Commentationes Mathematicae Universitatis Carolinae

11,4 (1970)

ON GENERALIZED LAMBERT SUMMABILITY

Patrick CASSENS and Francis REGAN

Oswego N.Y. - St.Louis,Mo.

Summability conditions on the sequence $\{a_n\}$ from the series

(1) $$F(x) = \sum_{n=1}^{\infty} a_n b_n \frac{x^n}{1 - b_n x^n}$$

where $\{a_n\}$ and $\{b_n\}$ are complex sequences with 1 the least upper bound of $\{ |b_n|^{1/n} \}$ have been studied in [3] and [6]. This paper will prove an extension of Hardy's theorem ([6],pp.194-196) by showing that the (C, p) summability of $\sum a_n$ implies summability of $\sum a_n$ in a generalized Lambert method. The series (1) is a generalized Lambert series; and we say $F(x)$ is represented by the F-series [4].

1. This section establishes the notation which will be employed throughout the paper.

Let n be a natural number, e and f be non-negative integers; and for a fixed natural number k define for all integers m such that $0 \le m \le k-1$,

$$s_n^m = \sum_{j=0}^{n} a_{kj-m}$$

and

$$G_m(e,f;x,y) = n^e x_m^f y^{fn}/(1 - x_m y^n)^{1+e} .$$

Let a sequence $\{a_i\}$ be given. For any real value of p, $S_n^p(a_i)$ denotes the Cesàro sum of order p and A_i^p the binomial coefficient of order p .

For q a non-negative integer and $g(x)$ differentiable at least q times, define $H^q g(x) = (x\, d/dx)^q g(x)$, where the operator $(x\, d/dx)^q$ is defined so that $(x\, d/dx)^0 g(x) = g(x)$, $(x\, d/dx)^1 g(x) = x \cdot d\,g(x)/dx$ and for $q \geq 1$, $(x\, d/dx)^q = x \cdot d(H^{q-1} g(x))/dx$.

The differences of order p, Δ^p belonging to a given sequence $\{a_n\}$ are given by $\Delta^1 a_n = a_n - a_{n+1}$ and for $p \geq 2$, $\Delta^p a_n = \Delta^1(\Delta^{p-1} a_n)$.

<u>Remark</u>. For these differences the following hold

(i) $\Delta^p a_m = \Sigma_{j=0}^{p} A_j^{-p-1} a_{m+j} = \Sigma_{j=0}^{p} (-1)^p C_{p,j}\, a_{m+j}$

and

(ii) $\Delta^p(a_m b_m) = \Sigma_{q=0}^{p} C_{p,q}\, \Delta^q a_m\, \Delta^{p-q} b_{p+q}$.

2. Formulation of results. We state here the main theorems which are to be proved.

<u>Theorem 1</u>. Let $F(x)$ be represented by the F -series; let k be a positive integer for which there exist k non-negative integers $p_0, p_1, \ldots, p_{k-1}$ such that

1. $\Sigma_{j=1}^{\infty} a_{k_j}/kj = s \quad (C, p_0)$;

2. for $t = 1, 2, \ldots, k-1$, $S_m^{p_t}(s_m^t) = o(m^{p_t})$.

Then, if either $b_{k_j} = 1$ for every $j \leq N$ and suitable N or $b_{k_j} \neq 1$ for every $j \leq N$ and suitable N, z^* is any primitive k-th root of 1 and

$b_n = o(n^{-p-q+3})$,

$$\lim_{r \to 1^-} (1-r)^{1+q} H^q F(r x^*) = s \cdot J \cdot q! ,$$

where $J = 1$ in the former case and $J = 0$ in the latter case.

Theorem 2. Let

1. $\sum a_j = s \quad (C, p)$ for some non-negative integer p,

2. $\{J_n(x)\}$ be a sequence of functions of the real variable x on the interval $(0, h)$ for positive h such that

(i) $\lim_{x \to 0+} J_n(x) = c$, c a constant, for every $n = 1, 2, \ldots, N$, for suitable N,

(ii) for all x in $(0, h)$, $\lim_{n \to \infty} n^p J_n(x) = 0$,

(iii) for all x in $(0, h)$, there exists a K independent of x such that

$$\sum n^p |\Delta^{p+1} J_n(x)| < K .$$

Then $\sum a_n J_n(x)$ converges for all x in $(0, h)$ and

$$\lim_{x \to 0+} \sum a_n J_n(x) = s \cdot J_0(0+) .$$

Comments. Theorem 1 is an extension of Hardy's theorem that (C, p) summability of $\sum a_n$ implies Lambert summability of $\sum a_n$ to the same sum. This is the restricted case $q = 0$, $b_n = 1$ for all n and p_t is a fixed natural number for $t = 0, 1, \ldots, k-1$. The case where all b_n are 1 has been considered previously [3]. Theorem 2 is an extension of Bromwich's theorem ([1],p.358) and is useful in establishing Theorem 1.

2. Proof of Theorem 2. To simplify notation let b_n denote $J_n(x)$ for x in $(0, h)$.

From the properties of Cesaro sums of order p we obtain

(2) $\sum_{j=0}^{n} S_j^{p-1}(a_m)\Delta^p b_j = \sum_{j=0}^{n} S_j^p(a_m)\Delta^{p+1} b_j + S_n^p(a_m)\Delta^p b_{n+1}$

and

(3) $\sum_{j=0}^{n} a_j b_j = \sum_{j=0}^{n} S_j^0(a_m)\Delta^1 b_j + S_n^0(a_m) b_{n+1}$.

If p is a positive integre, iteration of (2) with (3) establishes

$$\sum_{j=0}^{n} a_j b_j = \sum_{j=0}^{n} S_j^p(a_m)\Delta^{p+1} b_j + R_n$$

where

$$R_n = \sum_{j=0}^{p} S_n^j(a_m)\Delta^j b_{n+1} \; .$$

From Remark (i)

$$R_n = \sum_{j=0}^{p} \sum_{i=0}^{j} (-1)^i C_{j,i}\, S_n^j(a_m) b_{n+i+1} \; .$$

Since $S_n^j(a_m) = O(n^j)$, $|S_n^j(a_m)| < K n^j$ for some positive K , and

$$|R_n| \leq \sum_{j=0}^{p} \sum_{i=0}^{j} K \cdot C_{j,i}\, n^j\, |b_{n+i+1}|$$

$$\leq K \sum_{j=0}^{p} \sum_{i=0}^{p} C_{j,i}\,(n+i+1)^j |B_{n+i+1}| \; .$$

But, $(n+i+1)^j\,|b_{n+i+1}| \to 0$ as $n \to \infty$ by hypothesis and $C_{j,i} \leq C_{p,i}$ for $0 \leq j \leq p$. Therefore, $R_n = o(1)$.

Consequently, from Remark (ii),

(4) $\sum_{j=0}^{n} a_j b_j = \sum_{j=0}^{n} S_j^p(a_m)\Delta^{p+1} b_j + o(1)$.

Thus

(5) $$\sum a_j b_j$$

and

(6) $$\sum S_j^p(a_m)\Delta^{p+1} b_j$$

converge and diverge together.

Since $S_j^p(a_m)$ is $o(j^p)$,

$$\Sigma_{j=0}^{n} | S_j^p(a_m) \Delta^{p+1} b_j | \leq K \Sigma_{j=0}^{n} j^p | \Delta^{p+1} b_j |$$

for some positive K . But,

$$\Sigma_{j=0}^{\infty} j^p | \Delta^{p+1} b_j |$$

is convergent by hypothesis. Thus, series (6) converges. This fact implies series (5) converges, and the first part of the theorem is established.

Since the validity of (4) depends only on conditions 1 and 2(ii) of Theorem 2, the following lemma is easily obtainable from (4) and the comment that in the special case $a_0 = 1$, $a_n = 0$ for $n \geq 1$ we have for integral $p \geq -1$ that for all n , $S_n^p(a_m) = A_n^p = C_{n+p,n}$, $\Sigma a_m = 1$ (C, p) and $\Sigma a_m b_m = b_0$.

<u>Lemma.</u> Let $\{ J_n(x) \}$ satisfy condition 2(ii) of Theorem 2. Then

$$\Sigma_{j=0}^{\infty} C_{j+p,j} \Delta^{p+1} J_n(x) = J_0(x) .$$

Combining $\Sigma a_m b_m = b_0$, statement (4) and the facts that series (5) and (6) are convergent we find that

$$(7) \qquad \Sigma a_n b_n - s \cdot b_0 = \Sigma [S_n^p(a_j) - s \cdot C_{n+p,n}] \Delta^{p+1} b_n .$$

But, if s is the Cesàro limit in condition 1 of this theorem,

$$\lim_{n \to \infty} [S_n^p(a_j) / C_{n+p,n}] = s$$

and

$$\lim_{n \to \infty} [C_{n+p,n} / n^p] = 1/p! ,$$

Thus, for every $e > 0$, there exists an integer N_e such that for $n > N_e$,

$$|S_n^p(a_j) - s \cdot C_{n+p,n}| < e \cdot n^p \;;$$

and there exists a constant K such that for all n

$$|S_n^p(a_j)| < K \cdot n^p \text{ and } |s| C_{n+p,n} < K \cdot n^p \;.$$

Hence,

$$(8) \quad |\Sigma a_n b_n - s \cdot b_0| \leq \Sigma [|S_n^p(a_j) - s \cdot C_{n+p,n}||\Delta^{p+1} b_n|]$$

$$< 2K \Sigma_{n=0}^{N_e} (n+1)^p |\Delta^{p+1} b_n|$$

$$+ e \Sigma_{m=N_e+1}^{\infty} n^p |\Delta^{p+1} b_n| \;.$$

But,

$$\Sigma_{j=0}^{p+1} (-1)^j b_{n+j} \cdot C_{p+1,j} = \Delta^{p+1} b_n \;;$$

and for all $n \leq N$ for N suitably large, $\lim_{x \to 0+} b_n = c$ by hypothesis. Since $\Sigma_{j=0}^{p+1} (-1)^j c \cdot C_{p+1,j} = 0$, for every $n \leq N$ and N suitably large, we have

$\lim_{x \to 0+} \Delta^{p+1} b_n = 0 .$

Consequently,

$$\limsup_{x \to 0+} |\Sigma a_n b_n - s \cdot b_0| \leq e \cdot K \;,$$

which implies

$$\lim_{x \to 0+} [\Sigma a_n b_n - s \cdot b_0] = 0 \;.$$

The conclusion of Theorem 2 follows.

3. Before proving Theorem 1 we note the following three theorems.

Theorem A. ([5],p.100) If $r \geq p > -1$ and $\Sigma a_n = s \;\; (C, p)$, then $\Sigma a_n = s \;\; (C, r)$.

Theorem B. [2] Let $p > -1$ and $q > 0$.

1. If $S_n^p(a_j) = o(n^d)$ for some positive d, then $S_n^{p+q}(a_j) = o(n^{d+q})$ and $S_n^{p-q}(a_j) = o(n^d)$.

2. If $S_n^p(b_j^m) = o(n^d)$ for some positive d, then $a_{kn-m} = o((kn-m)^d)$ where for a fixed positive integer k we define, for all integers m such that $0 \leq m \leq k-1$, $b_j^m = \Sigma_{i=1}^j a_{ki-m}$.

Theorem C. [2] Let $F(x)$ be defined by the F-series; let k be a positive integer such that there exists a positive d for which $S_n^p(b_j^m) = o(n^d)$, for some $p > -1$ and all integers m satisfying $0 \leq m \leq k-1$. Let q be a positive integer, $M(q,1) = M(q,q) = 1$ and for $q > 2$ and $2 \leq w \leq q-1$,

$$M(q,w) = wM(q-1,w) + (q-w+1)M(q-1,w-1).$$

Then for $|x| < 1$,

$$H^q F(x) = \Sigma_{w=1}^q M(q,w) \Sigma_{m=0}^{k-1} \Sigma_{j=1}^{\infty} a_{kj-m} G_{kj-m}(q,w;b;x).$$

4. Proof of Theorem 1. Let q be a positive integer; let p be an integer and $p \geq p_t$ for all integers t satisfying $0 \leq t \leq k-1$; then, from Theorems A and B we may replace each p_t by p. From Theorem C

$$H^q F(x) = \Sigma_{w=1}^q M(q,w) \Sigma_{m=0}^{k-1} \Sigma_{j=1}^{\infty} a_{kj-m} G_{kj-m}(q,w;b;x).$$

Since $\Sigma_{w=1}^q M(q,w) = q!$ for all integers $q \geq 1$ ([3],p.431), this theorem will be demonstrated if the following are true:

(a) For any integer w satisfying $1 \leq w \leq q$,

$$\lim_{r \to 1^-} (1-r)^{1+q} \sum_{j=1}^{\infty} a_{kj} G_{kj}(q, w; b, rx^*) = s \cdot J ;$$

(b) For all integers w and t satisfying $1 \leq w \leq q$ and $1 \leq t \leq k-1$,

$$\lim_{r \to 1^-} (1-r)^{1+q} \sum_{j=1}^{\infty} a_{kj-t} G_{kj-t}(q, w; b, rx^*) = 0 .$$

Proof of (a). For integral w such that $1 \leq w \leq q$

$$\sum_{j=1}^{\infty} a_{kj} G_{kj}(q, w; b, rx^*) = \sum_{j=1}^{\infty} (kj) c_j G_{kj}(q, w; b, r)$$

for $c_j = a_{kj}/(kj)$ since x^* is a primitive k-th root of 1.

Because

$$\lim_{r \to 1^-} [(1-r)^{1+q} k^{1+q}/(1-r^{k})^{1+q}] = 1 ,$$

$$\lim_{r \to 1^-} [(1-r)^{1+q} \sum_{j=1}^{\infty} (kj) c_j G_{kj}(q, w; b, r)]$$

(9) $$= \lim_{r \to 1^-} \sum_{j=1}^{\infty} c_j b_{kj}^{w} \frac{(1-r^{k})^{1+q} j^{1+q} r^{wkj}}{(1 - b_{kj} r^{kj})^{1+q}}$$

provided the limit in (9) exists. To prove the existence of this limit we let $e^{-x} = r^{k}$,

$$J_j(x) = b_{kj}^{w} \frac{(1-e^{-x})^{1+q} j^{1+q} e^{-wjx}}{(1 - b_{kj} e^{-jx})^{1+q}}$$

and show $\sum c_j J_j(x)$ satisfies the conditions of Theorem 2.

Condition 1 of Theorem 2 is satisfied by hypothesis. For those j for which $b_{kj} = 1$, $J_j(0^+) = 1$, while if j is such that $b_{kj} \neq 1$, $J_j(0^+) = 0$. Therefore, by restricting all b_{kj} for some initial segment of $\{b_{kj}\}$, $j = 1, 2, \ldots$ to either be 1 or different from 1, condition 2(i) of Theorem 2 is satisfied.

For $h > 0$ and x in $(0, h)$ we have

(10) $$|j^{p} J_j(x)| \leq |b_{kj}^{w}| j^{q+p+1} e^{-wxj} = o(1)$$

since $w \geq 1$, $j \geq 1$ and $|b_{kj}| \leq 1$; and condition 2(ii) holds.

Previous techniques ([3] and [6]) for establishing conditions similar to 2(iii) require expansion of $[b_{kj}^{w} e^{(1+q-w)xj}] / [e^{xj} - b_{kj}]^{1+q}$ into partial fractions in e^{xj} This can be done only for b_{kj} a constant. Then, if b_{kj} is a real constant, K, the transformation $e^{-wt} = K \cdot e^{-xj}$ reduces our problem to that in [3]. If b_{kj} is complex, the sequence $\{\Delta^{p+1} J_j(x)\}$ contains complex expressions; and it is not possible to establish 2(iii) by the methods given here.

Consequently, we impose $b_n = o(n^{-(p+q+3)})$ which implies there exists a constant K such that

(11) $$|b_n| < K \cdot n^{-(p+q+3)} .$$

From (10) and (i) of the remark we have

(12) $$n^{p} |\Delta^{p+1} J_n(x)| < n^{p} (2n)^{1+q} \Sigma_{m=0}^{p+1} C_{p+1,m} |b_{k(n+m)}|^{w}$$

for n sufficiently large. From (11) and (12) there exists a constant K^1 independent of x such that $\Sigma n^{p} |\Delta^{p+1} J_n(x) < K'$, and (a) is proved.

Proōf of (b). For all integers w and t satisfying $1 \leq w \leq q$ and $1 \leq t \leq k-1$, we show

(13) $$\lim_{x \to 1^-} (1-x)^{1+q} \Sigma_{j=1}^{\infty} a_{kj-t} G_{kj-t}(q, w; b, xz^*) = 0 .$$

It follows from Theorem B that $S_m^r(s_j^t) = o(m^r)$ implies $a_{kj-t} = o((kj-t)^r)$, and there exists a constant K_1 such that

(14) $$|a_{kj-t}| < K_1 (kj-t)^r .$$

But, for all values of r and j,

$$[1 - b_{kj-t}(rz^*)^{kj-t}]^{1+q}$$

is bounded from zero since $|b_{kj-t}| \leq 1$, z^* is a primitive k-th root of 1 and $1 \leq t \leq k-1$. Therefore, for all j and for all $r < 1$, $|rz^*| < 1$, and there exists a constant $M > 0$ such that

(15) $$|a_{kj-t}\, G_{kj-t}(q,w;b,rz^*)| < M|a_{kj-t}||b_{kj-t}|^w(kj-t)^q .$$

From (11),(14) and $|b_{kj-t}| \leq 1$ we find the left side of (15) is less than

$$M\cdot K_1\cdot(kj-t)^r\cdot K\cdot(kj-t)^{-(r+q+3)}\cdot(kj-t)^q = (M\cdot K_1\cdot K)(kj-t)^{-3} .$$

Hence (b) is proved. This completes the proof of Theorem 1.

R e f e r e n c e s

[1] T.J.I'A. BROMWICH: On limits of certain infinite series and integrals, Math.Ann.65(1908),350-369.

[2] Patrick CASSENS and Francis REGAN: Hadamard operators applied to generalized Lambert series, to appear in Math.Notae.

[3] Patrick CASSENS, Francis REGAN and Daniel J. TROY: On Lambert series, J.Analyse Math.21(1968),423-431.

[4] J.M. FELD: The expansion of analytic functions in generalized Lambert series,Ann.of Math.33(1932),

139-143.

[5] G.H. HARDY: Divergent series, Oxford at Clarendon Press, London, 1949.

[6] G.H. HARDY: Note on Lambert series, Proc.London Math. Soc.13(1913),192-198.

State University of New York College
at Oswego, Oswego, New York 13126

Saint Louis University
St. Louis, Missouri 63103

(Oblatum 21.7.1970)