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ON INTERPRETABILITY IN SET THEORIES

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Denote by ZF the Zermelo-Fraenkel set theory (with regularity but without choice) and by GB the Gödel-Bernays set theory (the same restriction). Both theories are supposed to be formulated as formal systems with one sort of variables and one binary predicate $\in$. Every ZF-formula can be considered as a particular GB-formula by means of an obvious relative interpretation.

In a discussion with Professor G. Kreisel in summer 1969 I formulated the following

Problem: Does for every ZF-formula φ relative interpretability of (ZF, φ) in ZF imply relative interpretability of (GB, φ) in GB ?

Denoting, for every theory T which is either an extension of ZF or an extension of GB, by $\mathcal{J}_T$ the set of all ZF-formulas such that (T, φ) is relatively interpretable in T , our problem reads: Is $\mathcal{J}_{ZF} \subseteq \mathcal{J}_{GB}$?

We shall prove a theorem which implies the negative answer of our problem. The theorem also implies that $\mathcal{J}_{ZF}$ is not recursively enumerable (whereas $\mathcal{J}_{GB}$ is, which is easy to show). I discussed the problem with Professors G. Kreisel, J.R. Shoenfield and R. Solovay; I thank them for

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their interest and for the encouraging advice to look for a counterexample. Discussions with my wife on her work [4] were not only an exciting pleasure for me but also helped me to find a solution.

First, let us recall some known facts on finitary relative consistency proofs useful in the sequel and yielding a background of our problem. Presupposed is the knowledge of the notion of a relative interpretation in the sense of Tarski [7] and some familiarity with Feferman's fundamental work [2].

<u>Lemma 1</u>. For every ZF-formula φ, $ZF \vdash \varphi$ iff $GB \vdash \varphi$; equivalently, for every ZF-formula φ, $\mathit{Con}(ZF, \varphi)$ iff $\mathit{Con}(GB, \varphi)$.

See [6] for a finitary proof; in fact, Shoenfield constructs a primitive recursive function associating with every ZF-formula φ and every GB-proof of φ a ZF-proof of φ.

Although we shall be dealing with set theories, we shall explicitly use only variables ranging over the set of natural numbers; the letters $x, y, \ldots$ will be used for this purpose. $\xi(x)$ is an arbitrary but fixed bi-numeration of the set of axioms of ZF in ZF. If φ is a ZF-formula then $\xi \cup \{\bar{\varphi}\}$ means the formula $\xi(x) \vee x \approx \bar{\varphi}$ which bi-numerates the axioms of (ZF, φ) in ZF.

<u>Lemma 2</u>. For each ZF-formula φ, $\varphi \in \mathcal{J}_{ZF}$ iff $ZF \vdash \dot{\mathit{Con}}_{\xi \cup \{\bar{\varphi}\} \restriction n}$ for every n.

See [2] Theorem 8.10 (and also 6.3, 6.9 and 5.9) for the proof of the implication $\Longrightarrow$ (cf. also [5], footnote 22). The converse implication is easy to prove using reflexivity of (ZF, φ) and observing that

$$ZF \vdash [(\underset{\cdot}{Con}_{\xi \cup \{\overline{\varphi}\} \upharpoonright \overline{m}})^* \rightarrow \underset{\cdot}{Con}_{\xi \cup \{\overline{\varphi}\} \upharpoonright \overline{m}}]$$

($*$ denotes the image of the respective formula in the interpretation in question).

Hence, having proved $ZF \vdash \underset{\cdot}{Con}_{\xi \cup \{\overline{\varphi}\} \upharpoonright \overline{m}}$ for every m, we have the following: (i) (ZF, φ) is relatively interpretable in ZF, (ii) consequently, φ is relatively consistent w.r.t. ZF and (iii) is relatively consistent w.r.t. GB. But the question remains whether (GB, φ) is relatively interpretable in GB and we are led to our problem whether $\mathcal{J}_{ZF} \subseteq \mathcal{J}_{GB}$.

A counterexample is a ZF-formula φ such that (ZF, φ) is relatively interpretable in ZF, but (GB, φ) is not relatively interpretable in GB. Such a φ is consistent with GB, and also $\neg \varphi$ is consistent with GB, for otherwise the identical interpretation of GB would be an interpretation of (GB, φ) in GB.

<u>Theorem</u>. Suppose that ZF is ω-consistent. Let W be a recursively enumerable set of ZF-formulas such that, for every φ, $\varphi \in W$ implies $Con(ZF, \varphi)$. Then there is a φ such that $\varphi \in \mathcal{J}_{ZF} - W$. In fact, there is a primitive recursive function associating with every RE-formula $\vartheta(x)$ a formula φ such that, if W is the set numerated by $\vartheta(x)$ in ZF and if every element of W is a ZF-formula consistent with ZF, then $\varphi \in \mathcal{J}_{ZF} - W$.

Proof. Let $W = \{n ; (\exists m) A(m, n)\}$ where A is primitive recursive. Let $\alpha(x, y)$ be a PR-formula

such that $\alpha(x, y)$ bi-numerates A in ZF and $\bigvee_x \alpha(x, y)$ numerates W in ZF. (Cf.[2] 3.11.)
Using the diagonal lemma 5.1 [2] we can construct a ZF-formula φ such that

$$ZF \vdash \varphi \leftrightarrow \bigwedge_x (\alpha(x, \bar{\varphi}) \rightarrow \neg Con_{\S \cup \{\bar{\varphi}\} \restriction x}).$$

(a) $Con(ZF, \varphi)$. Otherwise we have $ZF \vdash \bigvee_x \alpha(x, \bar{\varphi})$ and therefore $\varphi \in W$, which implies $Con(ZF, \varphi)$.

(b) $\varphi \notin W$. Otherwise we have $A(m, \varphi)$ for some m; then $ZF \vdash \alpha(\bar{m}, \bar{\varphi})$ and $(ZF, \varphi) \vdash \neg Con_{\S \cup \{\bar{\varphi}\} \restriction \bar{m}}$. But since (ZF, φ) is consistent and reflexive (see [2], p.89) we have $(ZF, \varphi) \vdash Con_{\S \cup \{\bar{\varphi}\} \restriction \bar{m}}$ which contradicts the consistency of (ZF, φ).

(c) $\varphi \in \mathcal{I}_{ZF}$. We show $ZF \vdash Con_{\S \cup \{\bar{\varphi}\} \restriction \bar{m}}$ for every m; then $\varphi \in \mathcal{I}_{ZF}$ by Lemma 2. Since $(ZF, \varphi) \vdash Con_{\S \cup \{\bar{\varphi}\} \restriction \bar{m}}$ by the reflexivity, it suffices to show $(ZF, \neg\varphi) \vdash Con_{\S \cup \{\bar{\varphi}\} \restriction \bar{m}}$. But $\neg\varphi$ is equivalent in ZF to $\bigvee_x (\alpha(x, \bar{\varphi})\ \&\ Con_{\S \cup \{\bar{\varphi}\} \restriction x})$.
Now for each m we have $ZF \vdash \neg \alpha(\bar{m}, \bar{\varphi})$ since $\varphi \notin W$ by (b) and since α bi-numerates A in ZF.
Hence we have

$$(ZF, \neg\varphi) \vdash \bigvee_x (x > \bar{m}\ \&\ Con_{\S \cup \{\bar{\varphi}\} \restriction x})$$

for each m, which implies $(ZF, \neg\varphi) \vdash Con_{\S \cup \{\bar{\varphi}\} \restriction \bar{m}}$.
This completes the proof.

Corollary 1. If ZF is ω-consistent then $\mathcal{I}_{ZF} - \mathcal{I}_{GB} \neq 0$. For, evidently, $\varphi \in \mathcal{I}_{GB}$ implies $Con(ZF, \varphi)$ and $\mathcal{I}_{GB}$ is recursively enumerable. (A formula φ belongs to $\mathcal{I}_{GB}$ iff there are two GB-formulas defining classes and membership in the sense of the interpretations and, in addition, GB-proofs of the interpretations of all the finitely many - 15, say - axioms of (GB, φ).

<u>Corollary 2</u>. Let GB_1 be a consistent finitely axiomatized extension of GB (for example, by adding the axiom of existence of measurable cardinals, assuming that this extension is consistent). If ZF is ω-consistent then $\mathcal{J}_{ZF} - \mathcal{J}_{GB_1} \neq 0$.

<u>Corollary 3</u>. If ZF is ω-consistent then $\mathcal{J}_{ZF}$ is not recursively enumerable. (By the theorem, every recursively enumerable subset of $\mathcal{J}_{ZF}$ is a proper subset.)

Discussion. (1) A historical remark. The Cohen's pioneering proof of the independence of the continuum hypothesis (CH) can be understood as a proof that, for every n, $ZF \vdash Con_{\underline{\mathfrak{F}} \cup \{\neg \overline{CH}\} \restriction \overline{n}}$ (see [1]) and therefore yields a relative interpretation of $(ZF, \neg CH)$ in ZF. But it follows from our theorem that a relative interpretation of $(ZF, \neg CH)$ in ZF does not automatically yield an interpretation of $(GB, \neg CH)$ in GB. Such an interpretation was constructed in [8] by exploring the Cohen's proof (see also various relative interpretations of GB + additional axiom in GB constructed in [9] using the notion of Boolean valued models). It can be said that construction of a relative interpretation is the most natural kind of a relative consistency proof; but perhaps it is the matter of one's taste. (In fact, Vopěnka constructed a parametrical relative interpretation called a parametric syntactic model in [3]; but if (GB, φ) has a parametric relative interpretation in GB such that the range of parameters is described by a ZF-formula, then (GB, φ) has a (non-parametric) relative interpretation in GB, see [3], Theorem 4.)

(2) Is $\mathcal{I}_{GB} \subseteq \mathcal{I}_{ZF}$? It is true that if (GB, φ) has a "nice" relative interpretation in GB then $\varphi \in \mathcal{I}_{ZF}$. E.g. it suffices that M^* is absolute from below (i.e. $GB \vdash M^*(X) \rightarrow M(X)$) and, in addition, both $M^*(a)$ and $M^*(a)$ & $M^*(b)$ & $a \in^* b$ are equivalent in GB to some ZF-formulas. (Here X is a class variable and a, b are set variables.) One can formulate more general conditions, but the problem in full generality seems to be open.

(3) By Lemma 2, $\mathcal{I}_{ZF}$ is a Π^0_2 set and by Corollary 3, it is not a Σ^0_1 set. I do not know whether $\mathcal{I}_{ZF}$ is a Π^0_1 set and/or a Δ^0_2 set.

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