Commentationes Mathematicae Universitatis Carolinae

12, 1 (1971)

ON SOLUTIONS OF NONAUTONOMOUS LINEAR DELAYED DIFFERENTIAL EQUATIONS WHICH ARE DEFINED AND BOUNDED FOR $t \to -\infty$

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Let M_m be the space of real square matrices of order m, R - the real line, R^+ - the positive half-line (closed), R^- - the negative half-line, $A : R^- \to M_m$, $B \to R^- \to M_m$ locally integrable. For $y \in R^m$ denote by $|y|$ the Euclidean norm of y and for $C \in M_m$ put $|C| = \sup_{|y| \leq 1} |Cy|$. For $\gamma \in R^+$ let $Z(\gamma)$ be the set of such solutions $x : R^- \to R^m$ of

(1) $$\frac{dx}{dt}(t) = A(t)x(t) + B(t)x(t-1)$$

that

(2) $$\sup_{t \leq 0} e^{\gamma t} |x(t)| < \infty .$$

Obviously $Z(\gamma)$ is a linear manifold.

<u>Theorem</u>. Assume that $|B|^2$ is locally integrable and that

(3) $$\sup_{t \leq 0} \int_{t-1}^{t} |A(\tau)| d\tau < \infty, \quad \sup_{t \leq 0} \int_{t-1}^{t} |B(\tau)|^2 d\tau < \infty .$$

Then the dimension of $Z(\gamma)$ is finite. Moreover, there exists $\vartheta : (R^+)^3 \to R^+$ such that if

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AMS Primary 34A30,34K05,34K25 Ref.Ž. 7.926, 7.929, 7.928

(4) $\sup\limits_{t \leq 0} \int_{t-1}^{t} |A(\tau)| d\tau \leq a, \quad \sup\limits_{t \leq 0} \int_{t-1}^{t} |B(\tau)|^2 d\tau \leq b^2$,

then

(5) $$\dim Z(\gamma) \leq \theta(a, b, \gamma) .$$

<u>Note 1</u>. $\theta(a, b, \gamma)$ may be calculated (of course not the best one). Thus it may be shown that

(i) $\dim Z(\gamma) \leq m$ if

$$e^{(m+1)\gamma}[1+4e^{2a} \max(1, b^2)]^{m/2} e^{a} b < 1 ,$$

(ii) $\dim Z(\gamma) \leq m+1$ if

$$e^{(m+2)\gamma}[1+4e^{2a} \max(1, b^2)]^{m/2} e^{2a} b^2 < 1 ,$$

(iii) if $e^{a} b \geq 1$ and $e^{\gamma}(1+ae^{a}) b \to \infty$,

then

$$\theta(a, b, \gamma) \approx 2m\, e\, \pi^{-2} e^{2\gamma}(1+a e^{a})^2 b^2 .$$

<u>Note 2</u>. The above theorem is related to applications of Theory of Invariant Manifolds to Delayed Differential Equations (cf. [1],[2],[3]). Let us review some results which may be obtained for (1). For this purpose extend A and B to R putting $A(t) = 0 = B(t)$ for $t > 0$.

<u>Proposition</u>. Assume that A fulfils (4) and that B instead of (4) fulfils

(6) $$\sup\limits_{t} \int_{t-1}^{t} |B(\tau)| d\tau \leq \beta$$

and that there exists $L > 0$ such that

(7) $e^{a}(e^{a}+L)^2 \beta \leq L$,

(8) $(e^{a}+1) e^{a}(e^{a}+L)\beta < 1$.

Let $\mathcal{U}$ be a fundamental matrix of

$$(9) \qquad \frac{dx}{dt}(t) = A(t)\,x(t) \ .$$

Then there exists $\mathcal{Q} : R^* \to M_n$, continuous, $|\mathcal{Q}(t)| \leqq L$ for $t \in R$ such that every solution of

$$(10) \quad \frac{dx}{dt}(t) = (A(t) + B(t)[\mathcal{U}(t-1)\,\mathcal{U}^{-1}(t) + \mathcal{Q}(t)])\,x(t)$$

fulfils (1). Moreover, solutions of (10) belong to $Z(\gamma)$ with $\gamma = a + \log[1 + \beta(e^a + L)]$ (so that $\dim Z(\gamma) \geqq n$).

As $\int_{t-1}^{t} |B(\tau)|\,d\tau \leqq (\int_{t-1}^{t} |B(\tau)|^2 d\tau)^{1/2}$, Proposition may be applied, if B fulfils (4) and if (7) and (8) hold, β being replaced by b .

Fix a and choose L , e.g. $L = e^a$. Find such a b that (7) and (8) are fulfilled for β being replaced by b and that the inequality in (i), Note 1 is fulfilled with $\gamma \geqq a + \log[1 + b(e^a + L)]$. Then it may be concluded that $\dim Z(\gamma) = n$ (provided that A and B fulfil (4)).

R e f e r e n c e s

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(Oblatum 13.10.1970)

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