (Prf_\alpha(\overline{\rho}_\alpha, y) \rightarrow \bigvee_{z<y} Prf_\alpha(\overline{\sim\rho_\alpha}, z)).$$

We shall write $R_\alpha(y)$ instead of $Prf_\alpha(\overline{\rho}_\alpha, y) \rightarrow \bigvee_{z<y} Prf_\alpha(\overline{\sim\rho_\alpha}, z)$, so that we have $\vdash_{\mathcal{P}} \rho_\alpha \leftrightarrow \bigwedge_y R_\alpha(y)$. Further, let us mention that $R_\alpha(y)$ is a PR-formula in $\mathcal{P}$, whenever α is.

1.12. Denotation. For arbitrary formulas $\varphi_i \in Fm_K$ $(i = 0, \dots, m-1, m > 0)$ we write $\bigwedge\!\!\bigwedge_{i<m} \varphi_i$ instead of $\varphi_0 \wedge \dots \wedge \varphi_{m-1}$. Similarly, $\bigvee\!\!\bigvee_{i<m} \varphi_i$ is an abbreviation for $\varphi_0 \vee \dots \vee \varphi_{m-1}$.

1.13. Theorem. Let $\mathcal{A} = \langle A, K \rangle$ be a consistent axiomatic theory. If $\mathcal{P} \subseteq \mathcal{A}$, $\alpha \in Fm_{K_0}$ and if α bi-numerates A in $\mathcal{P}$ then

(i) $\not\vdash_{\mathcal{A}} \varphi_\alpha$,

(ii) $\not\vdash_{\mathcal{A}} \sim \varphi_\alpha$.

Proof. (i) Let $\vdash_{\mathcal{A}} \varphi_\alpha$ and let d be a proof of φ_α in $\mathcal{A}$. Then

$$\vdash_{\mathcal{A}} \bigvee_{x<\bar{d}} Prf_\alpha(\overline{\sim \varphi_\alpha}, x) .$$

By Lemma 3.1 [1], the last assertion is equivalent to the following one:

(1) $$\vdash_{\mathcal{A}} \bigvee\!\!\bigvee_{i<d} Prf_\alpha(\overline{\sim \varphi_\alpha}, \bar{i}) .$$

Since $\mathcal{A}$ is consistent and $\vdash_{\mathcal{A}} \varphi_\alpha$ we have $\not\vdash_{\mathcal{A}} \sim \varphi_\alpha$. Since α bi-numerates A in $\mathcal{P}$, Prf_α bi-numerates $Prf_{\mathcal{A}}$ in $\mathcal{P}$ (by 4.4 [1]). It follows that Prf_α bi-numerates $Prf_{\mathcal{A}}$ in $\mathcal{A}$ since $\mathcal{A}$ is a consistent extension of $\mathcal{P}$. Consequently,

(2) $$\vdash_{\mathcal{A}} \bigwedge\!\!\bigwedge_{i<d} \sim Prf_\alpha(\overline{\sim \varphi_\alpha}, \bar{i}) .$$

(1) and (2) give a contradiction in $\mathcal{A}$. We obtain $\not\vdash_{\mathcal{A}} \varphi_\alpha$.

(ii) Suppose $\vdash_{\mathcal{A}} \sim \varphi_\alpha$ and let d be a proof of $\sim \varphi_\alpha$ in $\mathcal{A}$. Then

(3) $$\vdash_{\mathcal{A}} \bigvee_{y<\bar{d}} Prf_\alpha(\bar{\varphi}_\alpha, y) \qquad \text{i.e.}$$
$$\vdash_{\mathcal{A}} \bigvee\!\!\bigvee_{i<d} Prf_\alpha(\bar{\varphi}_\alpha, \bar{i}) .$$

Analogously as in (i) we obtain

(4) $$\vdash_{\mathcal{A}} \bigwedge\!\!\bigwedge_{i<d} \sim Prf_\alpha(\bar{\varphi}_\alpha, \bar{i}) .$$

(4) together with (3) is a contradiction in $\mathcal{A}$. We have proved $\nvdash_{\mathcal{A}} \sim \varphi_\alpha$.

1.14. <u>Theorem</u>. Let $\mathcal{A} = \langle A, K \rangle$ be an axiomatic theory such that $\mathcal{P} \subseteq \mathcal{A}$ and let α be a PR -formula in $\mathcal{P}$ such that α bi-numerates A in $\mathcal{P}$. Then

(i) $\vdash_{\mathcal{P}} Pr_\alpha(\overline{\sim \varphi_\alpha}) \longrightarrow \sim Con\,\alpha$,

(ii) $\vdash_{\mathcal{P}} Pr_\alpha(\overline{\varphi_\alpha}) \longrightarrow \sim Con\,\alpha$.

Proof. Evidently, it is sufficient to show

(i)′ $\vdash_{\mathcal{M}} Pr_\alpha(\overline{\sim \varphi_\alpha}) \longrightarrow \sim Con_\alpha$,

(ii)′ $\vdash_{\mathcal{M}} Pr_\alpha(\overline{\varphi_\alpha}) \longrightarrow \sim Con_\alpha$.

(i)′ We proceed in $\mathcal{M}$. Suppose $Pr_\alpha(\overline{\sim \varphi_\alpha})$, i.e. $\bigvee_x Prf_\alpha(\overline{\sim \varphi_\alpha}, x)$. Further assume Con_α. By 1.7 we have $\bigvee_x Prf_\alpha(\widetilde{Prf_\alpha(\overline{\sim \varphi_\alpha}, x)})$.

Evidently $\vdash_{\mathcal{P}} \bigwedge_x [\sim \varphi_\alpha \wedge Prf_\alpha(\overline{\sim \varphi_\alpha}, x) \rightarrow \bigvee_{y<x} Prf_\alpha(\overline{\varphi_\alpha}, y)]$

and so $\vdash_{\mathcal{P}} Pr_\alpha(\overline{\bigwedge_x [\sim \varphi_\alpha \wedge Prf_\alpha(\overline{\sim \varphi_\alpha}, x) \rightarrow \bigvee_{y<x} Prf_\alpha(\overline{\varphi_\alpha}, y)]})$.

Hence our assumption $Pr_\alpha(\overline{\sim \varphi_\alpha})$ implies the following in $\mathcal{M}$ (cf. Lemma 1.5):

$$\bigvee_x Prf_\alpha(\overline{\sim \varphi_\alpha}, x) \wedge Pr_\alpha(\widetilde{\bigvee_{y<x} Prf_\alpha(\overline{\varphi_\alpha}, y)}) .$$

Using Theorem 1.8 and the assumption Con_α we obtain

$$\bigvee_x [Prf_\alpha(\overline{\sim \varphi_\alpha}, x) \wedge \bigvee_{y<x} Prf_\alpha(\overline{\varphi_\alpha}, y)$$

and consequently $\sim Con_\alpha$, which is a contradiction in $\mathcal{M}$.

The proof of (ii)′ is analogous.

1.15. <u>Remark</u>. Since the implication

$$\sim Con_\alpha \longrightarrow (Pr_\alpha(\overline{\sim \varphi_\alpha}) \wedge Pr_\alpha(\overline{\varphi_\alpha}))$$

is evidently provable in $\mathcal{P}$, we obtain in fact the following

$$\vdash_{\mathcal{P}} Pr_\alpha(\overline{\sim \varphi_\alpha}) \longleftrightarrow \sim Con_\alpha ,$$

$$\vdash_{\mathcal{P}} Pr_\alpha(\overline{\varphi_\alpha}) \leftrightarrow \sim Con_\alpha .$$

1.16. <u>Definition</u>. Let $\alpha \in Fm_{K_0}$, $Fr(\alpha) = \{x\}$ and let $\varphi_i \in St_K$ for $i = 0, \dots, k$. Using Lemma 1.9 and Lemma 1.1 we define a formula $\mu_\alpha \in Fm_{K_0}$ such that

$$\vdash_{\mathcal{P}} \mu_\alpha \leftrightarrow \bigwedge_y (\bigvee\!\!\bigvee_{i<k+1} Prf_\alpha(\overline{\varphi_i} \dot{\rightarrow} \overline{\mu_\alpha}, y) \rightarrow$$

$$\rightarrow \bigvee_{z<y} \bigvee\!\!\bigvee_{i<k+1} Prf_\alpha(\overline{\varphi_i} \dot{\rightarrow} \dot{\sim} \overline{\mu_\alpha}, z))^{(M)} .$$

1.17. <u>Remark</u>. The formula μ_α evidently depends on the choice of the formulas $\varphi_0, \dots, \varphi_k$. Therefore we ought to write $\mu_\alpha^{\varphi_0, \dots, \varphi_k}$. But we shall omit the indices because there will be no danger of confusion. We have the following obvious fact:

$$\vdash_{\mathcal{P}} \mu_\alpha \leftrightarrow \bigwedge_y (\bigvee\!\!\bigvee_{i<k+1} Prf_\alpha(\overline{\varphi_i \rightarrow \mu_\alpha}, y) \rightarrow$$

$$\rightarrow \bigvee_{z<y} \bigvee\!\!\bigvee_{i<k+1} Prf_\alpha(\overline{\varphi_i \rightarrow \sim \mu_\alpha}, z)) .$$

We shall write $M_\alpha(y)$ instead of

$\bigvee\!\!\bigvee_{i<k+1} Prf_\alpha(\overline{\varphi_i \rightarrow \mu_\alpha}, y) \rightarrow \bigvee_{z<y} \bigvee\!\!\bigvee_{i<k+1} Prf_\alpha(\overline{\varphi_i \rightarrow \sim \mu_\alpha}, z)$

so that we have $\vdash_{\mathcal{P}} \mu_\alpha \leftrightarrow \bigwedge_y M_\alpha(y)$.

Further, let us mention that $M_\alpha(y)$ as a PR -formula in $\mathcal{P}$ whenever α is.

1.18. <u>Theorem</u>. Let $\mathcal{A} = \langle A, K \rangle$ be an axiomatic theory such that $\mathcal{P} \subseteq \mathcal{A}$ and let α be an element of Fm_{K_0} which bi-numerates A in $\mathcal{P}$. Further, let $\varphi_i \in St_K$ and let $\mathcal{A}_i = \mathcal{A} + \{\varphi_i\}$ be a consistent axiomatic theory for $i = 0, \dots, k$. Let μ_α be defined as in Definition 1.16. Then, for each $i = 0, \dots, k$,

(i) $\nvdash_{\mathcal{A}_i} \mu_\alpha$,

(ii) $\nvdash_{\mathcal{A}_i} \sim \mu_\alpha$.

<u>Remark</u>. Under the conditions of Theorem 1.18 we shall say that μ_α is defined with respect to the theories $\mathcal{A}_i$ $(i = 0, \ldots, k)$.

Proof. (i) Let be $\vdash_{\mathcal{A}_j} \mu_\alpha$, i.e. $\vdash_{\mathcal{A}} \varphi_j \rightarrow \mu_\alpha$, for some j $(0 \leq j \leq k)$. Under this assumption there exist numbers p_1 and p_2 such that $p_2 \leq k$, $\mathrm{Prf}_{\mathcal{A}}(\varphi_{p_2} \rightarrow \rightarrow \mu_\alpha, p_1)$ and for arbitrary $i = 0, \ldots, k$ and d it follows $d \geq p_1$, whenever $\mathrm{Prf}_{\mathcal{A}}(\varphi_i \rightarrow \mu_\alpha, d)$, By 4.4. [1] Prf_α bi-numerates $\mathrm{Prf}_{\mathcal{A}}$ in $\mathcal{P}$ and, consequently, we have

$$\vdash_{\mathcal{P}} \mathrm{Prf}_\alpha(\overline{\varphi_{p_2} \rightarrow \mu_\alpha}, \overline{p}_1) .$$

Further, we have

$$\vdash_{\mathcal{A}_{p_1}} \bigvee_{x < \overline{p}_1} \bigvee\!\!\!\bigvee_{i < k+1} \mathrm{Prf}_\alpha(\overline{\varphi_i \rightarrow \sim \mu_\alpha}, x) .$$

Using Lemma 3.1 [1], we have

$$\vdash_{\mathcal{A}_{p_2}} \bigvee\!\!\!\bigvee_{\substack{j < p_1 \\ i < k+1}} \mathrm{Prf}_\alpha(\overline{\varphi_i \rightarrow \sim \mu_\alpha}, \overline{j}) .$$

Prf_α bi-numerates $\mathrm{Prf}_{\mathcal{A}}$ in $\mathcal{A}_{p_2}$, because $\mathcal{A}_{p_2}$ is a consistent extension of $\mathcal{A}$. Consequently, there exist numbers r_1 and r_2 such that $r_1 < p$, $r_2 \leq k$ and $\mathrm{Prf}_{\mathcal{A}}(\varphi_{r_2} \rightarrow \sim \mu_\alpha\, r_1)$. Therefore we have

$$\vdash_{\mathcal{A}_{r_2}} \bigvee_{y < \overline{r}_1} \bigvee\!\!\!\bigvee_{i < k+1} \mathrm{Prf}_\alpha(\overline{\varphi_i \rightarrow \mu_\alpha}, y) ,$$

$$\vdash_{\mathcal{A}_{r_2}} \bigvee\!\!\!\bigvee_{\substack{j < r_1 \\ i < k+1}} \mathrm{Prf}_\alpha(\overline{\varphi_i \rightarrow \mu_\alpha}, \overline{j}) .$$

Using the same consideration as before, we can conclude that there exist numbers s_1 , s_2 such that $0 \leq s_1 < r_1 < p_1$, $0 \leq s_2 \leq k$ and $\mathrm{Prf}_{\mathcal{A}}(\varphi_{s_2} \rightarrow \mu_\alpha, s_1)$. On the other hand, from the definition of p_1 , we have that $p_1 \leq \leq s_1$. This is a contradiction and (i) is proved.

(ii) Let $\vdash_{\mathcal{A}_j} \sim \mu_\alpha$, i.e. $\vdash_{\mathcal{A}} \varphi_j \rightarrow \sim \mu_\alpha$ for some j $(0 \leq j \leq k)$. Let d be a proof in $\mathcal{A}$ of the implication $\varphi_j \longrightarrow \sim \mu_\alpha$. If we set $r_1 = d$ and $r_2 = j$ we have $Prf_{\mathcal{A}} (\varphi_{r_2} \rightarrow \sim \mu_\alpha , r_1)$. We can continue exactly as in the end of the proof of (i). The existence of numbers s_1 and s_2 such that $s_2 \leq k$ and $Prf_{\mathcal{A}} (\varphi_{s_2} \longrightarrow \mu_\alpha , s_1)$ reduces case (ii) to case (i).

(c) Concerning the lattice theory

We take as known the fundamental definitions and theorems of the lattice theory (see e.g.[2]). In this section we only list the notions we shall use and remember two simple assertions that are closely related to the problems of this paper.

Let $K_1 = \{ r_{1,0} , r_{1,1} , f_{1,2} , f_{1,3} \}$. For arbitrary $\xi , \eta \in Tm_{K_1}$ we set $\xi \approx \eta = r_{1,0} \ulcorner \xi , \eta \urcorner$, $\xi \leq \eta = r_{1,1} \ulcorner \xi , \eta \urcorner$, $\xi \cap \eta = f_{1,2} \ulcorner \xi , \eta \urcorner$, $\xi \cup \eta = f_{1,3} \ulcorner \xi , \eta \urcorner$. We shall write $\xi < \eta$ as an abbreviation of the formula $\xi \leq \eta \wedge \sim (\xi \approx \eta)$.

Let S be a set containing the following formulas:

$$\bigwedge_x \bigwedge_y (x \cap y \approx y \cap x) ; \quad \bigwedge_x \bigwedge_y (x \cup y \approx y \cup x) ;$$

$$\bigwedge_x \bigwedge_y \bigwedge_z ((x \cap y) \cap z \approx x \cap (y \cap z)) ; \bigwedge_x \bigwedge_y \bigwedge_z ((x \cup y) \cup z \approx x \cup (y \cup z)) ;$$

$$\bigwedge_x \bigwedge_y (x \cap (x \cup y) \approx x) ; \quad \bigwedge_x \bigwedge_y (x \cup (x \cap y) \approx x) ;$$

$$\bigwedge_x \bigwedge_y (x \leq y \longleftrightarrow x \cap y \approx x) .$$

The set S_d contains in addition the following two formulas :

$$\bigwedge_x \bigwedge_y \bigwedge_z (x \cap (y \cup z) \approx (x \cap y) \cup (x \cap z)) ,$$

$$\bigwedge_x \bigwedge_y \bigwedge_z (x \cup (y \cap z) \approx (x \cup y) \cap (x \cup z)) .$$

The theory $\mathcal{S} = \langle S, K_1 \rangle$ is called the lattice theory and the theory $\mathcal{S}_d = \langle S_d, K_1 \rangle$ is called the distributive lattice theory. We shall use the Tarski's notions of satisfaction and model in the same way as Feferman does (cf.[1]).

A structure $\underline{M} = \langle M, G \rangle$ which is a model of $\mathcal{S} = \langle S, K_1 \rangle$ is called a lattice (similarly for distributive lattices). We write also $\langle M, \leq, \cap, \cup \rangle$ instead of $\langle M, G \rangle$, where $\leq$ is $G(\leq)$, $\cap$ is $G(\cap)$ and $\cup$ is $G(\cup)$.

Suppose $\varphi \in Fm_{\alpha_1}$; an ordered k-tuple $\langle a_0, \ldots, a_{k-1} \rangle$ of elements of M is said to satisfy φ in $\underline{M}$ (denotation: $\underline{M} \models \varphi [a_0, \ldots, a_{k-1}]$) if every assignment W such that $W(i_n) = a_n$ for $n = 0, \ldots, k-1$ satisfies φ in $\underline{M}$, where $Fv(\varphi) = \{v_{i_0}, \ldots, v_{i_{k-1}}\}$, $i_0 < \ldots < i_{k-1}$.

The notions of a sublattice and of an isomorphism have there usual meanings. If $\underline{M} = \langle M, G \rangle$ is a lattice and if $a, b \in M$, $a \leq b$, then we define the segment $\langle a ; b \rangle$ determined by a, b putting $\langle a; b \rangle = \{u \in M ; a \leq u \leq b\}$.

Evidently, a segment $\langle a; b \rangle$ determines a sublattice of $\underline{M}$. This lattice will be denoted also by $\langle a ; b \rangle$ if there will be no danger of confusion. If $\underline{M}$ is distributive then $\langle a ; b \rangle$ is also distributive.

1.19. Theorem. ([2],p. 70). Let $\underline{M} = \langle M, \leq, \cap, \cup \rangle$ be a distributive lattice and let a, b, c, d be elements of M such that $a < b$, $c \cap d = a$ and $c \cup d = b$. Then the function $f(x) = d \cup x$ is an isomorphism of $\langle a; c \rangle$ and $\langle d; b \rangle$.

1.20. Theorem. Let $\underline{M}$ and $\underline{M}'$ be lattices and let f be an isomorphism of $\underline{M}$ and $\underline{M}'$. Let $\varphi \in Fm_{K_1}$, $Fv(\varphi) = \{v_{i_0}, \ldots, v_{i_{n-1}}\}$ and let $\langle a_0, \ldots, a_{n-1} \rangle$ be an n-tuple of elements of M. Then $\underline{M} \models \varphi[a_0, \ldots, a_{n-1}]$ if and only if

$\underline{M}' \models \varphi[f(a_0), \ldots, f(a_{n-1})]$.

This holds for arbitrary relational structures. The proof is done by induction on formulas.

II. The lattice of bi-numerations of arithmetic

2.1. Assumptions. In this section, $\mathcal{A} = \langle A, K \rangle$ denotes an arbitrary fixed axiomatic theory such that

(1) A is primitive recursive,

(2) $\mathcal{A}$ is consistent,

(3) $\mathcal{P} \subseteq \mathcal{A}$.

Evidently, the set P of axioms of Peano arithmetic $\mathcal{P}$ is primitive recursive and consequently $\mathcal{A} = \mathcal{P}$ satisfies the assumptions (1) and (3).

We restrict ourselves to the study of PR-bi-numerations of A (cf. the Introduction). We recall Theorem 3.11 [1] from which follows that a set is primitive recursive if and only if it is bi-numerable in $\mathcal{Q}$ by a PR-formula. Moreover it follows that it is immaterial whether we speak of PR-bi-numerations in $\mathcal{Q}$ or in a consistent extension

$\mathcal{B}$ of $\mathcal{Q}$. Hence we can simply speak of PR -bi-numerations.

2.2. Definition. $\mathcal{B}in$ is the set of all PR -formulas in $\mathcal{P}$ bi-numerating A .

Evidently $\mathcal{B}in$ is non-empty.

2.3. Definition (7.1 [1]). Let $\mathcal{B} = \langle B, K \rangle$, $K_0 \subseteq K$ and suppose that α, $\alpha' \in Fm_{K_0}$, $Fv(\alpha) = Fv(\alpha') = \{x\}$. We put

(i) $\alpha \leq_{\mathcal{B}} \alpha'$ if $\vdash_{\mathcal{B}} \dot{Con}_{\alpha'} \longrightarrow \dot{Con}_{\alpha}$;

(ii) $\alpha <_{\mathcal{B}} \alpha'$ if $\alpha \leq_{\mathcal{B}} \alpha'$ but $\alpha' \not\leq_{\mathcal{B}} \alpha$;

(iii) $\alpha =_{\mathcal{B}} \alpha'$ if simultaneously $\alpha \leq_{\mathcal{B}} \alpha'$ and $\alpha' \leq_{\mathcal{B}} \alpha$.

2.4. Definition. $\underline{\mathcal{B}in} = \langle \mathcal{B}in , \leq_{\mathcal{A}} , =_{\mathcal{A}} \rangle$; i.e. $\underline{\mathcal{B}in}$ is the structure with the field $\mathcal{B}in$ and two binary relations $=_{\mathcal{A}}$ and $\leq_{\mathcal{A}}$.

Obviously, $\underline{\mathcal{B}in}$ is a (partially) ordered set with non-absolute equality. An ordered set in the usual sense results by factorisation:

2.5. Definition. Let $\alpha \in \mathcal{B}in$. We denote by $[\alpha]$ the set of all $\beta \in \mathcal{B}in$ such that $\alpha =_{\mathcal{A}} \beta$.

Let α , $\beta \in \mathcal{B}in$. We put $[\alpha] \leq_{\mathcal{A}} [\beta]$ if $\alpha \leq_{\mathcal{A}} \beta$. (This denotation cannot cause any confusion.)

$[\mathcal{B}in]$ is a set of all $[\alpha]$ where $\alpha \in \mathcal{B}in$,

$[\underline{\mathcal{B}in}] = \langle [\mathcal{B}in] , \leq_{\mathcal{A}} \rangle$.

$[\mathcal{B}in]$ is a (partially) ordered set. We shall freely use both the $\underline{\mathcal{B}in}$ symbolism and the $[\underline{\mathcal{B}in}]$ symbolism, because they are closely related, as it is well known.

Feferman proved that $\underline{\mathcal{B}in}$ has neither a minimal nor a maximal element:

2.6. <u>Theorem</u> (7.4 [1]). Suppose that $\mathcal{A}$ is reflexive. Then for every $\alpha \in \mathcal{B}in$ there is an $\alpha' \in \mathcal{B}in$ such that

$$\alpha' <_{\mathcal{A}} \alpha .$$

2.7. <u>Corollary</u>. If $\mathcal{A}$ is reflexive then $[\mathcal{B}in]$ is infinite.

2.8. <u>Theorem</u> (7.5 [1]). Suppose that $\mathcal{A}$ is ω -consistent. Then for every $\alpha \in \mathcal{B}in$ there is $\alpha' \in \mathcal{B}in$ such that

$$\alpha <_{\mathcal{A}} \alpha' .$$

2.9. <u>Corollary</u>. If $\mathcal{A}$ is ω -consistent then $[\mathcal{B}in]$ is infinite.

Considering the proofs of Theorems 2.6 and 2.8 one could conjecture that $\alpha \leq_{\mathcal{A}} \alpha'$ if and only if $\vdash_{\mathcal{A}} \bigwedge_x (\alpha(x) \longrightarrow \longrightarrow \alpha'(x))$. If $\vdash_{\mathcal{A}} \bigwedge_x (\alpha(x) \longrightarrow \alpha'(x))$ then really $\alpha \leq_{\mathcal{A}} \alpha'$. But we show in the following example that the converse is not true. In fact, we define formulas $\alpha', \alpha'' \in \in \mathcal{B}in$ such that

$$\alpha'' <_{\mathcal{A}} \alpha' ,$$

$$\not\vdash_{\mathcal{A}} (\bigwedge_x \alpha''(x) \longrightarrow \alpha'(x)) .$$

2.10. <u>Example</u>. Suppose that $\mathcal{A}$ is ω -consistent and let α, α' be elements of $\mathcal{B}in$ such that $\alpha <_{\mathcal{A}} \alpha'$ and $\vdash_{\mathcal{A}} \bigwedge_x (\alpha(x) \longrightarrow \alpha'(x))$ (the existence is guaranteed by the proof of 7.5 [1]). Put $\mathcal{B}_1 = \mathcal{A} + \{\underset{\cdot}{Con}_{\alpha'}\}$, $\mathcal{B}_2 = \mathcal{A} + \{\sim \underset{\cdot}{Con}_{\alpha'} \wedge \underset{\cdot}{Con}_{\alpha}\}$. Both $\mathcal{B}_1$ and $\mathcal{B}_2$ are consistent. Let μ_α be defined with respect to $\mathcal{B}_1$ and $\mathcal{B}_2$ (cf. 1.18). Further, put

$$\alpha''(x) = \alpha(x) \vee \bigvee_{y<x} \sim M_\alpha(y) \wedge (x \approx \overline{\rho_{\alpha'}} \wedge \underset{\cdot}{pr}_y \underset{\cdot}{\approx} \underset{\cdot}{pr}_y)^{(M)} .$$

Evidently $\alpha'' \in \mathit{Bin}$. Since $\vdash_{\mathcal{A}} \mathit{Con}_{\alpha'} \longrightarrow$
$\longrightarrow \sim \mathit{Pr}_{\alpha'}(\overline{\sim \varphi_{\alpha'}})$ and $\vdash_{\mathcal{A}} \bigwedge_x (\mathit{Pr}_\alpha(x) \longrightarrow \mathit{Pr}_{\alpha'}(x))$,
we have $\vdash_{\mathcal{A}} \mathit{Con}_{\alpha'} \longrightarrow \sim \mathit{Pr}_\alpha(\overline{\varphi_{\alpha'}})$, which implies
$\vdash_{\mathcal{A}} \mathit{Con}_{\alpha'} \longrightarrow \mathit{Con}_{\alpha''}$. On the other hand, $\vdash_{\mathcal{B}} (\mathit{Con}_\alpha \wedge$
$\wedge\, \mu_\alpha) \longrightarrow \mathit{Con}_{\alpha''}$ and $\not\vdash_{\mathcal{A}} (\sim \mathit{Con}_{\alpha'} \wedge \mathit{Con}_\alpha) \rightarrow \sim \mu_\alpha$
and consequently $\not\vdash_{\mathcal{A}} \mathit{Con}_{\alpha''} \longrightarrow \mathit{Con}_{\alpha'}$. We have proved
$\alpha'' <_{\mathcal{A}} \alpha'$. Further, we have

$$\vdash_{\mathcal{A}} (\mathit{Con}_{\alpha'} \wedge \sim \mu_\alpha) \rightarrow (\sim \mathit{Pr}_{\alpha'}(\overline{\varphi_{\alpha'}}) \wedge \mathit{Pr}_{\alpha''}(\overline{\varphi_{\alpha'}})).$$

Since $\not\vdash_{\mathcal{A}} \mathit{Con}_{\alpha'} \longrightarrow \mu_\alpha$ we have $\not\vdash_{\mathcal{A}} \mathit{Pr}_{\alpha''}(\overline{\varphi_{\alpha'}}) \rightarrow$
$\longrightarrow \mathit{Pr}_{\alpha'}(\overline{\varphi_{\alpha'}})$, which implies $\not\vdash_{\mathcal{A}} \bigwedge_x (\alpha''(x) \rightarrow \alpha'(x))$.

On the other hand, we have the following:

2.11. <u>Theorem</u>. For each $\alpha, \beta \in \mathit{Bin}$, $\alpha \leq_{\mathcal{A}} \beta$ if and only if there is a $\beta' \in \mathit{Bin}$ such that

(1) $\beta =_{\mathcal{A}} \beta'$,

(2) $\vdash_{\mathcal{A}} \bigwedge_x (\alpha(x) \rightarrow \beta'(x))$.

Proof. Let $\alpha, \beta \in \mathit{Bin}$ and suppose $\alpha \leq_{\mathcal{A}} \beta$. It is sufficient to set

$$\beta'(x) = \alpha(x) \vee \mathit{Fm}_K^{(\mathcal{M})}(x) \wedge \bigvee_{y<x} \mathit{Prf}_\beta(\overline{0 \approx 1}, y).$$

The converse is trivial.

Let us ask if the set Bin is ordered by $\leq_{\mathcal{A}}$ densely. The positive answer is given by the following:

2.12. <u>Theorem</u>. For each $\alpha_1, \alpha_2 \in \mathit{Bin}$ if $\alpha_1 <_{\mathcal{A}} \alpha_2$ then there is an $\alpha \in \mathit{Bin}$ such that $\alpha_1 <_{\mathcal{A}} \alpha <_{\mathcal{A}} \alpha_2$.

Proof. Let $\mathcal{B} = \mathcal{A} + \{\sim \mathit{Con}_{\alpha_2} \wedge \mathit{Con}_{\alpha_1}\}$ and put $\beta(x) = \alpha(x) \vee x \approx \overline{\sim \mathit{Con}_{\alpha_2} \wedge \mathit{Con}_{\alpha_1}}$. Evidently, β is a PR -formula in $\mathcal{P}$ and bi-numerates the set $B = A \cup \{\sim \mathit{Con}_{\alpha_2} \wedge \mathit{Con}_{\alpha_1}\}$. The assumption $\alpha_1 <_{\mathcal{A}} \alpha_2$ implies that $\mathcal{B} = \langle B, K \rangle$ is consistent. Let φ_β be defined by 1.10. We have

$$(1) \quad \nvdash_{\mathcal{A}} (\sim \mathrm{Con}_{\alpha_2} \wedge \mathrm{Con}_{\alpha_1}) \rightarrow \rho_\beta ,$$

$$(2) \quad \nvdash_{\mathcal{A}} (\sim \mathrm{Con}_{\alpha_2} \wedge \mathrm{Con}_{\alpha_1}) \rightarrow \sim \rho_\beta .$$

Put $\alpha(x) = \alpha_1(x) \vee \mathrm{Fm}_K^{(\mathcal{M})}(x) \wedge \bigvee_{y_1, y_2 < x} \sim R_\beta(y_1) \wedge \mathrm{Prf}_{\alpha_2}(\overline{0 \approx 1}, y_2)$.

Evidently, $\alpha \in \mathit{Bin}$ and $\alpha_1 \leq_{\mathcal{A}} \alpha \leq_{\mathcal{A}} \alpha_2$. Further, by the definition of α ,

$$(3) \quad \vdash_{\mathcal{P}} (\sim \mathrm{Con}_{\alpha_2} \wedge \sim \rho_\beta) \rightarrow \sim \mathrm{Con}_{\alpha} ,$$

$$(4) \quad \vdash_{\mathcal{P}} (\mathrm{Con}_{\alpha_1} \wedge \rho_\beta) \rightarrow \mathrm{Con}_{\alpha} .$$

(3) and (1) imply $\nvdash_{\mathcal{A}} \mathrm{Con}_{\alpha} \longrightarrow \mathrm{Con}_{\alpha_2}$, i.e. $\alpha_2 \not\leq_{\mathcal{A}} \alpha$,

(4) and (2) imply $\nvdash_{\mathcal{A}} \mathrm{Con}_{\alpha_1} \longrightarrow \mathrm{Con}_{\alpha}$, i.e. $\alpha \not\leq_{\mathcal{A}} \alpha_1$.

It is well known that every countable, linearly and densely ordered set M without maximal and minimal elements is homogeneous (i.e. for each $x , y \in M$ there is an automorphism of M which maps x to y). If [$\underline{\mathit{Bin}}$] were linearly ordered, the problem of "indescribability" (assuming reflexivity and ω -consistency of $\mathcal{A}$) would be completely settled. But in [$\underline{\mathit{Bin}}$] there are incomparable elements.

2.13. Definition. Let $\alpha , \beta \in \mathit{Bin}$. We put $\alpha \parallel_{\mathcal{A}} \beta$ and $[\alpha] \parallel_{\mathcal{A}} [\beta]$ if simultaneously $\alpha \not\leq_{\mathcal{A}} \beta$ and $\beta \not\leq_{\mathcal{A}} \alpha$.

2.14. Theorem. Let $\mathcal{A}$ be reflexive and ω -consistent. Then for each $\alpha \in \mathit{Bin}$ there is an $\alpha' \in \mathit{Bin}$ such that $\alpha \parallel_{\mathcal{A}} \alpha'$.

Proof. By 2.6, there is an $\alpha_1 \in \mathit{Bin}$ such that $\alpha_1 <_{\mathcal{A}}$ $<_{\mathcal{A}} \alpha$. Put $\mathcal{B}_1 = \mathcal{A} + \{\mathrm{Con}_{\alpha}\}$ and $\mathcal{B}_2 = \mathcal{A} + \{\sim \mathrm{Con}_{\alpha} \wedge \mathrm{Con}_{\alpha_1}\}$. Both $\mathcal{B}_1$ and $\mathcal{B}_2$ are consistent. Let μ_α be defined with respect to $\mathcal{B}_1$ and $\mathcal{B}_2$. Put

$$\alpha'(x) = \alpha_1(x) \vee \dot{F}m_K^{(M)}(x) \wedge \bigvee_{y<x} \sim M_\alpha(y) .$$

Evidently $\alpha' \in Bin$. We shall prove $\alpha' \parallel_{\mathcal{A}} \alpha$. Since $\vdash_{\mathcal{P}} (\mu_\alpha \wedge \dot{Con}_{\alpha_1}) \to \dot{Con}_{\alpha'}$, and $\nvdash_{\mathcal{A}} (\sim \dot{Con}_\alpha \wedge \dot{Con}_{\alpha_1}) \to \sim \mu_\alpha$, we have $\nvdash_{\mathcal{A}} \dot{Con}_{\alpha'} \to \dot{Con}_\alpha$, i.e. $\alpha \not\leq_{\mathcal{A}} \alpha'$. Since $\vdash_{\mathcal{P}} \sim \mu_\alpha \to \sim \dot{Con}_{\alpha'}$ and $\nvdash_{\mathcal{A}} \dot{Con}_\alpha \to \mu_\alpha$, we have $\nvdash_{\mathcal{A}} \dot{Con}_\alpha \to \dot{Con}_{\alpha'}$, i.e. $\alpha' \not\leq_{\mathcal{A}} \alpha$.

The following theorem is a simultaneous generalization of 2.12 and 2.14:

2.15. <u>Theorem</u>. Let $n \in \omega$, $\beta_1, \dots, \beta_n \in Bin$, $\alpha_1, \alpha_2 \in Bin$ and $\alpha_1 <_{\mathcal{A}} \alpha_2$. Suppose $\beta_i \not\leq_{\mathcal{A}} \alpha_1$ and $\beta_i \not\leq_{\mathcal{A}} \alpha_2$ for $i = 1, \dots, n$. Then there is an $\alpha \in Bin$ such that

(1) $\alpha_1 <_{\mathcal{A}} \alpha <_{\mathcal{A}} \alpha_2$ and

(2) $\beta_i \parallel_{\mathcal{A}} \alpha$ for each $i = 1, \dots, n$.

<u>Proof</u>. Let $\mathcal{D}_i = \mathcal{A} + \{\dot{Con}_{\alpha_1} \wedge \sim \dot{Con}_{\beta_i}\}$ ($i = 1, \dots, n$), $\mathcal{D}_{n+i} = \mathcal{A} + \{\dot{Con}_{\beta_i} \wedge \sim \dot{Con}_{\alpha_2}\}$ ($i = 1, \dots, n$) and $\mathcal{D}_{2n+1} = \mathcal{A} + \{\sim \dot{Con}_{\alpha_2} \wedge \dot{Con}_{\alpha_1}\}$. Evidently, each $\mathcal{D}_j$ ($j = 1, \dots, 2n+1$) is consistent. Define μ_{α_1} with respect to the theories $\mathcal{D}_j$ ($j = 1, \dots, 2n+1$). We have

(1) $\nvdash_{\mathcal{A}} (\dot{Con}_{\alpha_1} \wedge \sim \dot{Con}_{\beta_i}) \to \sim \mu_{\alpha_1}$ ($i = 1, \dots, n$),

(2) $\nvdash_{\mathcal{A}} (\dot{Con}_{\beta_i} \wedge \sim \dot{Con}_{\alpha_2}) \to \mu_{\alpha_1}$ ($i = 1, \dots, n$),

(3) $\nvdash_{\mathcal{A}} (\sim \dot{Con}_{\alpha_2} \wedge \dot{Con}_{\alpha_1}) \to \sim \mu_{\alpha_1}$,

(4) $\nvdash_{\mathcal{A}} (\sim \dot{Con}_{\alpha_2} \wedge \dot{Con}_{\alpha_1}) \to \mu_{\alpha_1}$.

Put

$$\alpha(x) = \alpha_1(x) \vee \dot{F}m_K^{(M)}(x) \wedge \bigvee_{v_1, v_2 < x} (\sim M_{\alpha_1}(v_1) \wedge \dot{P}rf_{\alpha_2}(\overline{0 \neq 1}, v_2)).$$

Evidently, $\alpha \in Bin$ and $\alpha_1 \leq_{\mathcal{A}} \alpha \leq_{\mathcal{A}} \alpha_2$. We have

(5) $\vdash_{\mathcal{P}} (\dot{Con}_{\alpha_1} \wedge \mu_{\alpha_1}) \to \dot{Con}_\alpha$,

(6) $\vdash_{\mathcal{P}} (\sim \dot{Con}_{\alpha_2} \wedge \sim \mu_{\alpha_1}) \to \sim Con_\alpha$.

(1) and (5) give $\nvdash_{\mathcal{A}} \dot{Con}_\alpha \to \dot{Con}_{\beta_i}$, i.e. $\beta_i \not\leq_{\mathcal{A}} \alpha$,

for each $i = 1, \ldots, n$. (2) and (6) give $\vdash_{\mathcal{A}} Con_{\beta_i} \rightarrow Con_{\alpha}$, i.e. $\alpha \not\leq_{\mathcal{A}} \beta_i$, for each $i = 1, \ldots, n$. The inequalities $\alpha_1 <_{\mathcal{A}} \alpha <_{\mathcal{A}} \alpha_2$ can be proved using (3) and (4) as in the proof of 2.12.

2.16. Corollary. Let $\mathcal{A}$ be reflexive and ω -consistent. Then for each $n \in \omega$ and arbitrary $\beta_1, \ldots$ $\ldots, \beta_n \in \mathcal{B}in$ there is an $\alpha \in \mathcal{B}in$ such that $\alpha \parallel_{\mathcal{A}} \beta_i$ for each $i = 1, \ldots, n$.

Proof. Put

$$\alpha_1'(x) = \beta_1(x) \wedge \ldots \wedge \beta_n(x) ,$$

$$\alpha_2'(x) = \beta_1(x) \vee \ldots \vee \beta_n(x) .$$

Evidently, α_1' , $\alpha_2' \in \mathcal{B}in$ and $\alpha_1' \leq_{\mathcal{A}} \beta_i \leq_{\mathcal{A}} \alpha_2'$ for each $i = 1, \ldots, n$. Choose an $\alpha_1 <_{\mathcal{A}} \alpha_1'$ (it exists by 2.6) and an $\alpha_2 >_{\mathcal{A}} \alpha_2'$ (it exists by 2.8). Theorem 2.15 gives the result.

2.17. Corollary. Under conditions of Corollary 2.16, each $\beta \in \mathcal{B}in$ belongs to some infinite set of mutually incomparable elements.

Proof. We put $\beta_1 = \beta$. If $\beta_1, \ldots, \beta_n$ are defined, we define β_{n+1} in the same way as α was defined in the preceding corollary.

In the proof of 2.16 we used the fact that in $\mathcal{B}in$ every n -tuple of elements has upper and lower boundaries. Now we ask whether suprema and infima exist. Theorems 2.19 and 2.21 answer this question affirmatively. One could hypothesize that, given $\alpha_1, \alpha_2 \in \mathcal{B}in$, $\alpha_1 \vee \alpha_2$ is the supremum and $\alpha_1 \wedge \alpha_2$ is the infimum. The next ~~example~~ shows that the hypothesis is false. We construct

$\alpha_1, \alpha_2 \in \mathcal{B}in$ such that $\alpha_1 =_{\mathcal{A}} \alpha_2$ but $\not\vdash_{\mathcal{A}} Con_{\alpha_1} \longrightarrow Con_{\alpha_1 \vee \alpha_2}$. In other words,

$$\alpha_1 \vee \alpha_2 >_{\mathcal{A}} \alpha_1 =_{\mathcal{A}} \alpha_2 =_{\mathcal{A}} sup(\alpha_1, \alpha_2).$$

2.18. Example. Let $\mathcal{A}$ be ω-consistent and suppose $\alpha \in \mathcal{B}in$. Let $B = A \cup \{Con_\alpha\}$ and let $\beta(x) = \alpha(x) \vee \vee x \approx \overline{Con_\alpha}$. Evidently, $\mathcal{B} = \langle B, K\rangle$ is consistent and $\beta(x)$ is a PR-formula in $\mathcal{P}$ bi-numerating B.

Put

$$\alpha_1(x) = \alpha(x) \vee \bigvee_{y<x} [\sim R_\beta(y) \wedge (x \approx \overline{\sim \varphi_\alpha} \wedge vr_y \approx vr_y)^{(\mathcal{M})}],$$

$$\alpha_2(x) = \alpha(x) \vee \bigvee_{y<x} [\sim R_\beta(y) \wedge (x \approx \overline{\varphi_\alpha} \wedge vr_y \approx vr_y)^{(\mathcal{M})}].$$

Evidently, $\alpha_1, \alpha_2 \in \mathcal{B}in$. We have $\vdash_{\mathcal{A}} Con_\alpha \leftrightarrow \sim Pr_\alpha(\overline{\varphi_\alpha})$ and $\vdash_{\mathcal{A}} Con_\alpha \leftrightarrow \sim Pr_\alpha(\overline{\sim \varphi_\alpha})$. Hence $\alpha =_{\mathcal{A}} \alpha_1 =_{\mathcal{A}} \alpha_2$. Since $\not\vdash_{\mathcal{A}} Con_\alpha \longrightarrow \varphi_\beta$ and $\vdash_{\mathcal{A}} \sim \varphi_\beta \longrightarrow (Pr_{\alpha_1}(\overline{\sim\varphi_\alpha}) \wedge Pr_{\alpha_2}(\overline{\varphi_\alpha}))$, we obtain $\not\vdash_{\mathcal{A}} Con_\alpha \longrightarrow Con_{\alpha_1 \vee \alpha_2}$.

One also could construct $\alpha_1, \alpha_2 \in \mathcal{B}in$ such that $\alpha_1 =_{\mathcal{A}} \alpha_2$ but $\alpha_1 \wedge \alpha_2 <_{\mathcal{A}} \alpha_1 =_{\mathcal{A}} \alpha_2 =_{\mathcal{A}} inf(\alpha_1, \alpha_2)$.

2.19. Theorem. In $[\mathcal{B}in]$ every pair $[\alpha_1], [\alpha_2]$ has the infimum.

Proof. Let $\alpha_1, \alpha_2 \in \mathcal{B}in$. We put

$$\alpha_1'(x) = \alpha_1(x) \vee Fm_K^{(\mathcal{M})}(x) \wedge \bigvee_{y<x} Prf_{\alpha_1}(\overline{0 \approx 1}, y),$$

$$\alpha_2'(x) = \alpha_2(x) \vee Fm_K^{(\mathcal{M})}(x) \wedge \bigvee_{y<x} Prf_{\alpha_2}(\overline{0 \approx 1}, y).$$

Evidently, $\alpha_1', \alpha_2' \in \mathcal{B}in$ and $\alpha_1' =_{\mathcal{A}} \alpha_1$ and $\alpha_2' =_{\mathcal{A}} \alpha_2$. Set $\alpha(x) = \alpha'(x) \wedge \alpha_2'(x)$. We shall prove that $[\alpha]$ is the infimum of $[\alpha_1]$ and $[\alpha_2]$. Evidently $\alpha \leq_{\mathcal{A}} \alpha_1$ and $\alpha \leq_{\mathcal{A}} \alpha_2$ and therefore $\vdash_{\mathcal{A}} (Con_{\alpha_1} \vee Con_{\alpha_2}) \rightarrow Con_\alpha$. Conversely,

$\vdash_{\mathcal{A}} (\sim Con_{\alpha_1} \wedge \sim Con_{\alpha_2}) \rightarrow \sim Con_{\alpha}$, because $\vdash_{\mathcal{A}} (\sim Con_{\alpha_1} \wedge \sim Con_{\alpha_2}) \rightarrow \bigvee_{y} \bigwedge_{x>y} Fm_K^{(M)}(x) \rightarrow (\alpha_1'(x) \wedge \alpha_2'(x))$. Let $\beta \in Bin$, $\beta \leq_{\mathcal{A}} \alpha_1$, $\beta \leq_{\mathcal{A}} \alpha_2$ and suppose $\alpha \leq_{\mathcal{A}} \beta$. Then $\vdash_{\mathcal{A}} (Con_{\beta} \leftrightarrow Con_{\alpha})$, i.e. $\alpha =_{\mathcal{A}} \beta$, because $\vdash_{\mathcal{A}} Con_{\beta} \leftrightarrow (Con_{\alpha_1} \vee Con_{\alpha_2})$.

By the proof of Theorem 2.19, the following holds.

2.20. <u>Corollary.</u> For each α_1 , α_2 , $\alpha \in Bin$, $[\alpha]$ is the infimum of $[\alpha_1]$ and $[\alpha_2]$ if and only if $\vdash_{\mathcal{A}} Con_{\alpha} \leftrightarrow (Con_{\alpha_1} \vee Con_{\alpha_2})$.

2.21. <u>Theorem</u>. In $[Bin]$ every pair of elements of Bin has the supremum.

<u>Proof</u>. Let α_1 , $\alpha_2 \in Bin$ and let $\alpha' \in Bin$ such that $\alpha' \leq_{\mathcal{A}} \alpha_1$ and $\alpha' \leq_{\mathcal{A}} \alpha_2$. Put

$$\alpha(x) = \alpha'(x) \vee Fm_K^{(M)}(x) \wedge \bigvee_{y<x} Prf_{\alpha_1}(\overline{0 \approx 1}, y) \vee Prf_{\alpha_2}(\overline{0 \approx 1}, y).$$

We shall prove that $[\alpha]$ is the supremum. Evidently, $\alpha \in Bin$, $\alpha \geq_{\mathcal{A}} \alpha_1$ and $\alpha \geq_{\mathcal{A}} \alpha_2$ and therefore $\vdash_{\mathcal{A}} Con_{\alpha} \rightarrow (Con_{\alpha_1} \wedge Con_{\alpha_2})$. On the other hand, $\vdash_{\mathcal{A}} (Con_{\alpha_1} \wedge Con_{\alpha_2}) \rightarrow Con_{\alpha}$, because we have $\vdash_{\mathcal{A}} (Con_{\alpha_1} \wedge Con_{\alpha_2}) \rightarrow \bigwedge_{x} (\alpha(x) \rightarrow \alpha'(x))$ and $\vdash_{\mathcal{A}} (Con_{\alpha_1} \rightarrow \rightarrow Con_{\alpha'})$. Let $\beta \in Bin$, $\beta \geq_{\mathcal{A}} \alpha_1$, $\beta \geq_{\mathcal{A}} \alpha_2$ and suppose $\beta \leq_{\mathcal{A}} \alpha$. Then $\vdash_{\mathcal{A}} (Con_{\beta} \leftrightarrow Con_{\alpha})$, i.e. $\beta =_{\mathcal{A}} \alpha$, because $\vdash_{\mathcal{A}} Con_{\beta} \leftrightarrow (Con_{\alpha_1} \wedge Con_{\alpha_2})$.

By the proof of Theorem 2.21, the following holds:

2.22. <u>Corollary</u>. For each α_1 , α_2 , $\alpha \in Bin$, $[\alpha]$ is the supremum of $[\alpha_1]$, $[\alpha_2]$ if and only if $\vdash_{\mathcal{A}} Con_{\alpha} \leftrightarrow (Con_{\alpha_1} \wedge Con_{\alpha_2})$.

2.23. <u>Denotation</u>. The supremum of $[\alpha_1]$, $[\alpha_2] \in [Bin]$ will be denoted by $[\alpha_1] \cup [\alpha_2]$, the infimum by

$[\alpha_1] \cap [\alpha_2]$. This is a correct denotation, since $[\underline{\mathcal{Bin}}]$ is a partially ordered set and therefore suprema and infima are uniquely determined.

We shall now modify (extend) Definition 2.5. In the remainder of the paper, the symbol $[\underline{\mathcal{Bin}}]$ will be used in the sense of the following definition.

2.24. Definition. $[\underline{\mathcal{Bin}}] = \langle [\mathcal{Bin}], \leq_{\mathcal{A}}, \cap, \cup \rangle$, where $\cap$ and $\cup$ are defined as in 2.23.

By Theorems 2.19, 2.21, 2.6 and 2.8, we have the following:

2.25. Theorem. $[\underline{\mathcal{Bin}}]$ is a lattice. If $\mathcal{A}$ is reflexive, then the lattice $[\underline{\mathcal{Bin}}]$ has no least element, if $\mathcal{A}$ is ω -consistent, then the lattice $[\underline{\mathcal{Bin}}]$ has no greatest element.

2.26. Definition. For each $\varphi \in St_K$ let $[\varphi]$ be the set of all $\psi \in St_K$ for which $\vdash_{\mathcal{A}} \varphi \leftrightarrow \psi$. Let $\varphi, \psi \in St_K$. We put $[\varphi] \leq_{\mathcal{A}} [\psi]$ if $\vdash_{\mathcal{A}} \psi \rightarrow \varphi$. We define $[\varphi] \cup [\psi] = [\varphi \wedge \psi]$, $[\varphi] \cap [\psi] = [\varphi \vee \psi]$, $[St_K] = \{[\varphi]; \varphi \in St_K\}$ and $[\underline{\mathcal{A}}] = \langle [St_K], \leq_{\mathcal{A}}, \cap, \cup \rangle$.

It is well known that $[\mathcal{A}]$ is a Boolean algebra.

2.27. Theorem. The function which associates with every $[\alpha] \in [\mathcal{Bin}]$ the class $[\mathcal{Cn}_\alpha]$ is an isomorphical embedding of the lattice $[\underline{\mathcal{Bin}}]$ into the Boolean algebra $[\underline{\mathcal{A}}]$.

Proof. By Definitions 2.24 and 2.26 and Corollaries 2.20 and 2.22.

2.28. Corollary. $[\underline{\mathcal{Bin}}]$ is a distributive lattice.

(To be continued.)

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