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ON NONPLANAR GRAPHS WITH THE MINIMUM NUMBER OF VERTICES AND A GIVEN GIRTH

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By the girth of a graph G we mean according to H.-J. Voss [2] the length of the shortest circuit included in the graph G. According to the well known theorem of G. Kuratowski [1] an arbitrary graph is nonplanar if and only if it includes a subgraph which is homeomorphic with the complete graph K_5 (Fig.1) or the regular bicomplete graph $K_{3,3}$ (Fig. 2).

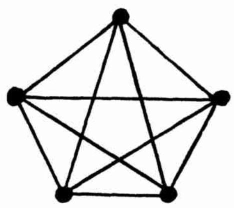

Fig. 1

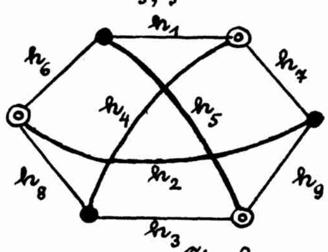

Fig. 2

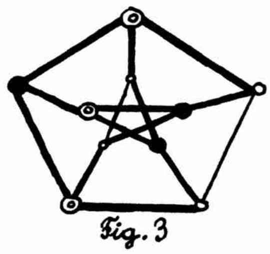

Fig. 3

For example the so called Petersen graph P (Fig.3) which is not a planar graph contains a subgraph homeomorphic with the graph $K_{3,3}$ (in Fig.3 the edges of this subgraph are denoted by thick lines).

The graph K_5 is a nonplanar graph with a girth $t(K_5) = 3$; the graph $K_{3,3}$ is a nonplanar graph with a girth $t(K_{3,3}) = 4$. Petersen's graph P is a nonplanar graph with a girth $t(P) = 5$.

Now the natural question arises: Which is the minimum number v_m $(m \geqq 4)$ of vertices of nonplanar graphs G which have a girth $t(G) = m$.

The answer is given in

<u>Theorem 1</u>. The minimum number v_m $(m \geqq 4)$ of vertices of all nonplanar graphs G which have a girth $t(G) = m$ is equal to

$$v_m = \left[\frac{9(m-1)}{4}\right] + d_m , \qquad (m \geqq 4)$$

where

$$d_m = 0 \quad \text{if} \quad m \not\equiv 3 \pmod 4 ;$$

$$d_m = 1 \quad \text{if} \quad m \equiv 3 \pmod 4 .$$

<u>Proof</u>: a) First we shall show that

$$v_m \leqq w_m = \left[\frac{9(m-1)}{4}\right] + d_m .$$

Therefore we shall construct a nonplanar graph G_m of the given girth m which has exactly w_m vertices. The number w_m can be expressed in the form

$$w_n = 6 + 9\left[\frac{n-4}{4}\right] + r_n$$

where

$$r_n = 0 \quad \text{if} \quad n \equiv 0 \pmod 4;$$

$$r_n = 3 \quad \text{if} \quad n \equiv 1 \pmod 4;$$

$$r_n = 5 \quad \text{if} \quad n \equiv 2 \pmod 4;$$

$$r_n = 8 \quad \text{if} \quad n \equiv 3 \pmod 4.$$

Now let us construct the graph $K_{3,3}$ (Fig.2). On each of the edges h_i, where $i = 1, 2, \ldots, r_n$ we choose $\left[\frac{n}{4}\right]$ new vertices. On each of the remaining edges h_j ($j = r_n + 1, r_n + 2, \ldots, 9$) let us choose $\left[\frac{n-4}{4}\right]$ new vertices. In this way we obtain the graph G_n which has w_n vertices. The graph $K_{3,3}$ contains only quadrangles and hexagons. The quadrangles of the graph $K_{3,3}$ turn into polygons with at least n vertices in the graph G_n (see Table 1). From the hexagons of graph $K_{3,3}$ circuits of a shorter length than $6\left[\frac{n}{4}\right]$ cannot develop in graph G_n. Which is always at least n for $n \neq 7$, $n \geqq 4$. If $n = 7$, then every circuit of the graph G_n which develops from the hexagon of graph $K_{3,3}$ has the length of at least 11. Besides the circuits which have developed from quadrangles and hexagons in the graph $K_{3,3}$ there are no other circuits in the graph G_n.

So the inequality $v_n \leqq w_n$ is proved.

b) We shall prove the equation $w_n = v_n$. We can apparently suppose that the nonplanar graph G_n^* with a girth n which has the minimum number of vertices v_n is itself homeomorphic with the graph K_5 or $K_{3,3}$.

Table of lengths of circuits in the graph G_n which are induced by the quadrangles of the graph $K_{3,3}$.				
Quadrangles induced by edges	$n \equiv 0 \bmod 4$	$n \equiv 1 \bmod 4$	$n \equiv 2 \bmod 4$	$n \equiv 3 \bmod 4$
$h_1\, h_7\, h_2\, h_6$	n	$n+1$	n	$n+1$
$h_2\, h_9\, h_3\, h_8$	n	$n+1$	n	n
$h_1\, h_4\, h_8\, h_6$	n	n	n	$n+1$
$h_3\, h_4\, h_7\, h_9$	n	n	n	n
$h_1\, h_7\, h_9\, h_5$	n	n	n	n
$h_1\, h_8\, h_6\, h_5$	n	n	n	$n+1$
$h_1\, h_4\, h_3\, h_5$	n	$n+1$	$n+2$	$n+1$
$h_2\, h_7\, h_4\, h_8$	n	n	n	$n+1$
$h_2\, h_6\, h_5\, h_9$	n	n	n	n

Table 1

Let us first suppose that the graph G_n^* is homeomorphic with the graph K_5. Therefore we can construct the graph G_n^* from the graph K_5 so that we choose $v_n - 5$ new vertices on its edges. Then on every triangle of the graph K_5 we must choose at least $n - 3$ new vertices. In the graph K_5 there are, on the whole, 10 different triangles, while every edge belongs to three triangles. So the graph G_m^* develops from the graph K_5 by adding at least

$$\left[\frac{10(m-3)+2}{3}\right]$$

vertices. Therefore

$$\left[\frac{10(m-3)+2}{3}\right] \leqq v_m - 5 \leqq w_m - 5 = 1 + 9\left[\frac{m-4}{4}\right] + r_m .$$

Because the inequality

$$\left[\frac{10(m-3)+2}{3}\right] \leqq 1 + 9\left[\frac{m-4}{4}\right] + r_m$$

has no solution for $m \geqq 4$, it is therefore proved that the graph G_m^* cannot be homeomorphic with the graph K_5.

So the graph G_m^* is homeomorphic with the graph $K_{3,3}$. In other words it develops from the graph $K_{3,3}$ so that we choose $v_n - 6$ new vertices suitably on its edges. Simultaneously we must choose at least $m - 4$ new vertices on each quadrangle of the graph $K_{3,3}$. In the graph $K_{3,3}$ there are, on the whole, 9 different quadrangles, while each edge belongs to four quadrangles. The graph G_m^* therefore develops from the graph $K_{3,3}$ by adding at least

$$\left[\frac{9(m-4)+3}{4}\right]$$

vertices. Therefore

$$\left[\frac{9(m-4)+3}{4}\right] \leqq v_m - 6 \leqq w_m - 6 = 9\left[\frac{m-4}{4}\right] + r_m$$

holds. It is easy to find out that

$$9\left[\frac{n-4}{4}\right] + \kappa_n - \left[\frac{9(n-4)+3}{4}\right] = d_n \ .$$

Hence for $n \not\equiv 3 \pmod 4$ it follows that $v_n = w_n$ and for $n \equiv 3 \pmod 4$ it follows that either $v_n = w_n$ or $v_n = w_n - 1$. We shall show that $v_n \neq w_n - 1$ holds even for $n \equiv 3 \pmod 4$. Let us, on the contrary, suppose that $v_n = w_n - 1$. The edges of the graph $K_{3,3}$ which contains less than $\left[\frac{n}{4}\right]$ new vertices (i.e. vertices which must be added to the edges of graph $K_{3,3}$ for it to become graph G_n^*), induces in $K_{3,3}$ a subgraph Q which has at least two edges and does not contain a quadrangle. For should the graph Q contain a quadrangle F, then in the graph G_n^* there would exist a circuit of the length $n-3$, and that is a contradiction. It is easy to find out that the subgraph Q must be isomorphic with some subgraph which is induced by these sets of edges of the graph $K_{3,3}$ (see Fig.2):

$E_1 = \{h_1, h_2\}$, $\qquad E_5 = \{h_1, h_7, h_8\}$,

$E_2 = \{h_1, h_6\}$, $\qquad E_6 = \{h_1, h_2, h_3\}$,

$E_3 = \{h_1, h_6, h_7\}$, $\qquad E_7 = \{h_1, h_6, h_7, h_8\}$,

$E_4 = \{h_1, h_4, h_7\}$, $\qquad E_8 = \{h_1, h_4, h_6, h_7\}$,

$E_9 = \{h_1, h_3, h_6, h_7\}$, $\quad E_{12} = \{h_1, h_4, h_6, h_7, h_9\}$,

$E_{10} = \{h_1, h_3, h_6, h_9\}$, $\quad E_{13} = \{h_1, h_4, h_5, h_6, h_7\}$,

$E_{11} = \{h_1, h_3, h_6, h_7, h_9\}$, $\quad E_{14} = \{h_1, h_3, h_6, h_7, h_8, h_9\}$.

Let us denote by z_i $(i = 1, 2, \dots, 9)$ the number of new vertices which we must choose on the edge h_i of the graph $K_{3,3}$ if we want to obtain the graph G_n^*. Let us further denote

$$x_i = z_i - \left[\frac{n-4}{4}\right], \text{ if } z_i \notin Q,$$

$$y_i = z_i - \frac{n-4}{4}, \text{ if } z_i \in Q.$$

Obviously for all permissible i

(N) $$x_i > 0, \quad y_i \leq 0$$

holds. Further

(R) $$\sum_{z_i \notin Q} x_i + \sum_{z_i \in Q} y_i = 7$$

holds. Because on the edge of every quadrangle F of the graph $K_{3,3}$ there are at least $n - 4$ new vertices, the inequality

(F) $$\sum_{\substack{z_i \notin Q \\ z_i \in F}} x_i + \sum_{\substack{z_i \in Q \\ z_i \in F}} y_i \geq 3 .$$

also holds. If the quadrangle F is induced by the edges h_r, h_s, h_t, h_u then we shall further denote the inequality (F) shortly by $(rstu)$.

Now we shall show that all 14 possibilities for the graph Q lead to a contradiction.

1) Let the graph Q be induced by one of the sets of edges E_1, E_2, E_3, E_4, E_5, E_7, E_8. Then from the inequality (R), inequalities (N) and inequalities (1267),(1468),(2478) we obtain contradictory inequalities

$$6 \leqq 2x_3 + 2x_5 + 2x_9 \leqq 5 .$$

2) Let the graph Q be induced by the set of edges E_6. Then from the equality (R), inequalities (N) and inequalities (1267),(1345),(2389) we get the contradictory inequalities

$$6 \leqq x_4 + x_5 + x_6 + x_7 + x_8 + x_9 \leqq 5 .$$

3) Let the graph Q be induced by one of the sets E_9, E_{11}, E_{12}, E_{13}, E_{14}. Then from the equality (R), inequalities (N) and inequalities (1468),(1579),(3479),(3568) we get

$$2 \leqq 2x_2 \leqq 2$$

or $x_2 = 1$. Simultaneously the inequality (1267) must hold, i.e. the inequality

$$y_1 + x_2 + y_6 + y_7 \geqq 3 .$$

It is, however, with respect to the equality $x_2 = 1$, in contradiction with the inequalities (N).

4) Finally let the graph Q be induced by the set of edges E_{10} . Then from the equality (R), inequalities (N) and inequalities (1267),(1468),(1579),(2389),(3479), (3569) we get

$$3 \leqq x_2 + x_4 + x_5 \leqq 3$$

so that $x_2 = x_4 = x_5 = 1$. Simultaneously the inequality (1345) must hold, i.e. the inequality

$$y_1 + y_3 + x_4 + x_5 \geq 3 .$$

But that is, with regard to the equalities $x_4 = x_5 = 1$, in contradiction to the inequalities (N).
So the possibility $v_m = w_m - 1$ is excluded even for the case $m \equiv 3 \pmod 4$. So the whole theorem is proved.

From Theorem 1 the following simple result follows:

<u>Result</u>. If G is an arbitrary graph which has less than v_m vertices and has a girth m , then this graph is a planar one.

References

[1] KURATOWSKI G.: Sur le problème des courbes gauches en topologie, Fund.Math.15-16(1930),271-283.

[2] VOSS H.-J.: Some properties of graphs containing k independent circuits, Theory of Graphs, Proc.Colloq.Tihany,Hungary,1966,Akadémiai Kiadó,Budapest(1968),321-332.

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