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# SOME GENERALIZATIONS OF THE NOTIONS LIMIT AND COLIMIT

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In the following note we introduce some generalizations of the notions limit and colimit in the theory of categories. With their help we are able to model some non-categorial products, especially some of the "non-direct" products used in algebra.

Remark: Let $\mathcal{K}$ be a category. Then $|\mathcal{K}|$ is the class of objects of $\mathcal{K}$ ; a diagram $D$ in $\mathcal{K}$ is a functor from a small category into $\mathcal{K}$ . A bound of a diagram $D : \mathcal{D} \longrightarrow \mathcal{K}$ is $\langle X, \{\varphi_d\}_{d \in |\mathcal{D}|} \rangle$ , where $X \in |\mathcal{K}|$ ; $\varphi_d \in \mathcal{K}(X, D(d))$ and where $\delta \in \mathcal{D}(d_1, d_2) \Rightarrow \varphi_{d_2} = D(\delta) \circ \varphi_{d_1}$ . A co-bound of $D$ is $\langle Z, \{\chi_d\}_{d \in |\mathcal{D}|} \rangle$ , where $Z \in |\mathcal{K}|$ , $\chi_d \in \mathcal{K}(D(d), Z)$ and where $\delta \in \mathcal{D}(d_1, d_2) \Rightarrow \chi_{d_1} = \chi_{d_2} \circ D(\delta)$.

Definition: Let $\mathcal{K}$ be a category, $\mathcal{T}$ a class of collections of morphisms of $\mathcal{K}$ , let $D : \mathcal{D} \longrightarrow \mathcal{K}$ be a diagram. A bound of $D$ , $\langle X, \{\varphi_d\}_{d \in |\mathcal{D}|} \rangle$ , is a $\mathcal{T}$-bound of $D$ if $\{\varphi_d\}_{d \in |\mathcal{D}|} \in \mathcal{T}$. A $\mathcal{T}$-bound of $D$ , $\langle X, \{\varphi_d\}_{d \in |\mathcal{D}|} \rangle$ is a $\mathcal{T}$-limit of $D$ if for each $\mathcal{T}$-bound of $D$ , $\langle Y, \{\psi_d\}_{d \in |\mathcal{D}|} \rangle$ there exists unique $\xi \in \mathcal{K}$ with $\psi_d = \varphi_d \circ \xi$ for each $d \in D$ . Analogously define $\mathcal{T}$-co-bound and $\mathcal{T}$-colimit of $D$ .

Definition: Let $\mathcal{K}, D, \mathcal{D}$ be as above. A bound (co-bound) of $D$ , $\langle X, \{\varphi_d\}_{d \in |\mathcal{D}|} \rangle$ is said to be strict if for each bound (co-bound) of $D$ , $\langle Y, \{\psi_d\}_{d \in |D|} \rangle$, there exists at most one $\xi \in \mathcal{R}$ with $\psi_d = \varphi_d \circ \xi \cdot$ $\cdot$ $(\psi_d = \xi \circ \varphi_d)$ for each $d \in |\mathcal{D}|$ . A strict bound (co-bound) of $D$ , $\langle X, \{\varphi_d\}_{d \in |\mathcal{D}|} \rangle$ , is <u>quasilimit</u> (<u>quasicolimit</u>) of $D$ if for each strict bound (co-bound) of $D$ , $\langle Y, \{\psi_d\}_{d \in |\mathcal{D}|} \rangle$ , $\xi \in \mathcal{R}$ is an isomorphism as soon as $\varphi_d = \psi_d \circ \xi$ $(\varphi_d = \xi \circ \psi_d)$ holds for each $d \in |\mathcal{D}|$ .

<u>Definition</u>: Strict $\mathcal{T}$-bound, $\mathcal{T}$-quasilimit, as well as the dual notions, are obvious generalizations of the preceding definitions.

<u>Example 1</u>: Box topology.

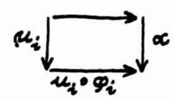
The box product of a collection of topological spaces $\{X_\iota\}_{\iota \in I}$ is their cartesian set product $\underset{\iota \in I}{X} X_\iota$ with the topology given by the collection of all open sets $\{\underset{\iota \in I}{X} \mathcal{U}_\iota ; \mathcal{U}_\iota$ open in $X_\iota$ for each $\iota \in I\}$ .

Let $\mathcal{R}$ be a complete category, let $\alpha \in \mathcal{R}(A, B)$ be a monomorphism. Define a class $\mathcal{T}_\alpha$ of collections of morphisms of $\mathcal{R}$ : $\{\varphi_\iota\}_{\iota \in I} \in \mathcal{T}_\alpha\ (\varphi_\iota \in \mathcal{R}(X_\iota, Y_\iota)) \Longleftrightarrow$

$$\Longleftrightarrow 1.\ X_i = X\ \forall i \in I\ \ 2.\ \forall \{u_\iota\}_{\iota \in I}\ (u_\iota \in \mathcal{R}(Y_\iota, B))\ \exists u \in \mathcal{R}(X, B)$$

such that


is pullback, where




is pullback for each $i \in I$ . (As $\alpha$ is mono, $\mu_i$ is mono and $\bigcap_{\iota \in I} \mu_\iota$ is correct.)

Now in case $\mathcal{K} = \mathbf{Top}$, $A$ is the one-point space on $\{0\}$, $B$ the space on $\{0, 1\}$ with $\overline{\{1\}} = \{1\}$ and $\overline{\{0\}} = \{0, 1\}$, $\alpha(0) = 0$ we get: The box product of topological spaces is just their $\mathcal{T}_\alpha$ -product.

Example 2: Weak direct product of universal algebras. Weak direct product of a collection of universal algebras of a given type $\{A_\iota\}_{\iota \in I}$ is any such a subalgebra $A$ of their direct product $\underset{\iota \in I}{\times} A_\iota$ that

$1. \{x_\iota\}_{\iota \in I} \in A, \{y_\iota\}_{\iota \in I} \in \underset{\iota \in I}{\times} A_\iota, card\{i \in I; x_i \neq y_i\} < \aleph_0 \Rightarrow \{y_\iota\}_{\iota \in I} \in A$

$2. \{x_\iota\}_{\iota \in I} \in A, \{y_\iota\}_{\iota \in I} \in A \Rightarrow card\{i \in I; x_i \neq y_i\} < \aleph_0$ .

Let $\mathcal{R}$ be a category of universal algebras of a type $\Delta$ and their homomorphisms. The weak direct product is the same as $\mathcal{T}$ -quasiproduct in $\mathcal{R}$ where

$\{\varphi_\iota\}_{\iota \in I} \in \mathcal{T} (\varphi_\iota \in \mathcal{R}(X_\iota, Y_\iota)) \Longleftrightarrow 1. X_i = X \ \forall i \in I$

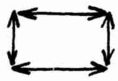
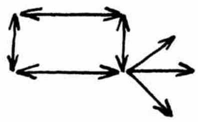
$2. \alpha, \beta \in \mathcal{R}(\mathbb{1}, X) \Longrightarrow card\{i \in I; \varphi_i \alpha \neq \varphi_i \beta\} < \aleph_0$ ,

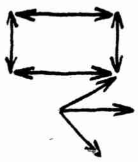
where $\mathbb{1}$ denotes the free algebra with one generator.

Example 3: Subdirect product of universal algebras. Subdirect product of a collection of universal algebras of a given type $\{A_\iota\}_{\iota \in I}$ is any such a subalgebra $A$ of $\underset{\iota \in I}{\times} A_\iota$ that $\pi_i(A) = A_i$ for each $i \in I$ ($\pi_i$ being the $i$ -th projection of $\underset{\iota \in I}{\times} A_\iota$ ). It is the same as the strict $\mathcal{T}$ -bound of the discrete diagram $\{A_\iota\}_{\iota \in I}$ , where $\mathcal{T}$ is the class of all collections of epimorphisms of $\mathcal{R}$ ( $\mathcal{R}$ see above).

Example 4: Quasicoproducts of connected graphs and topological spaces.

Denote Gra Con the category of connected graphs and their homomorphisms,Top Con the category of all connected topological spaces and their continous mappings. It is evident that the coproduct of any collection of graphs in Gra (or of topological spaces in Top) is disconnected and that there does not exist coproduct in Gra Con (TopCon). Still, there exist (but of oourse not generally unique) quasicoproducts in these categories and they give especially in case of two objects a natural factorization of the coproduct from Gra (Top): Let $A, B$ be connected graphs( topological spaces); we get just all quasicoproducts of $A$ and $B$ in Gra Con (Top Con) by choosing one point in each of the underlying sets of $A$ and $B$ and clewing $A$ with $B$ in these points.

For example A:  , B:  we get 

and  , the others being isomorphic to those two.

R e f e r e n c e s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