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S - CATEGORIES

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In this paper we generalize the concepts of projectively or inductively generated closure space, of proximity space and that of the uniform space. These concepts occur in the second edition of the Čech's book "Topological spaces" [1] (to appear this year). These concepts may be considered as generalizations of the product or the sum of objects being special cases of the limit of presheaves at the same time.

We start by giving some known definitions and less known designations.

Let $\mathcal{K}$ be a category. We shall write $X = \mathcal{D}\varphi$, $Y = \mathcal{E}\varphi$ if $\varphi \in \mathrm{Hom}_{\mathcal{K}}(X, Y)$. i_X is the identity morphism of the object X of $\mathcal{K}$.

The pair $\langle Y, \varphi \rangle$ is a subobject of an object X of $\mathcal{K}$ if $\varphi \in \mathrm{Hom}_{\mathcal{K}}(Y, X)$ is a monomorphism and if in every decomposition $\varphi = \varphi_1 \cdot \varphi_2$, where φ_1 is a monomorphism and φ_2 is a bimorphism, φ_2 is inversible (i.e. there is a φ_2' in $\mathcal{K}$ such that $\varphi_2 \cdot \varphi_2' = i_{\mathcal{E}\varphi_2}$, $\varphi_2' \cdot \varphi_2 = i_{\mathcal{D}\varphi_2}$). If $\mathcal{K}$ has the inversion property (i.e. if each bimorphism is inversible) then evidently $\langle Y, \varphi \rangle$ is a subobject of X if and only if $\varphi \in \mathrm{Hom}_{\mathcal{K}}(Y,X)$ is a monomorphism.

Dually the factor-object of an object X of $\mathcal{K}$ is defined.

Now, let $\mathcal{K}'$ be a subcategory of $\mathcal{K}$. We shall say that the pair $\langle X, \{\varphi_i \mid i \in I\}\rangle$ is the upper modification of the class $\{X_i \mid i \in I\}$ of objects of $\mathcal{K}$ in $\mathcal{K}'$ if X is an object of $\mathcal{K}'$, $\varphi_i \in \mathrm{Hom}_{\mathcal{K}}(X_i, X)$ for each $i \in I$ and if for any pair $\langle Y, \{\psi_i \mid i \in I\}\rangle$ such that Y is an object in $\mathcal{K}'$ and $\psi_i \in \mathrm{Hom}_{\mathcal{K}}(X_i, Y)$ for all $i \in I$ there is exactly one morphism φ of the category $\mathcal{K}'$ such that $\psi_i = \varphi \circ \varphi_i$ for each $i \in I$.

Dually, the lower modification in $\mathcal{K}'$ is defined.

The upper modification of the class $\{X_i \mid i \in I\}$ in $\mathcal{K}$ is called the sum of these objects (sign $\Sigma\{X_i \mid i \in I\}$) and the lower modification of this class in $\mathcal{K}$ is called the product (sign $\prod\{X_i \mid i \in I\}$).

For the upper modification or for the lower modification only the first member of the competent pair is sometimes taken.

The presheaf in $\mathcal{K}$ with carrier $\langle J, \rho\rangle$ is the family $\{\varphi_{ij} \mid \langle i, j\rangle \in \rho\}$ of morphisms of $\mathcal{K}$, where ρ is an order on the nonvoid set J (i.e. a transitive and reflexive relation on J), which fulfils the equalities $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ whenever $\langle i, j\rangle \in \rho$, $\langle j, k\rangle \in \rho$ and such that $\varphi_{ii} = 1_{\mathcal{D}\varphi_{ii}}$ for each $i \in I$. (Hence every presheaf is uniquely determined by some covariant functor $F : J_\rho \to \mathcal{K}$. Really, J_ρ is a category where the class of objects is the set J and the class of morphisms is ρ . The functor F is obvious. We shall not make difference between these definitions.)

If F is a presheaf in $\mathcal{K}$ with the carrier $\langle J, \rho\rangle$ (i.e. an object of the functor category $\mathcal{K}^{J_\rho}$) then the lower modification (sign $\varprojlim F$), the upper modification (sign $\varinjlim F$) resp., of F in $\mathcal{K}$ is called the projective limit of F , the inductive limit of F resp. (The definition is correct, as $\mathcal{K}$ is isomorphic in an obvious way to the subcategory of $\mathcal{K}^{J_\rho}$.

This isomorphism assigns to every object X of $\mathcal{K}$ the constant functor F which maps every $\langle i, j \rangle \in \varphi$ onto i_X .)

Now, we shall define some basic concepts. They all may be illustrated e.g. by taking for $\mathcal{K}$ the category of topological spaces and for $\mathcal{C}$ the category of sets. Other important examples (the generalized proximity spaces and the generalized uniform spaces) will be given in my next paper in CMUC 5,2 or 5,3.

<u>Definition 1</u>. A category $\mathcal{K}$ is called an <u>S-category over a category</u> $\mathcal{C}$ <u>with respect to a functor</u> (covariant or contravariant) T if $T : \mathcal{K} \to \mathcal{C}$ and if the following conditions are fulfilled:

(1) If $T\varphi = T\psi$ and $\mathcal{D}\varphi = \mathcal{D}\psi$, $\mathcal{E}\varphi = \mathcal{E}\psi$ then $\varphi = \psi$.

(2) For each morphism α of $\mathcal{C}$ is $T^{-1}[\alpha] \neq \emptyset$. Moreover, if $T X = A$, $T Y = B$, $\alpha \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, then there are morphisms $\varphi \in T^{-1}[\alpha]$, $\psi \in T^{-1}[\alpha]$ such that either $\mathcal{D}\varphi = X$, $\mathcal{E}\psi = Y$ if T is covariant or $\mathcal{E}\varphi = X$, $\mathcal{D}\psi = Y$ if T is contravariant.

(3) If φ is a morphism of $\mathcal{K}$ and $T\varphi = \alpha \circ \beta$ then there are morphisms $\varphi_1 \in T^{-1}[\alpha]$, $\varphi_2 \in T^{-1}[\beta]$ such that either $\varphi = \varphi_1 \circ \varphi_2$ if T is covariant or $\varphi = \varphi_2 \circ \varphi_1$ if T is contravariant.

(4) For each object A of $\mathcal{C}$ the class $T^{-1}[A]$ is a set which is complete with respect to the order

$$R_A = \{ \langle X, Y \rangle \mid T\varphi = i_A \quad \text{for some} \quad \varphi \in \mathrm{Hom}_{\mathcal{K}}(X,Y)\}$$

(i.e. each subset of $T^{-1}[A]$ has its sup and inf).

(5) If $\{ \varphi_i \mid i \in I \}$ is a nonvoid family of morphisms of $\mathcal{K}$ such that $T\varphi_i = T\varphi_j$ for each $\langle i, j \rangle \in I \times I$

then there are morphisms

$$\varphi \in \mathrm{Hom}_{\mathcal{K}}(\sup\{\mathfrak{D}\varphi_i \mid i \in I\}, \sup\{\mathcal{E}\varphi_i \mid i \in I\})$$

$$\psi \in \mathrm{Hom}_{\mathcal{K}}(\inf\{\mathfrak{D}\varphi_i \mid i \in I\}, \inf\{\mathcal{E}\varphi_i \mid i \in I\})$$

such that $T\varphi = T\psi = T\varphi_i$ for $i \in I$.

<u>Remark 1</u>. If $\mathcal{K}$ is an S-category over $\mathcal{C}$ with respect to T then it is clear that the dual category $\mathcal{K}^*$ is an S-category over $\mathcal{C}$ with respect to the dual functor T^*. In this sense every term and theorem in $\mathcal{K}$ has its dual one in $\mathcal{K}^*$.

Next, let the category $\mathcal{K}$ be an S-category over the category $\mathcal{C}$ with respect to the <u>covariant</u> functor T. (Hence φ is a monomorphism, an epimorphism, a bimorphism resp., of $\mathcal{K}$ if and only if $T\varphi$ has the same property in $\mathcal{C}$. A simple proof of this is carried out by using (1), (2) and (4) in definition 1.)

In this case there are subcategories $\mathcal{K}_1$, $\mathcal{K}_2$ of $\mathcal{K}$ both isomorphic to $\mathcal{C}$ with isomorphisms $T_1 = T/_{\mathcal{K}_1}$, $T_2 = T/_{\mathcal{K}_2}$ (hence there is an isomorphism $T': \mathcal{K}_1 \to \mathcal{K}_2$ such that $T_2 \circ T' = T_1$) such that each object X of $\mathcal{K}$ has its upper modification $\langle X_2, \varphi_2 \rangle$ in $\mathcal{K}_2$ and its lower modification $\langle X_1, \varphi_1 \rangle$ in $\mathcal{K}_1$. There is $T'X_1 = X_2$, $T\varphi_1 = T\varphi_2 = i_{TX}$. It is possible to define the S-category by this property.

<u>Theorem 1</u>. Let $\mathcal{K}'$ be a subcategory of $\mathcal{K}$, $T[\mathcal{K}'] = \mathcal{C}'$ and $\mathrm{Hom}_{\mathcal{K}'}(X, Y) = \mathrm{Hom}_{\mathcal{K}}(X, Y) \cap T^{-1}[\mathrm{Hom}_{\mathcal{C}'}(TX, TY)]$ (i.e. $\mathcal{K}'$ is the full subcategory of $T^{-1}[\mathcal{C}']$. If each object X of $T^{-1}[\mathcal{C}']$ has either the upper modification $\langle X_2, \varphi_2 \rangle$ in $\mathcal{K}'$ with $T\varphi_2 = i_{TX}$ or the lower modification $\langle X_1, \varphi_1 \rangle$

in $\mathcal{K}'$ with $T\varphi_1 = i_{TX}$ then $\mathcal{K}'$ is the S-category over $\mathcal{L}'$ with respect to $T/_{\mathcal{K}'}$.

Proof. It is sufficient to notice that for $I \neq \emptyset$ $X = \sup_R \{X_i \mid i \in I\}$ of objects of $\mathcal{K}'$ is either an object of $\mathcal{K}'$ (here $R = \bigcup\{R_A \mid A$ is an object of $\mathcal{L}\}$) and in this case $X = \sup_{R'} \{X_i \mid i \in I\}$ where $R' = R \cap ((T/_{\mathcal{K}'})^{-1}[TX] \times (T/_{\mathcal{K}'})^{-1}[TX])$ or X has its upper modification X_1 in $\mathcal{K}'$ and $X_1 = \sup_{R'} \{X_i \mid i \in I\}$. Similarly for inf.

Definition 2. We shall say that an object X of $\mathcal{K}$ is projectively generated by

$f \in \prod\{\mathrm{Hom}_{\mathcal{K}}(X_i, Y_i) \mid i \in I\}$ (sign $X = \underleftarrow{\mathrm{Lim}}\, f$)

or by a family $\{f\,i \mid i \in I\}$ (sign $X = \underleftarrow{\mathrm{Lim}}\{f\,i \mid i \in I\}$)

if $I \neq \emptyset$ and

$X = \max\{X' \mid$ there is a $\varphi_i \in \mathrm{Hom}_{\mathcal{K}}(X', Y_i)$ for each $i \in I$ such that $T\varphi_i = T\,f\,i\}$.

(X exists if and only if $T\,X_i = T\,X_j$ for all $\langle i, j\rangle \in I \times I$.)

Dually we define that an object Y is inductively generated by f (sign $Y = \underrightarrow{\mathrm{Lim}}\, f$ or $Y = \underrightarrow{\mathrm{Lim}}\{f\,i \mid i \in I\}$). (I.e. $Y = \min\{Y' \mid$ there is a $\varphi_i \in \mathrm{Hom}_{\mathcal{K}}(X_i, Y')$ for each $i \in I$ such that $T\varphi_i = T\,f\,i\}$; Y exists if and only if $T\,Y_i = T\,Y_j$ for all $\langle i, j\rangle \in I \times I$.)

Remark 2. It is evident that if $f \in \prod\{\mathrm{Hom}_{\mathcal{K}}(X_i, Y_i) \mid i \in I\}$, $g \in \prod\{\mathrm{Hom}_{\mathcal{K}}(X_i', Y_i) \mid i \in I\}$ and $T\,f\,i = T\,g\,i$ for each $i \in I$ then $\underleftarrow{\mathrm{Lim}}\, f = \underleftarrow{\mathrm{Lim}}\, g$ if either $\underleftarrow{\mathrm{Lim}}\, f$ or $\underleftarrow{\mathrm{Lim}}\, g$ exists. It follows that $\underleftarrow{\mathrm{Lim}}\, f = \inf\{\underleftarrow{\mathrm{Lim}}\, f\,i \mid i \in I\}$ if either $\underleftarrow{\mathrm{Lim}}\, f$ or $\inf\{\underleftarrow{\mathrm{Lim}}\, f\,i \mid i \in I\}$ exists. Similarly for inductively generated objects.

Theorem 2. Let $X = \underleftarrow{\mathrm{Lim}}\{\varphi_i \mid i \in I\}$, $\alpha \in \mathrm{Hom}_{\mathcal{L}}(TZ, TX)$. Then there is a $\psi \in \mathrm{Hom}_{\mathcal{K}}(Z, X) \cap T^{-1}[\alpha]$ if and only if there is a $g \in \prod\{\mathrm{Hom}_{\mathcal{K}}(Z, E\varphi_i) \mid i \in I\}$ such that $T\,g\,i =$

$= T\varphi_i \circ \alpha$.

Dually for $\varinjlim \{\varphi_i \mid i \in I\}$.

Proof. The necessity is obvious. We shall prove the sufficiency. Let g fulfil the condition of our theorem. By definition 1 (3) there are morphisms $\psi_i \in T^{-1}[\alpha]$ and $\varphi'_i \in T^{-1}[T\varphi_i]$ such that $g\,i = \varphi'_i \circ \psi_i$ for all $i \in I$. Hence there are morphisms $\psi'_i \in \mathrm{Hom}_{\mathcal{K}}(Z, \varprojlim \varphi'_i) \cap T^{-1}[\alpha]$ and by definition 1(5) there is a morphism $\psi \in \mathrm{Hom}_{\mathcal{K}}(Z, \inf\{\varprojlim \varphi'_i \mid i \in I\}) \cap T^{-1}[\alpha]$. By remark 2 $E\psi = X$.

Remark 3. It is almost self-evident that $\varprojlim \{\varphi_i \mid i \in I\}$, $\varinjlim \{\varphi_i \mid i \in I\}$ resp., is by theorem 2 completely characterized.

Theorem 3. Let $f \in \prod\{\mathrm{Hom}_{\mathcal{K}}(X_i, Y_i) \mid i \in I\}$, for each $i \in I$ be $g_i \in \prod\{\mathrm{Hom}_{\mathcal{K}}(Y_i, Z_j) \mid j \in J_i\}$ and $Y_i = \varprojlim g_i$. Let for each $\langle i, j\rangle \in \sum\{J_i \mid i \in I\}$ be $h\langle i, j\rangle = g_i[j] \circ f\,i$ (i.e. $h \in \prod\{\mathrm{Hom}_{\mathcal{K}}(X_i, Z_j) \mid \langle i, j\rangle \in \sum\{J_i \mid i \in I\}\}$) . Then $\varprojlim f = \varprojlim h$ if either $\varprojlim f$ or $\varprojlim h$ exists.

Dually for inductively generated objects.

Proof. Evidently, the existence of $\varprojlim f$ is equivalent to the existence of $\varprojlim h$. It is sufficient to prove

$\varprojlim f\,i = \varprojlim \{h\langle i, j\rangle \mid j \in J_i\}$ for each $i \in I$.

But by remark 2 for any $i \in I$

$\varprojlim \{h\langle i, j\rangle \mid j \in J_i\} = \inf\{\varprojlim h\langle i, j\rangle \mid j \in J_i\}$

and the equality $\varprojlim f\,i = \inf\{\varprojlim h\langle i, j\rangle \mid j \in J_i\}$ is obvious.

Remark 4. If $Y_i \neq \varprojlim g_i$ then we can prove only $\langle \varprojlim f, \varprojlim h\rangle \in R_{TX_i}$, $i \in I$, if these objects exist. But if $X_i^* = \varprojlim h$ for each $i \in I$ then $\varprojlim f = \varprojlim h$ holds also in this case.

Dually for inductively generated objects.

Theorem 4. Let $\{\varphi_{ij} \mid \langle i, j\rangle \in \rho\}$ be a presheaf in $\mathcal{K}$ with the carrier $\langle J, \rho\rangle$. Then $\langle X, \{\varphi_i \mid i \in J\}\rangle = \varprojlim \{\varphi_{ij} \mid \langle i, j\rangle \in \rho\}$ if and only if $\langle TX, \{T\varphi_i \mid i \in J\}\rangle = \varprojlim \{T\varphi_{ij} \mid \langle i, j\rangle \in \rho\}$ and $X = \varprojlim \{\varphi_i \mid i \in J\}$.

Especially this is true for the diagonal $\rho = \Delta_J$ (then $\varprojlim \{\varphi_{ij} \mid \langle i, j\rangle \in \rho\} = \prod \{\mathcal{D}\varphi_{ii} \mid i \in J\}$).

Dually for inductive limits.

Proof. Let $\langle X, \{\varphi_i \mid i \in J\}\rangle = \varprojlim \{\varphi_{ij} \mid \langle i, j\rangle \in \rho\}$ and let $h \in \prod \{\mathrm{Hom}_{\mathcal{L}}(A, \mathcal{D}\, T\varphi_{ii}) \mid i \in J\}$ such that for each $\langle i, j\rangle \in \rho$ is $h\, j = T\varphi_{ij} \circ h\, i$. Then by definition 1(2) there is such $f \in \prod \{\mathrm{Hom}_{\mathcal{K}}(\inf T^{-1}[A], \mathcal{D}\varphi_{ii}) \mid i \in J\}$ that $T\, f\, i = h\, i$ for each $i \in J$. It follows that $\langle TX, \{T\varphi_i \mid i \in J\}\rangle = \varprojlim \{T\varphi_{ij} \mid \langle i, j\rangle \in \rho\}$. It is obvious that $X = \varprojlim \{\varphi_i \mid i \in J\}$. On the other hand the sufficiency follows immediately from theorem 2.

Corollary. Let $f \in \prod \{\mathrm{Hom}_{\mathcal{K}}(X, Y_i) \mid i \in I\}$. Let $\langle Y, \{\varphi_i \mid i \in I\}\rangle = \prod \{Y_i \mid i \in I\}$ and let φ be such a morphism that $\varphi_i \circ \varphi = f\, i$ for each $i \in J$.

Then $\varprojlim f = \varprojlim \varphi$.

Dually for inductively generated objects.

Proof follows from theorem 3 and from the special case of the theorem 4.

Theorem 5. $\langle Y, \varphi\rangle$ is a subobject of an object X of $\mathcal{K}$ if and only if $\langle TY, T\varphi\rangle$ is a subobject of TX in $\mathcal{L}$ and $Y = \varprojlim \varphi$.

Dually for factor-objects.

Proof. Let $\langle Y, \varphi \rangle$ be a subobject of X. Then evidently $Y = \underleftarrow{\mathrm{Lim}}\, \varphi$. If $T\varphi = \alpha \circ \beta$ where β is a bimorphism in $\mathcal{C}$ and α is a monomorphism in $\mathcal{C}$ then by definition 1(3) $\varphi = \varphi_1 \circ \varphi_2$ where $\varphi_2 \in T^{-1}[\beta]$ is a bimorphism in $\mathcal{K}$ and $\varphi_1 \in T^{-1}[\alpha]$ is a monomorphism in $\mathcal{K}$. Hence φ_2 is inversible. It follows that β is inversible, too. So $\langle T Y, T\varphi \rangle$ is a subobject of $T X$ in $\mathcal{C}$. Dually for factor objects.

Now, let $\langle T Y, T\varphi \rangle$ be a subobject of $T X$ and $Y = \underleftarrow{\mathrm{Lim}}\, \varphi$. If $\varphi = \varphi_1 \circ \varphi_2$ where φ_2 is a bimorphism and φ_1 is a monomorphism then $T\varphi_2$ is inversible. By definition 1(3) is $1_{E\varphi_2} = \psi_2 \circ \psi_1$ where $\psi_2 \in T^{-1}[T\varphi_2]$, $\psi_1 \in T^{-1}[(T\varphi_2)^{-1}]$. But by remark 4 $Y = \underleftarrow{\mathrm{Lim}}\, \varphi_2$ and so there is a $\psi \in T^{-1}[1_{TY}]$ such that $\psi_2 = \varphi_2 \circ \psi$. Now, it is clear that φ_2 is inversible and $\varphi_2^{-1} = \psi \circ \psi_1$.

Corollary. If X is an object of $\mathcal{K}$, $\langle A, \alpha \rangle$ a subobject, factor-object resp., of $T X$ in $\mathcal{C}$ then there is a subobject, factor object resp., $\langle Y, \varphi \rangle$ of X in $\mathcal{K}$ such that $T Y = A$, $T\varphi = \alpha$.

Corollary. If $\mathcal{C}$ has the inversion property and if φ is a monomorphism of $\mathcal{K}$ then $\langle \underleftarrow{\mathrm{Lim}}\, \varphi, \varphi \rangle$ is a subobject of $E\varphi$.

Theorem 6. A category $\mathcal{K}$ is an S-category over a category $\mathcal{C}$ with respect to a covariant functor T if and only if the functor $T : \mathcal{K} \to \mathcal{C}$ fulfils the conditions (1),(2),(3) from definition 1 and the condition (4'): If F is a presheaf in $\mathcal{K}$ and if there is $\underleftarrow{\lim}\, T F$, $\underrightarrow{\lim}\, TF$ resp., in $\mathcal{C}$ then there exists $\underleftarrow{\lim}\, F$, $\underrightarrow{\lim}\, F$ resp., in $\mathcal{K}$ such that $T \varprojlim F = \varprojlim T F$, $T \varinjlim F = \varinjlim T F$ resp.

Proof. The necessity follows from theorem 4. We shall prove the sufficiency. Let (4´) be fulfilled. Let K be a nonvoid subset of $T^{-1}[A]$ for some object A of $\mathcal{C}$. $R_K = R_A \cap (K \times K)$ is the order on K. If $\varphi_{XY} \in \mathrm{Hom}_{\mathcal{K}}(X, Y) \cap T^{-1}[i_A]$ for $\langle X, Y \rangle \in R_K$ then $F = \{ \varphi_{XY} \mid \langle X, Y \rangle \in R_K \}$ is a presheaf in $\mathcal{K}$ with the carrier $\langle K, R_K \rangle$. Clearly $T F$ is the constant presheaf and so it has projective and inductive limit (this limit is A). By the condition (4´) there exist $\varprojlim F$ and $\varinjlim F$ which are elements of $T^{-1}[A]$. It is almost self-evident that $\varprojlim F = \inf K$, $\varinjlim F = \sup K$. Hence (4) is true. (5) follows from the proof of (4) and from the definition of mofidications.

Remark 5. We have seen that $\mathcal{K}$ keeps many of properties of $\mathcal{C}$. We can give further examples.

If $\mathcal{C}$ is an abelian category and if $T/_{\mathrm{Hom}_{\mathcal{K}}(X, Y)}$ is the group-homomorphism for each pair $\langle X, Y \rangle$ of objects of $\mathcal{K}$ then $\mathcal{K}$ is "almost abelian". Really, each morphism φ of $\mathcal{K}$ has its kernel and cokernel and $\langle \mathrm{Coim}\ \varphi, \mathrm{Im}\ \varphi \rangle \in R$ (for R see theorem 1, proof). But $\mathrm{Coim}\ \varphi$ and $\mathrm{Im}\ \varphi$ need not be isomorphic.

If $\mathcal{C}$ is a bicategory (see [2]) with S as the class of surjections and I as the class of injections then $\langle \mathcal{K}, S_1, I_1 \rangle$, $\langle \mathcal{K}, S_2, I_2 \rangle$ are also bicategories, where

$$S_1 = T^{-1}[S], \quad I_1 = \{ \varphi \mid T\varphi \in I,\ \mathcal{D}\varphi = \varprojlim \varphi \}$$
$$S_2 = \{ \varphi \mid T\varphi \in S,\ \mathcal{E}\varphi = \varinjlim \varphi \}, \quad I_2 = T^{-1}[I].$$

It is possible to find relations between projective (or injective) objects of $\mathcal{K}$ and $\mathcal{C}$.

R e f e r e n c e s :

[1] E. ČECH, Topological spaces (2nd edition, Prague, in preparation).

[2] Z. SEMADENI, Projectivity, injectivity and duality, Rozprawy matematyczne XXXV, 1963.