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THE LATTICE OF RADICAL FILTERS OF A COMMUTATIVE NOETHERIAN RING

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As it was shown by V. Dlab [2], there is a one-to-one correspondence between all radical filters and some sets of prime ideals of a commutative Noetherian ring (namely, the set of all prime ideals contained in $\mathcal{E}$ corresponds to any radical filter $\mathcal{E}$). In this brief note, there is given a new one-to-one correspondence between all radical filters and some sets of prime ideals of a commutative Noetherian ring Λ and it is shown that the lattice $\mathcal{L}$ of all radical filters of Λ is distributive. Further, some necessary and sufficient conditions for Λ , under which the lattice $\mathcal{L}$ is complementary, are given.

In what follows, Λ stands for an associative commutative Noetherian ring with unity. Recall that a (non-empty) family $\mathcal{E}$ of ideals of Λ is called a radical filter (commutativity is assumed!) if

(1) $I \in \mathcal{E}$, $I \subseteq J \Rightarrow J \in \mathcal{E}$,

(2) $I \subseteq J, J \in \mathcal{E}$ and $(I : \lambda) \in \mathcal{E}$ for any $\lambda \in J \Rightarrow$ $\Rightarrow I \in \mathcal{E}$, where $(I : \lambda) = \{\mu \in \Lambda , \mu\lambda \in I\}$.

Let us denote by $\mathcal{P}$ the set of all prime ideals of

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Λ and by $\mathfrak{M}$ the set of all maximal ideals of Λ. We call a subset $\mathfrak{N}$ of $\mathcal{P}$ a radical set, if any two elements of $\mathfrak{N}$ are incomparable (in the order of the inclusion). Let $\mathfrak{Q}$ be any (non-empty) set of ideals of Λ. The maximal elements of the set of all prime ideals which are contained in some ideal from $\mathfrak{Q}$ form a radical set - the radical set belonging to $\mathfrak{Q}$.

Lemma 1: Let $\mathfrak{N} \subseteq \mathcal{P}$ be a radical set. Then the set $\mathcal{C}_{\mathfrak{N}} = \{ I , I \nsubseteq N , \forall N \in \mathfrak{N} , I$ ideal in $\Lambda \}$ is the radical filter.

Proof: The property (1) is evident.
Proving (2) indirectly we shall show

(3) $I \notin \mathcal{C}_{\mathfrak{N}} \Rightarrow \forall J , J \in \mathcal{C}_{\mathfrak{N}} , I \subseteq J$, there exists $\lambda \in J$ with $(I : \lambda) \notin \mathcal{C}_{\mathfrak{N}}$.

Let us suppose $I \notin \mathcal{C}_{\mathfrak{N}}$. Then there exists $N \in \mathfrak{N}$ with $I \subseteq N$. For $J \in \mathcal{C}_{\mathfrak{N}}$ we have $J \dot{-} N \neq \emptyset$, hence we can take $\lambda \in J \dot{-} N$. Then $(I : \lambda) = \{\mu \in \Lambda , \mu\lambda \in I \subseteq N\} \subseteq (N : \lambda)$. But $(N : \lambda) = N$ because N is a prime ideal and $\lambda \notin N$ which finishes the proof of (3).

Lemma 2: Let $\mathfrak{N}_1 , \mathfrak{N}_2$ be two radical sets. Then $\mathcal{C}_{\mathfrak{N}_1} \subseteq \mathcal{C}_{\mathfrak{N}_2}$ if and only if to any $N_2 \in \mathfrak{N}_2$ there exists $N_1 \in \mathfrak{N}_1$ with $N_2 \subseteq N_1$. Consequently, $\mathcal{C}_{\mathfrak{N}_1} = \mathcal{C}_{\mathfrak{N}_2}$ if and only if $\mathfrak{N}_1 = \mathfrak{N}_2$.

Proof: At first, suppose that the condition holds. Then $I \in \mathcal{C}_{\mathfrak{N}_1} \Rightarrow I \nsubseteq N, \forall N \in \mathfrak{N}_1 \Rightarrow I \nsubseteq N , \forall N \in \mathfrak{N}_2 \Rightarrow I \in \mathcal{C}_{\mathfrak{N}_2}$.
Conversely, if there exists $N \in \mathfrak{N}_2$ which is not contained in any $N' \in \mathfrak{N}_1$, then $N \in \mathcal{C}_{\mathfrak{N}_1} \dot{-} \mathcal{C}_{\mathfrak{N}_2}$. For the proof of the last part let us note that if $\mathcal{C}_{\mathfrak{N}_1} = \mathcal{C}_{\mathfrak{N}_2}$, then to any $N_2 \in \mathfrak{N}_2$ there exists $N_1 \in \mathfrak{N}_1$ and,

further, $N_2' \in \mathcal{N}_2$ with $N_2 \subseteq N_1 \subseteq N_2'$. But $N_2 = N_2'$ for $\mathcal{N}_2$ being a radical set which implies $\mathcal{N}_2 \subseteq \mathcal{N}_1$. The inclusion $\mathcal{N}_1 \subseteq \mathcal{N}_2$ follows by symmetrical arguments.

Theorem 1: There is a one-to-one correspondence between all radical filters and all radical sets of prime ideals of Λ.

Proof: In view of Lemmas 1 and 2 it suffices to prove that to any radical filter $\mathcal{E}$, there exists a radical set $\mathcal{N}$ such that $\mathcal{E} = \mathcal{E}_{\mathcal{N}}$. Let $\mathcal{N}$ be the set of all maximal elements of the set of all ideals which do not belong to $\mathcal{E}$. It is easy to see that it suffices to show that $\mathcal{N}$ contains the prime ideals only. One can easily show that an ideal I is prime if and only if $(I : \lambda) = I$ for any $\lambda \in \Lambda \dot{-} I$. Let us take $I \in \mathcal{N}$ arbitrarily, and let us assume the existence of $\lambda \in \Lambda \dot{-} I$ with $(I : \lambda) \supsetneqq I$. By hypothesis (maximality of I) it is $(I : \lambda) \in \mathcal{E}$ and $J = \{I, \lambda\} \in \mathcal{E}$ (J is the ideal generated in Λ by I and λ). Writing any element $\varphi \in J$ in the form $\varphi = \alpha\lambda + \beta$, $\alpha \in \Lambda$, $\beta \in I$, we have $\mu\varphi = \alpha\mu\lambda + \mu\beta \in I$ for any $\mu \in (I : \lambda)$, hence $(I : \lambda) \subseteq (I : \varphi)$ for any $\varphi \in J$. Then $I \in \mathcal{E}$ by (1) and (2), which contradicts our hypothesis. Theorem 1 is therefore proved.

It is easy to see that the intersection of any set of radical filters is a radical filter so that the radical filters form a (complete) lattice which we denote by $\mathcal{L}$.

Theorem 2: Let $\mathcal{N}_1$, $\mathcal{N}_2$ be two radical sets of prime ideals. Then $\mathcal{E}_{\mathcal{N}_1} \wedge \mathcal{E}_{\mathcal{N}_2} = \mathcal{E}_{\mathcal{N}}$, where $\mathcal{N}$ is the radical set belonging to $\mathcal{N}_1 \cup \mathcal{N}_2$ and $\mathcal{E}_{\mathcal{N}_1} \vee \mathcal{E}_{\mathcal{N}_2} = \mathcal{E}_{\mathcal{N}}$ where $\mathcal{N}$ is the radical set belonging to the set

$$\mathcal{Q} = \{ N_1 \cap N_2 \, , \ N_1 \in \mathcal{N}_1 \, , \ N_2 \in \mathcal{N}_2 \} \ .$$

Proof: The proof for intersection is direct and we shall omit it. Proving the part for join, let us have $I \in \mathcal{E}_{\mathcal{N}_i}$, $i = 1, 2$. Then $I \nsubseteq N_i$ for any $N_i \in \mathcal{N}_i$, $i = 1, 2$ and therefore $I \nsubseteq N$ for any $N \in \mathcal{N}$ which denotes $I \in \mathcal{E}_{\mathcal{N}}$ and hence $\mathcal{E}_{\mathcal{N}_1} \vee \mathcal{E}_{\mathcal{N}_2} \subseteq \mathcal{E}_{\mathcal{N}}$. Conversely, let $\mathcal{E}_{\mathcal{N}'}$ be any radical filter containing $\mathcal{E}_{\mathcal{N}_1} \cup \mathcal{E}_{\mathcal{N}_2}$. Then from $\mathcal{E}_{\mathcal{N}_i} \subseteq \mathcal{E}_{\mathcal{N}'}$, $i = 1, 2$ and Lemma 2 it easily follows that to any $N' \in \mathcal{N}'$ there exist $N_i \in \mathcal{N}_i$, $i = 1, 2$ with $N' \subseteq N_1 \cap N_2$. Hence $N' \subseteq N$ for some $N \in \mathcal{N}$ owing to the definition of $\mathcal{N}$. Using Lemma 2 again, one gets $\mathcal{E}_{\mathcal{N}} \subseteq \mathcal{E}_{\mathcal{N}'}$ as was to be shown.

Theorem 3: The lattice $\mathcal{L}$ is distributive,

Proof: We shall prove the "cancellation form" of distributivity indirectly, namely $b \neq c$, $a \wedge b = a \wedge c \Rightarrow$ $\Rightarrow a \vee b \neq a \vee c$. Let us suppose we have three radical filters $\mathcal{E}_{\mathcal{N}_1}, \mathcal{E}_{\mathcal{N}_2}, \mathcal{E}_{\mathcal{N}_3}$ satisfying $\mathcal{E}_{\mathcal{N}_2} \neq \mathcal{E}_{\mathcal{N}_3}$ and

(4) $$\mathcal{E}_{\mathcal{N}_1} \wedge \mathcal{E}_{\mathcal{N}_2} = \mathcal{E}_{\mathcal{N}_1} \wedge \mathcal{E}_{\mathcal{N}_3} = \mathcal{E}_{\mathcal{N}} \ .$$

Let us put

$$\mathcal{N}_1' = \mathcal{N} \cap \mathcal{N}_1 \ ,$$

$$\mathcal{N}_2' = \mathcal{N}_3' = \mathcal{N} \div \mathcal{N}_1' \ ,$$

$$\mathcal{N}_1'' = \{ N \in \mathcal{N}_1 \, , \ \exists M \in \mathcal{N}_2' \ ; \ N \subsetneqq M \} \ ,$$

$$\mathcal{N}_2'' = \{ N \in \mathcal{N}_2 \, , \ \exists M \in \mathcal{N}_1 \ ; \ N \subseteq M \} \ ,$$

$$\mathcal{N}_3'' = \{ N \in \mathcal{N}_3 , \exists M \in \mathcal{N}_1 ; N \subseteq M \} .$$

One can easily see (by using Theorem 2 and (4)) that $\mathcal{N}_i'$ and $\mathcal{N}_i''$ are disjoint and $\mathcal{N}_i' \cup \mathcal{N}_i'' = \mathcal{N}_i$, $i = 1, 2, 3$.

In view of $\mathcal{L}_{\mathcal{N}_2} \neq \mathcal{L}_{\mathcal{N}_3}$ two cases can arise:

a) There exists $N_2 \in \mathcal{N}_2$ incomparable (in the inclusion) with any $N_3 \in \mathcal{N}_3$,

b) there exists $N_2 \in \mathcal{N}_2$, $N_3 \in \mathcal{N}_3$ with $N_2 \subsetneqq N_3$

(we omit the symmetrical two cases concerning $\mathcal{N}_2$ and $\mathcal{N}_3$)..

Ad a): For $N_2 \in \mathcal{N}_2'$ we have $N_2 \in \mathcal{N}_3' \subseteq \mathcal{N}_3$ - a contradiction. Hence $N_2 \in \mathcal{N}_2''$, i.e. there exists $M \in \mathcal{N}_1$, $N_2 \subseteq M$.

At first, $N_2 = M \cap N_2$, $M \in \mathcal{N}_1$, $N_2 \in \mathcal{N}_2$ implies $N_2 \notin \mathcal{L}_{\mathcal{N}_1} \vee \mathcal{L}_{\mathcal{N}_2}$ by Theorem 2. Secondly, $N_2 \subseteq M_1 \cap M_3$, $M_1 \in \mathcal{N}_1$, $M_3 \in \mathcal{N}_3$ implies $N_2 \subseteq M_3$, $M_3 \in \mathcal{N}_3$ - - a contradiction proving $N_2 \in \mathcal{L}_{\mathcal{N}_1} \vee \mathcal{L}_{\mathcal{N}_3}$. .

Ad b) : It is easy to see that $N_3 \in \mathcal{N}_3'$ gives $N_2 = N_3$ - a contradiction.

Hence $N_3 \in \mathcal{N}_3''$, i.e. there exists $M \in \mathcal{N}_1$ satisfying $N_3 \subseteq M$. For $N_3 \subseteq M_1 \cap M_2$, $M_1 \in \mathcal{N}_1$, $M_2 \in \mathcal{N}_2$ we have $N_2 \subsetneqq M_2$ - a contradiction. Hence $N_3 \in \mathcal{L}_{\mathcal{N}_1} \vee \mathcal{L}_{\mathcal{N}_2}$. Finally, $N_3 = M \cap N_3$, $M \in \mathcal{N}_1$ gives rise to $N_3 \notin \mathcal{L}_{\mathcal{N}_1} \vee \mathcal{L}_{\mathcal{N}_3}$ which completes the proof of Theorem 3,

<u>Theorem 4</u>: An element $\mathcal{L}_{\mathcal{N}}$ has a complement in $\mathcal{L}$ if and only if

a) $\mathcal{N}$ contains the maximal ideals only,

b) for any prime ideal P the set $\mathcal{M}_P$ of all ideals from $\mathfrak{M}$ containing P satisfies either $\mathfrak{M}_P \subseteq \mathfrak{N}$ or $\mathfrak{M}_P \cap \mathfrak{N} = \Phi$.

Proof: It is clear that the unit element of $\mathcal{L}$ is $\mathcal{E}_\Phi$ and the zero element is $\mathcal{E}_{\mathfrak{M}}$. Let us suppose that the conditions a) and b) are satisfied and let $\mathfrak{N}' = \mathfrak{M} \dot{-} \mathfrak{N}$. Then $\mathcal{E}_{\mathfrak{N}} \wedge \mathcal{E}_{\mathfrak{N}'} = \mathcal{E}_{\mathfrak{M}}$ by Theorem 2 and $\mathcal{E}_{\mathfrak{N}} \vee \mathcal{E}_{\mathfrak{N}'} = \mathcal{E}_\Phi$ by b) and Theorem 2.

Conversely, let $\mathcal{E}_{\mathfrak{N}}$ have a complement $\mathcal{E}_{\mathfrak{N}'}$ in $\mathcal{L}$. If $\mathfrak{N}$ contains an ideal N which is not in $\mathfrak{M}$, then there exists $M \in \mathfrak{M}$ with $N \subsetneqq M$. For $M \in \mathfrak{N}'$ we have $N \in \mathcal{E}_\Phi \dot{-} \mathcal{E}_{\mathfrak{N}} \vee \mathcal{E}_{\mathfrak{N}'}$ by Theorem 2 and for $M \notin \mathfrak{N}'$ we have $M \in \mathcal{E}_{\mathfrak{N}} \wedge \mathcal{E}_{\mathfrak{N}'} \dot{-} \mathcal{E}_{\mathfrak{M}}$ - a contradiction proving a). Finally, $\mathfrak{N}'$ must be a complement of $\mathfrak{N}$ in $\mathfrak{M}$ (intersection). If there exists $P \subseteq M \cap M'$, P prime, $M \in \mathfrak{N}$, $M' \in \mathfrak{N}'$, then $P \in \mathcal{E}_{\mathfrak{M}} \dot{-} \mathcal{E}_{\mathfrak{N}} \vee \mathcal{E}_{\mathfrak{N}'}$ - a contradiction proving b).

Theorem 5: The lattice $\mathcal{L}$ is complementary if and only if any prime ideal in Λ is maximal.

Proof: If $\mathcal{L}$ is complementary, then by a) Theorem 4 and Lemma 1 any prime ideal in Λ is maximal. The converse follows immediately from Theorem 4.

References

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