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ON THE CATEGORY OF FILTERS

Václav KOUBEK, Jan REITERMAN, Praha

In the present note the category of filters is studied. Denote by $\mathbb{F}_o$ the category, the objects pf which are ordered pairs $[A, \mathcal{F}]$, where A is a set and $\mathcal{F}$ a filter on A. The morphisms from $[A, \mathcal{F}]$ to $[B, \mathcal{G}]$ are all mappings $\alpha : A \rightarrow B$ with $\alpha^{-1}(G) \in \mathcal{F}$ for every $G \in \mathcal{G}$. Denote by $\mathbf{F}$ the category which we obtained from $\mathbf{F}_o$ by identifications of those mappings α, α' which are equal on a set $F \in \mathcal{F}$. Exact definition c.f. below. The note has four parts. The first contains the basic conventions and exact definition of the category $\mathbb{F}_*$. The second part contains the characterization of epimorphism and monomorphisms in $\mathbf{F}$. In the third part the concretizability of the category $\mathbf{F}$ is proved. The fourth part contains some examples of categories the concretizability of which follows immediately from the concretizability of the category $\mathbf{F}$.

1. Conventions from the set theory

If A, B are sets, f a mapping $f : A \rightarrow B$, and C a subset of A then f/C denotes the restriction of f to the domain C.

If A, B are sets and b_a is given for every $a \in A$, then the set of all b_a, $a \in A$ is denoted by $\{ b_a ; a \in A \}$; the mapping $a \to b_a$ is denoted by $\{ b_a \mid a \in A \}$.

Conventions from the category theory. If $\mathbb{K}$ is a category, then $\mathbb{K}^{\sigma}$ denotes the class of all its objects and $\mathbb{K}^{m}$ the class of all its morphisms. If $a, b \in \mathbb{K}^{\sigma}$ then $\mathbb{K}(a, b)$ denotes the set of all morphisms from a into b. If

$$a, b, c \in K^{\sigma}, \ f \in K(a,b), \ g \in K(b,c),$$

then the composition of f and g is denoted by $g \circ f$.

We recall the following definition: A category $\mathbb{K}$ is said to be concretizable if and only if there exists an isofunctor from $\mathbb{K}$ into $\mathbb{S}$, where $\mathbb{S}$ is the category of all sets and their mappings. It is well known that a category $\mathbb{K}$ is concretizable if and only if there exists a faithful functor from $\mathbb{K}$ into $\mathbb{S}$.

Definition of the category $\mathbb{F}$. Let $\mathcal{C}$ be the class of all ordered pairs $[A, \mathcal{F}]$, where A is a set and $\mathcal{F}$ is a filter on A. A triple $\langle \mathcal{F}, \mathcal{G}, \alpha \rangle$ will be called a morphism from $[A, \mathcal{F}]$ into $[B, \mathcal{G}]$ if and only if α is a mapping, $\alpha : A \to B$ such that

$$G \in \mathcal{G} \Rightarrow \alpha^{-1}(G) \in \mathcal{F} .$$

We define composition of two morphisms as follows:

$$\langle \mathcal{G}, \mathcal{H}, \beta \rangle \circ \langle \mathcal{F}, \mathcal{G}, \alpha \rangle = \langle \mathcal{F}, \mathcal{H}, \beta \circ \alpha \rangle .$$

Denote by $\mathbb{F}_o$ the category such that $\mathbb{F}_o^{\sigma} = \tilde{\mathcal{C}}$ and $\mathbb{F}_o^{m}$ is the class of all morphisms described above with the composition defined above. We define an equivalence on $\mathbb{F}$ as follows:

$$\langle \mathcal{F}_1, \mathcal{G}_1, \alpha_1 \rangle \sim \langle \mathcal{F}_2, \mathcal{G}_2, \alpha_2 \rangle \equiv (\mathcal{F}_1 = \mathcal{F}_2) \,\&$$

$$\& \,(\mathcal{G}_1 = \mathcal{G}_2) \,\&\, (\exists F \in \mathcal{F}_1)(\alpha_1 / F = \alpha_2 / F) \;.$$

It is easy to see that $\sim$ is a congruence on $\mathbb{F}_o^{m}$ and consequently it defines a factorcategory $\mathbb{F}$, morphisms of which are equivalence-classes of morphisms of $\mathbb{F}_o$ with respect to $\sim$. We shall denote the morphisms of the category $\mathbb{F}$ by $f, g, h \dots$.

We shall write $\alpha \in f$, whenever $\langle \mathcal{F}, \mathcal{G}, \alpha \rangle \in f$ and we shall say that the mapping α designates the morphism f .

2.

Lemma 1: A morphism $f \in \mathbb{F}([A, \mathcal{F}], [B, \mathcal{G}])$ is an epimorphism if and only if the following holds:

(1) $$(\forall \alpha \in f)(\forall F \in \mathcal{F})(\alpha(F) \in \mathcal{G}) \;.$$

Remark: The condition (1) is equivalent to the condition (1´):

(1´) $$(\exists \alpha \in f)(\forall F \in \mathcal{F})(\alpha(F) \in \mathcal{G}) \;.$$

Proof of the remark is evident.

Proof of Lemma 1: Let us assume that the condition holds and f is not an epimorphism, i.e.

$$(\exists [C, \mathcal{H}] \in \mathbb{F}^{\sigma})(\exists g, h \in \mathbb{F}([B, \mathcal{G}], [C, \mathcal{H}])(g \neq h, g \circ f = h \circ f).$$

The last equality implies

$(\forall \alpha \in f)(\forall \beta \in g)(\forall \gamma \in h)(\exists F \in \mathcal{F})(\beta \circ \alpha / F = \gamma \circ \alpha / F).$

It means that $\beta/\alpha(F) = \gamma/\alpha(F)$, consequently $h = g$ which is a contradiction.

Let us assume that the condition (1) does not hold. Then there exists $F \in \mathcal{F}$ such that $\alpha(F) \notin \mathcal{G}$. On the other hand the set $B - \alpha(F)$ is not a member of $\mathcal{G}$ because $(\alpha^{-1}(B - \alpha(F))) \cap F = \emptyset$.

Denote:

$$\mathcal{G}_1 = \{G \cap \alpha(F);\ G \in \mathcal{G}\},$$

$$\mathcal{G}_2 = \{G \cap (B - \alpha(F));\ G \in \mathcal{G}\}.$$

It is easy to see that $\mathcal{G}_1$ (or $\mathcal{G}_2$) is a filter on a set $\alpha(F)$ (or $\beta - \alpha(F)$ respectively). Let $C = C_1 \cup C_2 \cup C_3$, where C_i are disjoint sets such that

$$card\ C_1 = card\ \alpha(F),$$

$$card\ C_2 = card\ C_3 = card\ (B - \alpha(F)).$$

Let $\omega : \alpha(F) \to C_1$, $\pi_1 : (B - \alpha(F)) \to C_2$, $\pi_2 : (B - \alpha(F)) \to C_3$ be arbitrary bijective mappings.

Define the filter $\mathcal{H}$ on the set C as follows:

$$(X \in \mathcal{H}) \equiv (\omega^{-1}(X \cap C_1) \in \mathcal{G}_1 \ \&\ \pi_1^{-1}(X \cap C_2) \in \mathcal{G}_2 \ \&\ \pi_2^{-1}(X \cap C_3) \in \mathcal{G}_2).$$

The mappings $\varepsilon, \mu : B \to C$ defined by

$$\varepsilon/\alpha(F) = \mu/\alpha(F) = \omega,\quad \varepsilon/(B - \alpha(F)) = \pi_1,\ \mu/(B - \alpha(F)) = \pi_2$$

designate the morphisms g, h such that $g \neq h$,

$g \circ f = h \circ f$.

Consequently, f is not an epimorphism.

Lemma 2: A morphism $f \in \mathbb{F}([A,\mathcal{F}], [B,\mathcal{G}])$ is a monomorphism if and only if the following holds:

(2) $(\forall \alpha \in f)(\exists F \in \mathcal{F})(\forall x, y \in F)(x \neq y \Rightarrow \alpha(x) \neq \alpha(y))$.

Remark: The condition (2) is equivalent to the condition (2′):

(2′) $(\exists \alpha \in f)(\exists F \in \mathcal{F})(\forall x, y \in F)(x \neq y \Rightarrow \alpha(x) \neq \alpha(y))$.

Proof of the remark is evident.

Proof of Lemma 2: Clearly, if (2) is satisfied then f is a monomorphism. Let us assume that the condition (2) does not hold, i.e.

$$(\exists \alpha \in f)(\forall F \in \mathcal{F})(\exists a_F, b_F \in F)(a_F \neq b_F \,\&\, \alpha(a_F) = \alpha(b_F)).$$

Put $C = \{[a_F, b_F]; \ F \in \mathcal{F}\}$. Let $\mathcal{H}$ be a filter on the set C a base of which is the set of all $\{[a_F, b_F]; \ F \subset G\}$, where $G \in \mathcal{F}$.

The mappings $\varepsilon, \mu : C \to A$ defined by

$$\varepsilon([a_F, b_F]) = a_F, \quad \mu([a_F, b_F]) = b_F$$

designate the morphisms g, h of $[C, \mathcal{H}]$ into $[A, \mathcal{F}]$ such that

$$g \neq h, \ f \circ g = f \circ h.$$

Consequently, the morphism f is not a monomorphism.

3.

Definition: Denote by $\mathcal{U}$ the full subcategory of $\mathbb{F}$ the objects of which are all $[A, \mathcal{F}]$ where $\mathcal{F}$ is an ultrafilter.

Convention: Let $\mathcal{T}$ be the class of all cardinal numbers. For every $t \in \mathcal{T}$ choose a set X_t with $card\, X_t = t$. The sets X_t will be fixed in the sequel.

Definition: For every object $[A, \mathcal{F}] \in \mathbf{F}^{\sigma}$ put $\min_{F \in \mathcal{F}} card\, F = \| [A, \mathcal{F}] \|$. The number $\| [A, \mathcal{F}] \|$ will be called essential cardinality of the filter $\mathcal{F}$.

Lemma 3: There exists a skeleton $\mathcal{U}_1$ of $\mathcal{U}$ with the following property: if $[A, \mathcal{F}] \in \mathcal{U}_1^{\sigma}$ then $A = X_{\|[A,\mathcal{F}]\|}$

Proof is evident.

Lemma 4: The category $\mathcal{U}$ is concretizable.

Proof: It is sufficient to prove that $\mathcal{U}_1$ is concretizable.

1) First we prove that:

$$[X_t, \mathcal{F}], [X_u, \mathcal{G}] \in \mathcal{U}_1^{\sigma};\ t < u \Rightarrow \mathcal{U}_1([X_t, \mathcal{F}], [X_u, \mathcal{G}]) = \emptyset .$$

Assume that there exist $f \in \mathcal{U}_1([X_t, \mathcal{F}], [X_u, \mathcal{G}])$. If $\alpha \in f$, $F \in \mathcal{F}$, then $\alpha(F) \in \mathcal{G}$. For, $\mathcal{G}$ is an ultrafilter and $\alpha^{-1}(X_u - \alpha(F)) \cap F = \emptyset$. Thus, $card\, \alpha(F) = u$ while $card\, F = t < u$. That is a contradiction.

2) Consequently,

$$\bigcup_{b \in \mathcal{U}_1^{\sigma}} \mathcal{U}_1(a, b) = \bigcup_{b \in \mathcal{U}_1^{\sigma}, \|b\| \leq \|a\|} \mathcal{U}_1(a, b) .$$

The right side hand is evidently a set, which implies that $\mathcal{U}_1$ is concretizable because we can use the

Mac-Lane's representation for the category $\mathcal{U}_1^*$ dual to $\mathcal{U}_1$.

<u>Definition:</u> Let $\mathbb{K}$ be arbitrary category. Define the category $H^{\mathbb{K}}$ as follows. The object of the category $H^{\mathbb{K}}$ are all sets of objects of the category $\mathbb{K}$. Let a, b be the objects of the category $H^{\mathbb{K}}$. Morphisms from a to b are exactly all collections $\{f_m \mid m \in a\}$ where $f_m \in \mathbb{K}(m, N_m)$, $N_m \in b$. We define the composition:

$$\{g_n \mid n \in b\} \circ \{f_m \mid m \in a\} = \{g_{N_m} \circ f_m \mid m \in a\} .$$

<u>Remark:</u> It is evident that $H^{\mathbb{K}}$ is a category.

<u>Lemma 5</u>: If the category $\mathbb{K}$ is concretizable then the category $H^{\mathbb{K}}$ is concretizable.

Proof is evident.

<u>Theorem</u>: The category $\mathbb{F}$ is concretizable.

<u>Proof:</u> 1) The category $H^{\mathcal{U}}$ is concretizable. 2) Now we shall construct a functor $\Psi : \mathbb{F} \longrightarrow H^{\mathcal{U}}$. For every $[A, \mathcal{F}] \in \mathbb{F}^{\sigma}$ define $\Psi[A, \mathcal{F}]$ as the set of all $[A, \mathcal{H}]$, where $\mathcal{H}$ is an ultrafilter on A and $\mathcal{F} \subset \mathcal{H}$ (i.e. $F \in \mathcal{F} \Rightarrow F \in \mathcal{H}$). If

$$f \in \mathbb{F}([A, \mathcal{F}], [B, \mathcal{G}]), \alpha \in f, [A, \mathcal{H}] \in \Psi[A, \mathcal{F}] ,$$

then the set $\{\alpha(H) ; H \in \mathcal{H}\}$ is a base of an ultrafilter on B which will be called $f(\mathcal{H})$. (The ultrafilter $f(\mathcal{H})$ does not depend on a choice of $\alpha \in f$.) Define:

$$\Psi(f) = \{f_{[A, \mathcal{H}]} \mid [A, \mathcal{H}] \in \Psi[A, \mathcal{F}]\} ,$$

where $f_{[A,\mathcal{H}]} \in \mathcal{U}([A,\mathcal{H}], [B, f(\mathcal{H})])$ such that $\alpha \in f_{[A,\mathcal{H}]}$ whenever α is a mapping $\alpha : A \to B$ with $\alpha \in f$.

3) Now we prove that Ψ is an isofunctor from $\mathbb{F}$ into $H^{\mathcal{U}}$. The mapping $\Psi / \mathbb{F}^{\sigma}$ is one-to-one because

$$\mathcal{F} = \bigcap_{[A,\mathcal{H}] \in \Psi[A,\mathcal{F}]} \mathcal{H}$$

for each filter $\mathcal{F}$ on A. We shall prove that for each $a, b \in \mathbb{F}^{\sigma}$, $a = [A, \mathcal{F}]$, $b = [B, \mathcal{G}]$, $\Psi/_{\mathbb{F}(a,b)}$ is one-to-one. Let f, g be two morphisms from a to b, $f \neq g$. Choose $\alpha \in f$, $\beta \in g$ and set

$$C = \{ x \in A ; \alpha(x) \neq \beta(x) \}.$$

Since $f \neq g$, $C \cap F \neq \emptyset$ holds for each $F \in \mathcal{F}$. Consequently, $\{C \cap F ; F \in \mathcal{F}\}$ is a base of a filter $\mathcal{O}\!\mathcal{L}$ on A. Let $\mathcal{H}$ be an ultrafilter on A with $\mathcal{H} \supset \supset \mathcal{O}\!\mathcal{L}$. Since $\mathcal{O}\!\mathcal{L} \supset \mathcal{F}$, $\mathcal{H} \in \Psi[A, \mathcal{F}]$, it is easy to see that $H \cap C \neq \emptyset$ for every $H \in \mathcal{H}$. Therefore $f_{[A,\mathcal{H}]} \neq g_{[A,\mathcal{H}]}$, consequently $\Psi(f) \neq \Psi(g)$.

4) The assertion of the theorem follows now immediately from 3) and 1).

4. Some examples

1) We recall: a directed set is an ordered pair $[A, \kappa]$, where A is a set and κ a partial order on A such that

$$(\forall a \in A)(\forall b \in A)(\exists c \in A)(a \varkappa c \ \& \ b \varkappa c) .$$

Let $[A_1, \varkappa_1], [A_2, \varkappa_2]$ be two directed sets. A triple $\langle \varkappa_1, \varkappa_2, \alpha \rangle$ will be called a morphism from $[A_1, \varkappa_1]$ into $[A_2, \varkappa_2]$ if and only if α is a $\varkappa_1 - \varkappa_2$ compatible mapping, $\alpha : A_1 \rightarrow A_2$, i.e. α is a mapping from A_1 into A_2 such that

$$a, b \in A_1 , \ a \varkappa_1 b \Longrightarrow \alpha(a) \varkappa_2 \alpha(b) .$$

We define the composition of two morphisms as follows:

$$\langle \varkappa_2, \varkappa_3, \beta \rangle \circ \langle \varkappa_1, \varkappa_2, \alpha \rangle = \langle \varkappa_1, \varkappa_3, \beta \circ \alpha \rangle .$$

It is clear that directed sets as objects with morphisms just described form a category. Denote this category by $\mathbb{R}_o$. Denote by $\mathbb{R}$ the factorcategory of $\mathbb{R}_o$ with respect to the congruence $\sim$ where $\sim$ is defined as follows:

$$\langle \varkappa_1, \varkappa_2, \alpha_i \rangle \in \mathbb{R}_o([A_1, \varkappa_1], [A_2, \varkappa_2]), \quad i = 1, 2 ,$$

$$\langle \varkappa_1, \varkappa_2, \alpha_1 \rangle \sim \langle \varkappa_1, \varkappa_2, \alpha_2 \rangle \equiv$$

$$\equiv (\exists x \in A_1)(\forall y \in A_1)(x \varkappa_1 y \Longrightarrow \alpha_1(y) = \alpha_2(y)) .$$

2) Denote by P the class of all triples $[t, T, \mathcal{T}]$ where $[T, \mathcal{T}]$ is a topological space and $t \in T$. A continuous mapping f from $[T, \mathcal{T}]$ into $[S, \mathcal{S}]$ will be called a morphism from $[t, T, \mathcal{T}]$ into $[s, S, \mathcal{S}]$ if and only if $f(t) = s$. The composition of morphisms

is the usual composition of mappings. Clearly, elements of $\mathbf{P}$ as objects and morphisms just described form a category. Denote by $\mathbf{T}_o$ this category. Denote by $\mathbf{T}$ the factorcategory of $\mathbf{T}_o$ with respect to the congruence $\sim$, where $\sim$ is defined as follows:

$$\alpha, \beta \in \mathbf{T}_o([t, T, \mathcal{T}], [s, S, \mathcal{S}]) \ ,$$

$$\alpha \sim \beta \equiv (\exists U \in \mathcal{U}_t^{\mathcal{T}})(\alpha / U = \beta / U) \ .$$

($\mathcal{U}_t^{\mathcal{T}}$ denote the system of all neighborhoods of the point t in the topology $\mathcal{T}$.)

3) Let $\mathcal{Q}$ be the class of all ordered pairs $[M, \mu]$, where M is a set and μ a non-trivial measure on M . If $[M, \mu] \in \mathcal{Q}$, let us denote by $\mathcal{D}\mu$ (or $\mathcal{D}_o \mu$) the system of all μ-measurable sets (or the system of all $N \subset M$ such that $\mu(N) = 0$, respectively). A mapping $\alpha : M_1 \longrightarrow M_2$ will be called a morphism from $[M_1, \mu_1]$ into $[M_2, \mu_2]$ if and only if

$$(N \in \mathcal{D}\mu_2 \Longrightarrow \alpha^{-1}(N) \in \mathcal{D}\mu_1) \& (N \in \mathcal{D}_o \mu_2 \Longrightarrow \alpha^{-1}(N) \in \mathcal{D}_o \mu_1).$$

The composition of morphisms is the usual composition of mappings. It is easy to see that elements of $\mathcal{Q}$ and morphisms just described form a category. Denote this category by $\mathbb{M}_o$. Denote by $\mathbb{M}$ the functorcategory of $\mathbb{M}_o$ with respect to congruence $\sim$, where $\sim$ is defined as follows:

$$[M, \mu], [N, \nu] \in \mathcal{Q}, \quad \alpha, \beta \in \mathbb{M}_o([M, \mu], [N, \nu]),$$

$(\alpha \sim \beta) \equiv (\alpha = \beta \quad \mu\text{-almost everywhere})$.

<u>Proposition</u>: The categories $\mathbb{R}$, $\mathbb{T}$, $\mathbb{M}$ are concretizable. It follows almost immediately from the fact that the category $\mathbb{F}$ is concretizable. The categories $\mathbb{R}$, $\mathbb{T}$, $\mathbb{M}$ can be represented as subcategories of the category $\mathbb{F}$.

Matematicko-fyzikální fakulta
Karlovy university
Praha 8 Karlín
Sokolovská 83, Czechoslovakia

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