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ON THE CHARACTERIZATION OF BANACH SPACES WITH THE STRONG KIRTZBRAUN-VALENTINE PROPERTY

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Let $X = (X, \rho)$ and $Y = (Y, \sigma)$ be metric spaces. If φ is a transformation of X into Y, then φ is said to be Lipschitzian with the constant λ, provided

$$\sigma(\varphi(x), \varphi(y)) \leq \lambda \rho(x, y)$$

for all $x, y \in X$. A Lipschitzian transformation φ with the constant $\lambda = 1$ is called a contraction.

The problem of extending of a Lipschitzian transformation A to Y (where A is a subspace of a space X) to a transformation of X to Y was studied by various authors. The existence of such an extension for $Y = E_1$ is proved by Banach in [2]. As a consequence of the result of Aronszajn and Panichpakdi [1] we get the existence of an extension for a hyperconvex space Y (i.e. spaces which have the following property: If $\mathcal{G} = \{\Omega(x_i, r_i) : i \in I\}$ is a system of σ-metric cells in Y such that for each $i \in I$, $j \in I$ there is $\sigma(x_i, x_j) \leq r_i + r_j$, then $\bigcap \mathcal{G} \neq \emptyset$).

Mc Shane [5], Kirtzbraun [4] and Valentine [6], [7] showed that this extension problem is associated, with the following intersection property.

A pair of metric spaces (X, ρ) and (Y, σ) is said to have a Valentine intersection property provided that:

If $\mathcal{F} = \{\Omega(x_i, r_i) : i \in I\}$

is a system of ρ -cells in X and

$$\mathcal{G} = \{ \Omega (y_i, r_i) : i \in I \}$$

a system of σ-cells in Y such that for each $i \in I$, $j \in I$ there is

$$\rho (x_i, x_j) \geq \sigma (y_i, y_j) ,$$

then

$$\cap \mathcal{F} \neq \emptyset \Rightarrow \cap \mathcal{G} \neq \emptyset .$$

In this paper we shall discuss contractions only, since the general Lipschitzian extension problem can be reduced to an adequate contraction problem (see [7]p.93) if Y is a normed linear space.

There is proved in the paper of Valentine [7], that the situation is the following one:

For any metric spaces X and Y the following two statements are equivalent:

(1) (X, Y) has the Valentine intersection property;

(2) for every $A \subset X$ and every contraction f of A into Y there exists an extension $F \supset f$ such that F is a contraction mapping X into Y.

There is also proved in [7] that for each of the following cases the Valentine intersection property is satisfied

(a) X is an arbitrary metric space, $Y = E_1$,

(b) $X = Y = E_n$;

(c) $X = Y = H$, H being a Hilbert space;

(d) $X = Y = S^n$, S^b being an n-dimensional Euclidean sphere.

In the cases (b) and (c) it may be proved that the extension F of a contraction f of $A \subset X$ into Y may be found in such a way that

$$\overline{\text{conv}}\ f(A) = \overline{\text{conv}}\ F(X);$$

where the symbol $\overline{\text{conv}}$ B denotes a closed convex hull of the set B .

In connection with the results above we formulate the following definitions:

A metric space X is said to have a Valentine intersection property if the pair (X, X) has the Valentine intersection property.

A metric linear space X is said to have a strong Valentine intersection property provided that:

if $\mathcal{F} = \{ \Omega (x_i, r_i) : i \in I \}$ and $\mathcal{G} = \{ \Omega (y_i, r_i), i \in I \}$

are systems of cells in X such that

$$\rho (x_i, x_j) \geqq \rho (y_i, y_j) \quad \text{for each } i, j \in I ,$$

then

$$\cap \mathcal{F} = \emptyset \Rightarrow (\cap \mathcal{G}) \cap \overline{\text{conv}} \bigcup_{i \in I} y_i \neq \emptyset .$$

It is easy to prove the following:

Let X be a Banach space. Then the following statements are equivalent:

(1) X has a strong Valentine intersection property;

(2) for each $A \subset X$ and each contraction f of A into X there exists an extension $F \supset f$ of f such that F is a contraction of X into itself and $\overline{\text{conv}}\, f(A) = \overline{\text{conv}}\, F(X)$.

The problem of characterization of all Banach spaces with the Valentine intersection property is still unsolved. The main result of this paper is the complete characterization of all Banach spaces with the strong Valentine intersection property. The situation is described by the following theorem:

Theorem: Let X be a real Banach space. The following statements are equivalent:

(1) X has a strong Valentine intersection property;

(2) for each $A \subset X$ and each contraction f of A into X there exists an extension $F \supset f$ such that F is a contraction of X into X and $\overline{\text{conv}}\, f(A) = \overline{\text{conv}}\, F(X)$;

(3) for each $A \subset X$ and each contraction f of A into X there exists an extension $F \supset f$ such that F is a contraction of X into X and $\overline{\text{sp}}\, f(A) = \overline{\text{sp}}\, F(X)$, where $\overline{\text{sp}}\, B$ denotes the closed linear hull of the set $B \subset X$;

(4) either X is an inner product space (i.e. Euclidean or Hilbert space) or X is a two-dimensional space ℓ_2^∞ whose unit sphere is a parallelogram.

Proof: Obvously (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3). The statement (4) $\Longrightarrow$ (1) is proved in [7] for the case that X is an inner product space. As an easy consequence of [1] we get immediately the validity of (1) in the space ℓ_2^∞.

Hence it remains to prove

the implication (3) $\Longrightarrow$ (4). Let X be a space with the property (3) and Z a two-dimensional subspace of X. It is clear that Z has the property (3), too. Let S be a unit cell in Z and let Σ be the unit sphere which is boundary of S. We distinguish the following two cases:

A. Let S be strictly convex (i.e. $x, y \in \Sigma$, $0 < \lambda < 1$, $\lambda x + (1 - \lambda) y \in \Sigma \Rightarrow x = y$). In this case we shall prove that Σ is an ellipse.

Let x_1 and x_2 be different points of Z and let y_1 and y_2 be points lying in different half-planes with the stright line $\overline{x_1, x_2}$ as the common boundary and which have the property

$$\| x_1 - y_1 \| = \| x_2 - y_1 \| , \| x_1 - y_2 \| = \| x_2 - y_2 \| .$$

Let $x_0 = \frac{1}{2}(x_1 + x_2)$ and let y_0 be the common point of $\overline{x_1, x_2}$ and $\overline{y_1, y_2}$.

Let us assume $y_0 \neq x_0$. In this case exactly one of the cells

$$x_1 + \| x_1 - x_0 \| S , \quad x_2 + \| x_2 - x_0 \| S$$

contain a point y_0. Let, e.g.,

$$y_0 \in x_1 + \| x_1 - x_0 \| S .$$

Therefore

$$\| x_1 - x_0 \| = \| x_2 - x_0 \| ,$$

$$(y_1 + \| y_1 - y_0 \| S) \cap (y_2 + \| y_2 - y_0 \| S) \cap (x_1 + \| x_1 - x_0 \| S) \neq \emptyset$$

and (since S is strictly convex) we get

$$(y_1 + \| y_1 - y_0 \| S) \cap (y_2 + \| y_2 - y_0 \| S) \cap (x_2 + \| x_2 - x_0 \| S) = \emptyset,$$

which fact is a contradiction with the Valentine intersection property of Z.

Therefore there is $y_0 = x_0$ and we get that, if x_1, x_2 are arbitrary points, then the centers of all cells containing x_1, x_2 are lying in the stright line. By means of elementary geometric considerations it may be proved that ellipse is the only one possible convex cell with this property.

B. Let S be not strictly convex. At first, let us mind that the corollary of the Valentine intersection property of Z is the following property of Z :

(A) If there exists a cell with a radius r in Z such that it contains the points x_i, $i = 1, \ldots, n$ and if

$\| x_i' - x_j' \| = \| x_i - x_j \|$ for each $i, j = 1, \ldots, n$,

then there exists a cell with the radius r containing all points x_i' , $i = 1, \ldots, n$.

At first we shall prove that S is a 2n-gon with sides of equal lentgh (in the sense of the norm in Z).

Let x_1, x_2 be endpoints of some maximal (straight line) segment of Σ. The cell $x_1 + \| x_2 - x_1 \| S$ has a center in the boundary of the cell $- x_1 + 2S$. Therefore the boundaries of these cells have two common points, one of these points being x_2. Let us denote x_3 the second one.

The points $x_1, x_2, - x_1$ are contained in S. The property (A) of Z implies the existence of a cell of radius 1, containing all points $x_1, x_3, - x_1$. Because S is the unique cell of radius 1, which contains both x_1 and $-x_1$, there is $x_3 \in S$; since $\| x_3 - (- x_1) \| = 2$, there is $x_3 \in \Sigma$. Obviously $\| \frac{1}{2} (x_3 + x_1) \| = 1$ and therefore the whole segment $\overline{x_1, x_3}$ lies in Σ, i.e. $\overline{x_1, x_3}$ is a part of some segment of Σ with a length at least $\| x_2 - x_1 \|$.

As a consequence we get the fact that S is a 2n-gon with sides of equal (Minkowski) length.

Now, we shall prove $n = 2$, i.e., S is a parallelogram. Let $n > 2$ and let x_1, x_2, x_3 be three consecutive vertices of S. It is easy to show that

$$\| x_1 - x_2 \| < \| x_1 - x_3 \| .$$

Let us consider the points

$$y_2 = \frac{1}{2} (x_1 + x_2) + \frac{\| x_3 - x_1 \|}{2 \| x_2 - x_1 \|} (x_2 - x_1), \quad y_1 = \frac{1}{2} (x_1 + x_2) - \frac{\| x_3 - x_1 \|}{2 \| x_2 - x_1 \|} (x_2 - x_1) .$$

Since $\| y_2 - x_2 \| < \frac{1}{2} \| x_2 - x_1 \|$, the point

$$z = - \frac{1}{2} (x_1 + x_2) - (y_2 - x_2)$$

lies in the interior of the segment $\overline{-x_1, -x_2}$ and therefore

$$\| y_2 + \frac{1}{2}(x_1 + x_2) \| = \| x_2 - z \| = 2 = \| x_3 - (-x_2) \| .$$

Further we are able to prove that the segments

$\overline{y_1, -\frac{1}{2}(x_1 + x_2)}$ and $\overline{x_1, -x_2}$

have the same length. The points $x_1, -x_2, x_3$ are contained in the cell of radius 1 , but the points $y_1, y_2,$ $-\frac{1}{2}(x_1 + x_2)$ are not contained in any cell of radius 1 ; which fact is a contradiction with the property (A) of the space Z .

Meanwhile the following statement was proved:

If Z is a two-dimensional subspace of X and if X has the strong Valentine intersection property, then the unit cell in Z is either an ellipse, or a parallelogram.

Let Σ be a unit sphere in a normed linear space with the strong Valentine property X . Let us denote by Δ the set of all intersections of Σ with the two-dimensional subspaces of X . Let ρ be the metric in X ; for $S \in \Delta$, $S' \in \Delta$ put

$$h(S, S') = \max \left(\sup_{x \in S} \rho(x, S'), \sup_{x' \in S'} \rho(x', S) \right).$$

The function h is a metric on Δ (so called Hausdorff metric). The mapping φ of $\Sigma \times \Sigma$ into Δ , which to every $(i, j) \in \Sigma \times \Sigma$ consignates the intersection Σ with the plane sp (i, j) is obviously continuous as the mapping into the metric space (Δ, h) . As the subsets

$\{ (i, j) \in \Sigma \times \Sigma : \varphi(i, j)$ is an ellipse $\}$ and

$\{ (i, j) \in \Sigma \times \Sigma : \varphi(i, j)$ is a parallelogram $\}$

are evidently open in $\Sigma \times \Sigma$, one of these sets must be empty.

Then for all spaces satisfying the condition (3) only one of the following situations can occur.

(α) The intersection Σ with every plane containing the origin is ellipse.

(β) The intersection Σ with every plane containing the origin is a parallelogram.

The situation (β) can occur only if X is a two-dimensional space $X = \ell_2^{\infty}$.

If the (α) occurs, very two-dimensional subspace of X is Euclidean and therefore in a consequence of [3] p.115 (JN_1) X is an inner product space.

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LA MÉTHODE DU REPÉRAGE DES SYSTÈMES DE SOUS - VARIÉTÉS

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R.N. Ščerbakov et ses collaborateurs (Tomsk, l'U.R.S.S.) ont élaboré dans une série des cas concrets la méthode du repérage de sous-variété dont la déscription générale est donnée dans le travail [6]. Nous présentons certaine extension resp. généralisation de cette méthode qui permet d'étudier simultanément une groupe de sous-variétés de dimension quelconque plongées dans une variété donnée. Un exemple de l'application concrète de notre méthode est montré dans la deuxième partie du présent exposé.

1. La partie générale

Dans un espace projectif P_n à n dimensions considérons un repère mobile R formé de n + 1 points analytiques linéairement indépendants $A_1, \ldots, A_{n+1}$. Le déplacement infinitésimal du repère R est défini par un système d'équations différentielles qui expriment les différentielles dA_i des points fondamentaux du repère R comme des combinaisons linéaires de ces points. Les coefficients ω du système mentionné sont des formes de Pfaff en différentielles des paramètres dont dépend la position du repère R dans l'espace P_n . Les formes ω satisfont aux équations de structure bien connues d'un espace projectif.

Nous entendons sous le nom <u>figure fondamentale</u> de l'espace P_n un ensemble formé par un nombre fini de sous-espaces linéaires de dimension quelconque de P_n. Nous admettons pour plus de simplicité que la figure fondamentale ne soit pas trop étendue en ce sens qu' elle peut être complétement déterminée par des points linéairement indépendants en nombre de $h \leqslant n$.

Les considérations suivantes sont consacrées à l'étude générale des variétés engendrées par les figures fondamentales par la méthode du repère mobile. Pour cela, nous introduisons d'une manière naturelle la notion de variété $\mathcal{V}_p$ à p dimensions de la classe $\mathcal{C}^k$ de figures fondamentales. En supposant que la figure fondamentale soit déterminée par les points analytiques $M_1, \ldots, M_h$ la variété $\mathcal{V}_p$ se trouve définie par un système de fonctions $M_1 = M_1(u^1, \ldots, u^p)$, $\ldots$, $M_h = M_h(u^1, \ldots, u^p)$ de la classe $\mathcal{C}^k$, soumises aux conditions habituelles.

A chaque figure fondamentale de $\mathcal{V}_p$ nous faisons correspondre un repère mobile R de manière que les points A_1, $\ldots$, A_h du repère en question se confondent avec les points $M_1, \ldots, M_h$. Les repères attachés d'une telle façon à la variété $\mathcal{V}_p$ dépendent de p paramètres principaux $u^1, \ldots, u^p$ et d'un certain nombre de paramètres secondaires [1]. La condition que la figure fondamentale reste fixe en position par rapport au mouvement du repère correspondant R se traduit par le fait qu'un nombre convenable q de composantes $\omega^1, \ldots, \omega^q$ du repère ne dépend que des différentielles des paramètres principaux. Donc, les formes en question sont des formes principales de la variété $\mathcal{V}_p$ et le système d'équations de Pfaff

(1) $$\omega^1 = 0, \ldots, \omega^q = 0$$

est complètement intégrable [3], parce qu'il s'agit du système des équations d'un sous-groupe stacionnaire de la figure fondamentale.

La variété $\mathcal{V}_p$ étant à p dimensions, on a $q \geqslant p$ et il en découle que les formes principales sont liées par $q - p$ rélations linéaires de la forme

$$(2) \qquad \omega^{\beta} = L^{\beta}_{\alpha}\, \omega^{\alpha} \qquad (\alpha = 1,\dots, p \ ; \ \beta = p+1,\dots,q) .$$

Les équations (2) servent de point de départ à l'étude de la variété $\mathcal{V}_p$. Les formes résiduelles, en nombre de $N - q$, où N désigne le nombre des composantes indépendantes d'un repère mobile général, peuvent s'écrire sous la forme

$$(3) \qquad \omega^{\gamma} = L^{\gamma}_{\alpha}\, \omega^{\alpha} + e^{\gamma} \qquad (\gamma = q + 1,\dots, N) ,$$

les e^{γ} étant des formes secondaires dont dépend la position du repère R attaché à la figure fondamentale de la variété $\mathcal{V}_p$. En annulant les formes secondaires en vertu des choix particuliers du repère R on réalise le passage successif au repère canonique de la variété $\mathcal{V}_p$. Ce procédé étant terminé tous le coefficient L^{β}_{α} , L^{γ}_{α} devient soit constants soit invariants de la variété $\mathcal{V}_p$ [1].

Cela étant, considérons une "sous-variété" $\mathcal{W}_m$ à m dimensions plongée dans la variété $\mathcal{V}_p$ et supposons qu'elle soit définie par un système d'équations de Pfaff de la forme

$$(4) \qquad \bar{\omega}^{1} = 0, \ \dots, \ \bar{\omega}^{p-m} = 0 ,$$

où $\bar{\omega}^{1} ,\dots, \bar{\omega}^{p-m}$ sont des combinaisons linéaires indépendantes des formes $\omega^1,\dots,\omega^p$. On dit que la sous-variété $\mathcal{W}_m$ est holonome, si le système (4) est complètement intégrable. La solution du système en question détermine dans le cas considéré un ensemble, dépendant de $p - m$ paramètres, de sous-variétés à m dimensions contenues dans $\mathcal{V}_p$. Au contraire, la sous-variété $\mathcal{W}_m$ s'appelle anholonome, si le système (4) n'est pas complètement intégrable; on peut trouver par ex. dans [6] une déscription plus détaillée des variétés anholonomes. Dans ce qui suit nous nous bornons au cas des sous-variétés holonomes.

<u>Le voisinage différentiel du premier ordre d'une sous-variété</u> $\mathcal{W}_m$ détermine certains objets géométriques en relation invariante avec $\mathcal{W}_m$. Il y a avantage, pour l'étude de la sous-variété $\mathcal{W}_m$, à choisir une partie convenable du repère R dans les objets mentionnés. Le choix considéré du repère mobile et ainsi l'adjonction du repère à une sous-variété concrète a pour conséquence que les formes secondaires indépendantes $\tilde{e}^1, \ldots, \tilde{e}^{m'}$, en nombre convenable m' , s'annullent identiquement. Les annulations spéciales des formes secondaires en question conduisent aux sous-variétés du type spécial. Au contraire, si l'on laisse de coté une annulation quelconque de ces formes et si l'on admet que les formes en question sont tout à fait libres, on peut associer la partie correspondante du repère R avec une sous-variété $\mathcal{W}_m$ arbitraire.

La déscription géométrique générale de la construction précédente est un peu indéterminée et elle dépend essentiellement de la forme de la figure fondamentale. Pour cette raison nous préféreons, dans des considérations suivantes, une déscription analytique de la construction indiquée.

La sous-variété $\mathcal{W}_m$ de $\mathcal{V}_p$ étant choisie d'une manière bien déterminée, nous pouvons supposer, sans restreindre la généralité, qu'il soit possible de choisir le repère mobile attaché à $\mathcal{W}_m$ de telle manière que la sous-variété $\mathcal{W}_m$ est donnée par les équations

(5) $$\omega^{m+1} = 0 , \ldots , \omega^p = 0 .$$

Nous allons rechercher l'influence des changements des paramètres secondaires sur les équations (5) de $\mathcal{W}_m$.

Le système (1) étant complètement intégrable les différentielles extérieures des formes ω^i $(i = 1, \dots, q)$ prennent la forme

(6) $d\omega^i = \varphi^i_j \wedge \omega^j \quad (i, j = 1, \dots, q)$

(d est le symbole pour la différentielle extérieure), où φ^i_j sont des formes de Pfaff convenables. On en obtient, en portant des relations (2) dans les premières p équations (6), les formules

(7) $d\omega^\alpha = \varphi^\alpha_{\alpha'} \wedge \omega^{\alpha'} + \varphi^\alpha_\beta \wedge L^\beta_{\alpha'} \omega^{\alpha'} = \vartheta^\alpha_{\alpha'} \wedge \omega^{\alpha'}$

$(\alpha, \alpha' = 1, \dots, p; \ \beta = p + 1, \dots, q)$,

où l'on a posé $\vartheta^\alpha_{\alpha'} = \varphi^\alpha_{\alpha'} + \varphi^\alpha_\beta L^\beta_{\alpha'}$. Il en résulte, d'après les formules bien connues

$$d\omega^\alpha = d\,\omega^\alpha(\delta) - \delta\omega^\alpha(d), \quad \vartheta^\alpha_{\alpha'} \wedge \omega^{\alpha'} = \begin{vmatrix} \vartheta^\alpha_{\alpha'}(d) & \omega^{\alpha'}(d) \\ \vartheta^\alpha_{\alpha'}(\delta) & \omega^{\alpha'}(\delta) \end{vmatrix}$$

et en vertu de $\omega^\alpha(\delta) = 0$, les équations suivantes

(8) $\delta\omega^\alpha(d) = \vartheta^\alpha_{\alpha'}(\delta)\,\omega^{\alpha'}(d)$.

Supposons que les formes secondaires $\vartheta^\alpha_{\alpha'}(\delta)$ soient linéairement indépendantes. Cela étant, le système des équations (5) est invariante, si les variations des formes $\omega^{m+1}, \dots$ $\dots, \omega^p$ peuvent être exprimées comme des combinaisons linéaires des mêmes formes, c'est-à-dire si $\vartheta^\tau_\sigma(\delta) = 0$ $(\sigma = 1, \dots, m; \ \tau = m + 1, \dots, p)$. Les formes secondaires en question, en nombre de $m' = m(p - m)$, jouent le rôle des formes $\tilde{e}^1, \dots, \tilde{e}^{m'}$ dont nous avons parlé auparavant.

Considérons maintenant un système $\mathcal{S}$ de sous-variétés ${}^1\mathcal{W}_{m_1}, \dots, {}^\ell\mathcal{W}_{m_\ell}$ dont les dimensions satisfont à la relation $m_1 + \dots + m_\ell = p$. Supposons qu'il soit possible de choisir le repère mobile R attaché à la variété $\mathcal{V}_p$ de manière que les sous-variétés particulières ${}^{1}\mathcal{W}_{m_1}, \ldots, {}^{\ell}\mathcal{W}_{m}$ sont exprimées par les équations différentielles suivantes

$$ {}^{1}\mathcal{W}_{m_1}: \ \omega^{m_1+1} = 0 \ , \ \ldots, \ \omega^{p} = 0 $$

$$ {}^{2}\mathcal{W}_{m_2}: \ \omega^{1} = 0, \ \ldots, \ \omega^{m_1} = 0, \ \omega^{m_1+m_2+1} = 0, \ \ldots $$
$$ \ldots, \omega^{p} = 0 $$

$$ {}^{\ell}\mathcal{W}_{m_\ell}: \ \omega^{1} = 0 \ , \ \ldots, \ \omega^{m_1+m_2+\ldots+m_{\ell-1}} = 0 \ . $$

En appliquant la considération précédente à chaque des sous-variétés du système $\mathcal{S}$ on obtient un système de $M = m_1' + \ldots + m'$ formes secondaires indépendantes. Si l'on annule toutes les formes en question, les objets déterminées par le voisinage différentiel du premier ordre de toutes les sous-variétés du système devient fixés en position. L'ensemble des formes secondaires ayant la propriété précédente s'appelle <u>système caractéristique</u> de $\mathcal{S}$ et le repère mobile R qui s'obtient en faisant toutes les autres formes secondaires soit nulles soit des combinaisons linéaires de formes du système caractéristique s'appelle <u>repère semicanonique de la variété $\mathcal{V}_p$ par rapport au système de sous-variétés $\mathcal{S}$</u> .

La construction du repère semicanonique de la variété $\mathcal{V}_p$ se réalise par la méthode du prolongement successif du système d'équations différentielles (2). On obtient ainsi les équa - tions de la forme

(9) $$ \delta B_\rho = f_\gamma (B_\rho) \, e^{\gamma} \ , $$

les B_ρ étant des coefficients de ω^α qui se produisent au cours du prolongement mentionné. En construisant le repère semicanonique il faut séparer des relations (9) des combinaisons qui ne dépendent pas essentiellement des formes $\tilde{e}^{\gamma'}$ du système caractéristique, cela veut dire les combinaisons qui ont la forme

$$(10) \qquad \delta g_\lambda (B_\rho) = f_{\gamma'} (B_\rho)\, \tilde{e}^{\gamma'} + f_{\gamma''} (B_\rho)\, \tilde{e}^{\gamma''} ,$$

$$f_{\gamma'} = \rho^{\mu}_{\gamma'} \, , \; g_\mu \, ,$$

les $\tilde{e}^{\gamma''}$ étant des formes qui n'appartient pas au système caractéristique. En partant de ces formes on peut réaliser les choix particuliers du repère attaché à la variété $\mathcal{V}_p$ jusqu'au moment où toutes les formes secondaires $\tilde{e}^{\gamma''}$ sont où nulles où des combinaisons linéaires de formes du système caractéristique. Dans ce cas, on peut ajouter les paramètres secondaires, en nombre de M, au nombre des fonctions inconnues dont dépend la variété $\mathcal{V}_p$ avec le système de sous-variétés $\mathcal{S}$ et considérer toutes les formes du système caractéristique comme nulles. Les relations (2),(3) prennent ensuite la forme

$$(11) \qquad \omega^{\varkappa} = L^{\varkappa}_{\alpha}\, \omega^{\alpha} \qquad (\alpha = 1, \ldots, p \; ; \; \varkappa = p + 1, \ldots, N)$$

et elles forment, avec les conditions d'intégrabilité correspondantes, le système d'équations différentielles qui définie la variété $\mathcal{V}_p$ en commun avec le système de sous-variétés $\mathcal{S}$. Nous appellerons le système en question <u>système fondamental d'équations différentielles de la variété $\mathcal{V}_p$ rapportée au système $\mathcal{S}$</u> . Le degré de généralité de la solution du système fondamental est plus grand de M fonctions arbitraires de p arguments que le degré de généralité de la variété $\mathcal{V}_p$.

Les coefficients $L^{\varkappa}_{\alpha}$ des équations (11) sont des invariants du repère semicanonique, ainsi que <u>des invariants du système $\mathcal{S}$ de sous-variétés de la variété $\mathcal{V}_p$</u> . Chaque relation entre les invariants $L_{\alpha}^{\varkappa}$ peut être régardée comme "une déscription naturelle" d'une classe de systèmes $\mathcal{S}$. Il se pose la question d'examiner l'existence et le degré de généralité de certaines classes spéciales de systèmes $\mathcal{S}$ et de construire les repères semicanoniques correspondants. En outre, quelques classes spéciales de systèmes $\mathcal{S}$ n'existent que sur les variétés $\mathcal{V}_p$ spéciales de sorte qu'il est possible de caractériser quelques classes particulières de variétés $\mathcal{V}_p$.

Nous dirigerons notre attention sur deux types importants du système $\mathcal{S}$ sur la variété $\mathcal{V}_p$.

Soit $m_1 = \dots = m = 1$ et alors $\ell = p$. Dans le cas considéré le système $\mathcal{S}$ se compose de p sous-variétés indépendantes de dimension 1 . Parce qu'un système de $p - 1$ équations de Pfaff linéairement indépendantes en différentielles de p variables est nécessairement complètement intégrable, il s'agit donc de la construction d'un repère semicanonique de la variété $\mathcal{V}_p$ rapportée à p systèmes indépendants à $p - 1$ paramètres de sous-variétés à une dimension de figures fondamentales.

Soit $m_1 = m$, $m_2 = p - m$ et alors $\ell = 2$. Le cas en question est une généralisation immédiate de la construction décrite par R.N. Ščerbakov [6], qui étudie une seule sous-variété $\mathcal{W}_m$ plongée dans une variété $\mathcal{V}_p$ donnée. Il est possible, en partant du voisinage de premier ordre de la sous-variété $\mathcal{W}_m$, d'attacher une partie convenable du repère R à la variété $\mathcal{W}_m$ en annullant certaines formes secondaires en nombre de $m(p - m)$. La construction du repère semicanonique de la variété $\mathcal{V}_p$ par rapport à la sous-variété $\mathcal{W}_m$

se réalise de cette manière que les formes secondaires mentionnées restent tout à fait indéterminées tandis que toutes les autres formes secondaires s'annullent identiquement au cours du prolongement successif du système initial de la forme (2). Cela exige, naturellement, d'attacher la seconde partie du repère mobile considéré à une sous-variété $\overline{\mathcal{W}}_{p-m}$ convenable qui n'est pas choisie d'une manière quelconque mais, au contraire, qui est "conjuguée" avec la sous-variété $\mathcal{W}_m$. Le système $\mathcal{S}$ formé de deux sous-variétés $\mathcal{W}_m$, $\overline{\mathcal{W}}_{p-m}$ et considéré par R.N. Ščerbakov par voie indirecte est donc spécial - "conjuguée", tandis que la méthode que nous avons exposée dans les considérations précédentes permet d'étudier un tel système dans le cas tout à fait général. D'ailleurs, la construction du repère semicanonique d'une surface par rapport à un réseau général dans un espace projectif à trois dimensions, qui fait l'objet de l'étude dans la seconde partie de cet article, est réalisée de telle manière que le repère général en question passe, dans le cas spécial d'un réseau conjugué, dans le repère considéré par R.N. Ščerbakov dans son Mémoire [5].

2. <u>Repère semicanonique d'une surface par rapport à un réseau de courbes</u> (par J. Kolář)

Dans ce qui suit, nous nous proposons d'effectuer des calculs concrets dans un des cas les plus simples. Plus précisement, nous allons construire, dans un espace projectif P_3 à trois dimensions, le repère semicanonique d'une surface $\mathcal{V}_2$ par rapport à un réseau général $\mathcal{S}$ de courbes. La figure fondamentale est donc formée par un seul sous-espace linéaire de dimension 0 et il s'agit du cas où $n = 3$, $p = 2$, $\ell = 2$, $m_1 = m_2 = 1$.

Considérons, dans l'espace P_3 en question, un repère mobile R formé de points A_0, A_1, A_2, A_3 linéairement indépendants et supposons que les coordonnées homogènes de ces points soient normalisées de telle façon que $[A_0A_1A_2A_3] = 1$. On a alors les équations

$$(1) \qquad dA_i = \omega_i^j A_j \qquad (i, j = 0,1,2,3)$$

dont les coefficients satisfont aux équations de structure

$$(2) \qquad \underline{d}\,\omega_i^j = \omega_i^k \wedge \omega_k^j, \quad (i,j,k = 0,1,2,3)$$

et à la condition suivante

$$(3) \qquad \omega_0^0 + \omega_1^1 + \omega_2^2 + \omega_3^3 = 0 .$$

Attachons à chaque point $P(u^1, u^2)$ de la surface $\mathcal{V}_2$ les repères R tels que le point A_0 se confond avec le point P. Les repères en question dépendent de deux paramètres principaux u^1, u^2 et de 12 paramètres secondaires $v^1, \dots, v^{12}$. Au cours de la construction du repère canonique de la surface tous les paramètres secondaires devient des fonctions des paramètres principaux. En construisant le repère semicanonique de la surface par rapport au réseau de courbes nous procédons par la méthode expliquée dans les considérations précédentes.

Le point A_0 étant fixe, les formes ω_0^1, ω_0^2, ω_0^3 s'annullent en vertu de $du^1 = du^2 = 0$. Il en résulte que les formes en question ne dépendent que des différentielles des paramètres principaux et qu'elles apparaissent ainsi comme formes principales de la surface. En choisissant les points A_1, A_2 dans le plan tangent de la surface au point A_0 on a

(4) $$\omega_0^3 = 0$$

et les formes principales $\omega^1 = \omega_0^1$, $\omega^2 = \omega_0^2$ sont linéairement indépendantes.

Cela étant, appliquons les équations de structure à la formule (4). On en obtient ensuite, en vertu de lemme de Cartan, les relations

(5) $$\omega_1^3 = a\,\omega^1 + b\,\omega^2\,, \qquad \omega_2^3 = b\,\omega^1 + c\,\omega^2\,.$$

En répétant le procédé précédant nous obtenons les équations

$$da + a(\omega_0^0 + \omega_3^3 - 2\,\omega_1^1) - 2\,b\,\omega_1^2 = e\omega^1 + f\,\omega^2,$$

(6) $$db + b(\omega_0^0 + \omega_3^3 - \omega_1^1 - \omega_2^2) - a\omega_2^1 - c\,\omega_1^2 = f\omega^1 + g\omega^2\,,$$

$$dc + c(\omega_0^0 + \omega_3^3 - 2\,\omega_2^2) - 2\,b\,\omega_2^1 = g\omega^1 + h\,\omega^2,$$

qui donnent, en désignant par δ le symbole de différentiation obtenue en laissant u^1, u^2 fixes, les formules suivantes

$$\delta a + a(e_0^0 + e_3^3 - 2\,e_1^1) - 2\,b\,e_1^2 = 0$$

(7) $$\delta b + b(e_0^0 + e_3^3 - e_1^1 - e_2^2) - a\,e_2^1 - c\,e_1^2 = 0$$

$$\delta c + c(e_0^0 + e_3^3 - 2\,e_2^2) - 2\,b\,e_2^1 = 0\,.$$

Le voisinage différentiel du premier ordre d'une courbe sur la surface détermine une droite située dans le plan tangent de la surface, à savoir sa tangente. Nous dirons que le repère R est attaché au réseau $\mathcal{S}$ situé sur la surface si, dans une position quelconque du point A_0 sur la surface, les arêtes $[A_0A_1]$, $[A_0A_2]$ du repère R se confondent avec les tangentes correspondantes des courbes formant le réseau $\mathcal{S}$ en question. Or, on a

$$\delta[A_0A_1] = (e_0^0 + e_1^1)\,[A_0A_1] + e_1^2\,[A_0A_2]$$

$$\delta[A_0A_2] = e_2^1 [A_0A_1] + (e_0^0 + e_2^2) [A_0A_2]$$

et il en découle que les formes e_1^2, e_2^1 forment le système caractéristique de formes secondaires. En annullant les formes mentionnées on attache le repère R à un réseau concret. Au contraire, au cours de la construction du repère semicanonique de la surface par rapport à un réseau $\mathcal{S}$ arbitraire les formes e_1^2, e_2^1 doivent rester tout à fait indéterminées.

En multipliant les équations (6) successivement par $-c$, $2b$, $-a$ et en faisant la somme des équations ainsi obtenues on a

(8) $$d(b^2 - ac) + 2(b^2 - ac)(\omega_0^0 + \omega_3^3 - \omega_1^1 - \omega_2^2) = \\ = 2F\omega^1 + 2G\omega^2 ,$$

où

(9) $$2F = 2bf - ag - ce , \quad 2G = 2bg - ah - cf .$$

Il en résulte que

$$\delta(b^2 - ac) + 2(b^2 - ac)(e_0^0 + e_3^3 - e_1^1 - e_2^2) = 0$$

et on a $b^2 - ac \neq 0$ pour des points non-paraboliques de sorte que, en nous bornant aux points en question, on peut poser

(10) $$b^2 - ac = 1 .$$

Nous avons ensuite, d'après (8) et (10),

(11) $$\omega_0^0 + \omega_3^3 - \omega_1^1 - \omega_2^2 = F\omega^1 + G\omega^2$$

et en prolongeant l'équation précédente nous obtenons les relations

(12) $$dF + F(\omega_0^0 - \omega_1^1) - G\omega_1^2 + 2\omega_1^0 - 2a\omega_3^1 - 2b\omega_3^2 = \\ = 2H\omega^1 + 2K\omega^2$$

$$dG + G(\omega_0^0 - \omega_2^2) - F\omega_2^1 + 2\omega_2^0 - 2b\omega_3^1 - 2c\omega_3^2 = \\ = 2K\omega^1 + 2L\omega^2 .$$

On a alors

$$\delta F + F(e_0^0 - e_1^1) - G\, e_1^2 + 2\, e_1^0 - 2\, a\, e_3^1 - 2\, b\, e_3^2 = 0$$

$$\delta G + G(e_0^0 - e_2^2) - F\, e_2^1 + 2\, e_2^0 - 2\, b\, e_3^1 - 2\, c\, e_3^2 = 0$$

et on en voit qu'il est possible de poser

(13) $$F = 0\ ,\quad G = 0\ .$$

En vertu de (13) les équations (3) et (11) donnent

(14) $$\omega_0^0 + \omega_3^3 = 0\ ,\qquad \omega_1^1 + \omega_2^2 = 0\ ,$$

et les formules (12) prennent la forme suivante

(15)
$$\omega_1^0 - a\omega_3^1 - b\,\omega_3^2 = H\omega^1 + K\omega^2$$
$$\omega_2^0 - b\omega_3^1 - c\,\omega_3^2 = K\omega^1 + L\omega^2\ .$$

Le prolongement ultérieur fournit les relations

$$dH + 2\,H(\omega_0^0 - \omega_1^1) - 2\,K\,\omega_1^2 + 2\,a\,\omega_3^0 - e\,\omega_3^1 - f\,\omega_3^2 = M\,\omega^1 + N\,\omega^2$$

$$dK + 2\,K\omega_0^0 - H\omega_2^1 - L\omega_1^2 + 2\,b\omega_3^0 - f\omega_3^1 - g\omega_3^2 = N\omega^1 + P\omega^2$$

$$dL + 2\,L(\omega_0^0 - \omega_2^2) - 2\,K\omega_2^1 + 2\,c\omega_3^0 - g\omega_3^1 - h\omega_3^2 = P\omega^1 + Q\omega^2\ .$$

Il en résultent les formules

$$\delta H + 2\,H(e_0^0 - e_1^1) - 2\,K\,e_1^2 + 2\,a\,e_3^0 - e\,e_3^1 - f\,e_3^2 = 0$$

$$\delta K + 2\,K\,e_0^0 - H\,e_2^1 - L\,e_1^2 + 2\,b\,e_3^0 - f\,e_3^1 - g\,e_3^2 = 0$$

$$\delta L + 2\,L(e_0^0 - e_2^2) - 2\,K\,e_2^1 + 2\,c\,e_3^0 - g\,e_3^1 - h\,e_2^2 = 0$$

qui permettent, en excluant les surfaces réglées, de poser

(17) $$H = 0\ ,\quad K = 0\ ,\quad L = 0\ .$$

Cela étant, les équations (15) et (16) ont, en vertu de (17), la forme suivante

(18) $$\omega_1^0 = a\omega_3^1 + b\,\omega_3^2$$

$$\omega_2^0 = b\omega_3^1 + c\omega_3^2$$

et

$$2\,a\omega_3^0 - e\omega_3^1 - f\omega_3^2 = M\omega^1 + N\omega^2$$

$$(19)\quad 2\,b\omega_3^0 - f\omega_3^1 - g\omega_3^2 = N\omega^1 + P\omega^2$$

$$2\,c\omega_3^0 - g\omega_3^1 - h\omega_3^2 = P\omega^1 + Q\omega^2 .$$

Les particularisations précédentes entraînent que les formes secondaires uniques qui restent encore non-nulles sont $e_0^0 = - e_3^3$, $e_1^1 = - e_2^2$, e_1^2 , e_2^1 . En changeant les paramètres secondaires qui restent à notre disposition les arêtes $[A_0A_3]$, $[A_1A_2]$ et le sommet A_3 du repère sont fixes en position. Nous déduirons leurs signification géométrique en comparant le repère considéré avec le repère canonique de la surface. Pour cela, nous pouvons annuler les formes secondaires en posant, d'après (7) et (20), $a = 0$, $c = 0$, $e = - 2$, $h = - 2$. Un calcul facile montre que le repère ainsi obtenu se confond avec le repère canonique de la surface [5]. Or, on sait que $[A_0A_3]$ et $[A_1A_2]$ sont les directrices de Wilczynski et A_3 le point d'intersection de la première directrice avec la quadrique de Lie. Mais, les objets en question ne se changent pas au cours de la particularisation auxiliaire et les considérations précédentes donnent alors leurs signification géométrique.

Pour faire suite à nos recherches précédentes nous allons normaliser les coordonnées du point A_0 de la même manière que dans le cas du repère canonique. Pour cela, nous déduirons tout d'abord les variations des fonctions e, f, g, h et nous obtiendrons, en vertu de (6), les relations

$$\delta e + e(e_0^0 - 3\, e_1^1) - 3\, f\, e_1^2 = 0$$

(20)

$$\delta f + f(e_0^0 - e_1^1) - e\, e_2^1 - 2\, g\, e_1^2 = 0$$

$$\delta g + g(e_0^0 + e_1^1) - 2\, f\, e_2^1 - h\, e_1^2 = 0$$

$$\delta h + h(e_0^0 + 3\, e_1^1) - 3\, g\, e_2^1 = 0\,.$$

En posant

(21) $V = e^2h^2 - 3\, f^2g^2 + 4\, e\, g^3 + 4\, f^3h - 6\, e\, f\, g\, h$

on obtient, d'après (20), $\delta V + 4\, e_0^0\, V = 0$. Les surfaces reglées étant exclues on a $V \neq 0$ et on peut normaliser le point A_0 et même le point A_3 de manière que $V = 16$ ce qui donne ensuite $e_0^0 = 0$. Remarquons que cette normalisation ne diffère pas de la normalisation employée à la construction du repère canonique de la surface.

En vertu de la particularisation précédente les formes secondaires e_1^2, e_2^1, $e_1^1 = - e_2^2$ restent libres et elles déterminent la position des points A_1, A_2, $A_1 + A_2$ sur la droite $[A_1A_2]$ fixe. Le repère construit est un repère semi-canonique de la surface par rapport à une trois-couche des courbes. Une telle trois-couche de courbes sur la surface étant donnée nous pouvons, à l'aide des paramètres secondaires réstants, fixer les points A_1, A_2, $A_1 + A_2$ sur les tangentes des courbes contenues dans les couches particulières. En proposant d'examiner une trois-couche arbitraire sur la surface, on peut considérer les paramètres secondaires en question comme fonctions arbitrairement choisie . Cela étant, les formes secondaires résiduelles s'annullent, toutes les formes du repère sont formes principales et tous les coefficients sont des invariants de la trois-couche de courbes sur la surface.

En considérant la construction du repère canonique du réseau $\mathcal{S}$ sur la surface il nous faut choisir encore le paramètre secondaire qui correspond à la forme e_1^1. Pour ce but nous allons procéder par la voie géométrique.

Les courbes asymptotiques sur la surface satisfont à l'équation différentielle

(22) $$[A_0 A_1 A_2 d^2 A_0] = a(\omega^1)^2 + 2\, b\, \omega^1 \omega^2 + c(\omega^2)^2 = 0$$

tandis que le réseau $\mathcal{S}$ est donné par l'équation $\omega^1 \omega^2 = 0$. Considérons le réseau de courbes qui sépare harmoniquement le réseau $\mathcal{S}$ et le réseau asymptotique et qui est déterminé par l'équation

(23) $$\begin{vmatrix} a\,\omega^1 + b\,\omega^2 & b\,\omega^1 + c\,\omega^2 \\ \omega^2 & \omega^1 \end{vmatrix} = a(\omega^1)^2 - c(\omega^2)^2 = 0.$$

Supposant en outre qu'aucune des deux couches du réseau $\mathcal{S}$ ne soit pas engendrée par les courbes asymptotiques de sorte que $a \neq 0$, $c \neq 0$. Il est aisé de voir que l'on peut choisir le point $A_1 + A_2$ sur la tangente d'une couche du réseau en question et que le choix considéré peut être réalisé en posant

(24) $$c = \varepsilon\, a\,, \quad \varepsilon = \pm 1$$

suivant que le réseau (23) est réel ou imaginaire.

On a ensuite, en vertu de (7) et (14) $e_1^1 = \frac{1}{2}\, \frac{b}{a}\,(\varepsilon\, e_2^1 - e_1^2)$ et ils ne restent que deux formes secondaires libres e_1^2, e_2^1. Un réseau $\mathcal{S}$ étant choisi sur la surface en question on peut annuler ces deux formes en choisissant les points A_1, A_2 sur les tangentes des courbes du réseau $\mathcal{S}$. Si l'on annulle les formes e_1^2, e_2^1 d'une manière tout à fait arbitraire, on obtient le repère semicanonique de la surface par rapport à un réseau $\mathcal{S}$ quelconque. Les équations différentielles correspondantes sont

$$dA_0 = \omega_0^0 A_0 + \omega^1 A_1 + \omega^2 A_2$$

$$dA_1 = (a\omega_3^1 + b\omega_3^2) A_0 + \omega_1^1 A_1 + \omega_1^2 A_2 + + (a\omega^1 + b\omega^2) A_3$$

(25)

$$dA_2 = (b\omega_3^1 + \varepsilon a\omega_3^2) A_0 + \omega_2^1 A_1 - \omega_1^1 A_2 + + (b\omega^1 + \varepsilon a\omega^2) A_3$$

$$dA_3 = \omega_3^0 A_0 + \omega_3^1 A_1 + \omega_3^2 A_2 - \omega_0^0 A_3$$

où, d'après (10) et (24),

(26) $$b^2 - \varepsilon a^2 = 1 .$$

Pour abréger, nous omettons les conditions d'intégrabilité des équations (25).

Pour $b = 0$, c'est-à-dire dans le cas spécial d'un réseau conjugué le repère (25) se confond avec le repère C considéré par R.N. Ščerbakov [5].

Tous les coefficients des équations (25) sont des invariants du réseau de courbes sur la surface. Nous donnerons leurs signification géométrique dans les cas les plus simples et nous poserons pour cela $\omega_i^j = a_i^j \omega^1 + b_i^j \omega^2$.

Les coefficients a, b liées par la rélation précédente (26), déterminent l'équation (22) et ils correspondent donc au rapport anharmonique des tangentes du réseau $\mathcal{S}$ et des tangentes asymptotiques.

La signification géométrique des coefficients a_1^2, b_1^2 , a_2^1 , b_2^1 est la suivante: Le plan tangent au point

$b_1^2 A_0 - A_1$, resp. $a_2^1 A_0 - A_2$ de la surface réglée , qui est engendrée par les tangentes $[A_0A_1]$, resp. $[A_0A_2]$ le long d'une des courbes de la couche $\omega^1 = 0$, resp. $\omega^2 = 0$, passe par la première directrice de Wilczynski. Le point $(b\, a_1^2 - a\, b_1^2) A_0 + a A_1$, resp. $(\varepsilon b\, b_2^1 - a\, a_2^1) A_0 + a A_2$ est le second foyer de la congruence $[A_0A_1]$, resp. $[A_0A_2]$.

Les coefficients a_3^1, b_3^2 ont la signification géométrique suivante: Le plan tangent au point $a_2^1 A_0 - A_3$, resp. $b_3^2 A_0 - A_3$ de la surface réglée engendrée par la première directrice de Wilczynkai le long d'une de courbes de la couche $\omega^2 = 0$, resp. $\omega^1 = 0$ passe par la droite $[A_0A_2]$, resp. $[A_0A_1]$.

L'étude plus détaillée du réseau de courbes sur la surface, fondée sur la méthode considérée dans ce travail, fait le contenu de l'article [7].

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ОБ ОДНОЗНАЧНОЙ ОПРЕДЕЛЕННОСТИ НЕКОТОРЫХ ПРОИЗВОЛЬНО ИСКРИВЛЕННЫХ ПОВЕРХНОСТЕЙ

К.К. МОКРИЩЕВ, Ростов на Дону

§ 1. Рассмотрим регулярную прверхность F

$$\bar{r} = \bar{r}(u^1, u^2), \qquad (u^1, u^2) \in \mathcal{D} .$$

Обозначим единичный вектор нормали, первый (метрический) и второй тензоры и среднюю кривизну этой поверхности соответственно через

$$\bar{\xi}\ , \quad g_{ik}\ , \quad B_{i,k}\ , \quad H\ .$$

Пусть $F^\times$ есть поверхность, изометричная F и отнесенная к тем же координатам u^1, u^2 (устанавливаемым по изометрии). Аналогичные с $F^\times$ связанные величины обозначим

$$\bar{\xi}^\times, \quad g_{ik}\ , \quad B^\times_{ik}\ , \quad H^\times .$$

Если в каждой точке $(u^1, u^2) \in \mathcal{D}$ будет

$$B^\times_{ik} = B_{ik}$$

тогда по основной теореме теории поверхностей [1] F и $F^\times$ будут или конгруентны, или симметричны.

Обозначим

$$A_{ik} = B_{ik} - B^\times_{ik}$$

Справедлива следующая, принадлежащая К.Р. Гротемейеру [2]

<u>Лемма</u>: если в каждой паре соответственных точек изометричных поверхностей F и $F^\times$ имеет место равенство $H = H^\times$, то справедливо соотношение

$$\| A_{ik} \| \leq 0$$

и из равенства $\| A_{ik} \| = 0$ следует $A_{ik} = 0$.

Если F и $F^{\times}$ будут замкнутыми рода нуль, то имеет место формула Г. Герлотца [3], которую можно рпедставить в виде

$$\iint_F \frac{\|A_{ik}\|}{g} P d\sigma = \iint_{F^{\times}} H^{\times} d\sigma^{\times} - \iint_F H d\sigma ,$$

где $g = g_{11} g_{22} - g_{12}^2 > 0$, $P = (\bar{\tau}\, \bar{\xi})$, $d\sigma = d\sigma^{\times}$ – соответственные элементы площади на F и $F^{\times}$.

§ 2. Будем рассматривать ограниченную замкнутую, регулярную поверхность вращения F, меридиан которой пересекает ось вращения только в двух точках S (южный полюс) и N (северный ролюс) и образует с нею в этих точках рпямой угол.

1. Предположим, что меридиан имеет только две точки перегиба A и B. Дугу AB меридиана, обращенную выпуклостью к оси вращения, будем называть участком вогнутости меридиана. На этом участке кривизна поверхности F –отрицательна, на параллелях, описиваемых точками A и B – равна нулю, а во всех остальных точках – положительна.

Могут представиться три случая.

1) Касательные a и b к меридиану соответственно в точках A и B пересекают ось вращения в точках A_1 и B_1 так, что $A_1 \check{C} A$ и $B_1 \check{C} B$, где $C \equiv a \times b$, а символ $A_1 \check{C} A$ означает, что точка C лежит между A_1 и A (рис. 1).

2) Касательные a и b пересекают ось вращения в точках A_1 и B_1 так, что $C \check{A}_1 A$ и $C \check{B}_1 B$ (рис. 2).

3) $A_1 \equiv B_1 \equiv C$, т-е $C \in Ox$.

В первом из этих случаев общую часть отрезков $A_1 B_1$ и SN обозначим через ω_1 и назовем отрезком глобальной жесткости, а поверхность F через F_1. Все внутенние точки отрезка ω_1 обладают тем свойством, что любой луч, исходящий из такой точки и встречающий меридиан, пересекнет его только в одной точке и образует с соответствующей нормалью к F_1, угол φ, удовлетворяющий или тольео условию $0 \le \varphi \le \frac{\pi}{2}$, или только условию $\frac{\pi}{2} \le \varphi \le \pi$.

Теорема 1. Если две трижды непрерывно дифференйируемые изометрические поверхности F_1 и F^* имеют в соответственных по изометрии точках $H_1 = H^*$, тогда эти поверхности или конгруентны, или симметричны.

Систему координат $Oxyz$ выберем так, чтобы $O \in \omega_1$ и ось Oz совпадала с осью вращения поверхности F_1. Отнесем поверхности F_1 и F^* к таким общим координатам u^1, u^2, что $u^1 = const$, будут параллелями на F_1, а $u^2 = const$. - ее меридианами. Пусть

$$\bar{r} = \bar{r}(u^1, u^2)$$

есть уравнение F_1. Предположим, что направления на линиях u^1 и u^2 выбраны так, что единичный вектор нормали $\bar{\xi}$ к F_1 направлен внутрь этой поверхности. Тогда

$$P = (\bar{r}\,\bar{\xi}) \le 0$$

во всех точках поверхности F_1. Формула Герглотца непосредственно дает

$$\iint\limits_{F_1} \frac{\|A_{ik}\|}{g} P d\sigma = 0 .$$

Отсюда, в силу леммы следует $A_{ik} = 0$, т - е $B^*_{ik} = B_{ik}$. Теорема доказана.

Эту теорему (как и все последующие) можно рассматривать как теорему о глобальной жесткости [4], или однозначной определенности [5] поверхности F_1.

II. Предположим терер, что меридиан поверхности F имеет не один, а n участков вогнутости и каждый из них обладает отрезком глобальной жесткости ω_k, $(k = 1, 2, \dots, n)$. Если общая часть всех этих отрезков есть отрезок ω_0 (отличный от нулевого), тогда поверхность F будем обозначать через F_n.

Теорема II. Если две трижды непрерывно дифференцируемые изометрические поверхности F_n и F^* имеют в соответственных по изометрии точках $H_n = H^*$, тогда эти поверхности или конгруентны, или симметричны.

Доказательство аналогично доказательству теоремы I.

III. Предположим, что меридиан поверхности F имеет только одну точку перегиба и "вмят" по направлению оси, т-е имеет форму, изображенную на рисунке 3.

Здесь дуга ABS, как и дуга NA – выпуклая. Точка B – точка максимума, A – точка перегиба.

В случае, когда касательня a к меридиану в точке A встречает отрезок SN во внутренней точке A_1, тогда поверхность F обозначаем F'. Отрезок A_1S будет отрезком глобальной жесткости. Имеет место

Теорема III. Если две трижды непрерывно дифференцируемые изометрические поверхности F' и F^* имеют в соответственных по изометрии точках $H' = H^*$, тогда эти поверхности или конгруентны, или симметричны.

IУ. Рассмотрим поверхнось F меридиан которой имеет две точки перегиба, но "вмят" у обоих полюсов, т-е имеет вид, изображенный на рис. 4.

Если общая часть отрезков A_1S и B_1N есть отрезок, отличный от нулевого, тогда F, обозначаем F''. Справедлива

Теорема IУ. Если две трижды непрерывно дифференцируемые изометрические поверхности F'' и F^* имеют в соответственных по изометрии точках $H = H^*$, тогда эти поверхниости или конгруентны или симметричны.

У. Комбинируя надлежащим образом рассмотренные выше случаи приходим к поверхности F^o, меридиан которой имеет, например, вид, изображенный на рис. 5. Нетрудно видеть, что если, любую внутреннюю точку отрезка B, C принять за начало координат, тогда функция P для всех точек поверхности F^o будет иметь значения одного знака и потому верна

Теорема У. Если две трижды непрерывно дифференцируемые изометрические поверхности F^o и F^* имеют в соответственных по изометрии точках $H^o = H^*$, тогда эти поверхности или конгруентны, или симметричны.

Отметим, наконец, что если вместо отрезков глобальной жесткости ввести надлежащим образом трехмерные области, то предыдущие теоремы можно распространить на некоторые поверхности не являющиеся поверхностями вращения.

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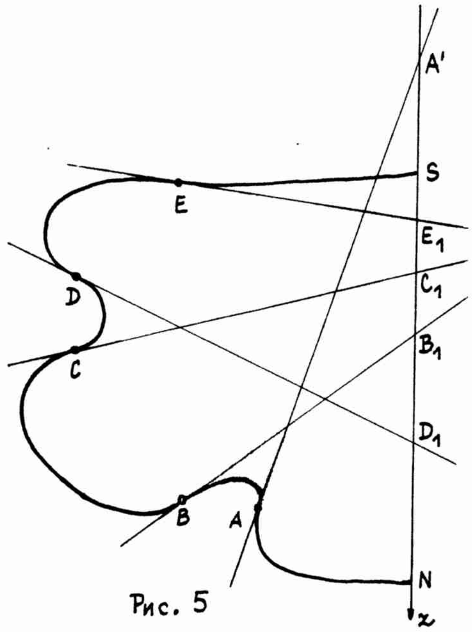

Рис. 5

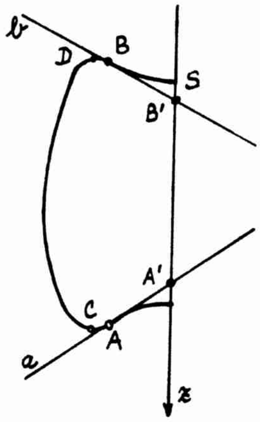

Рис. 4

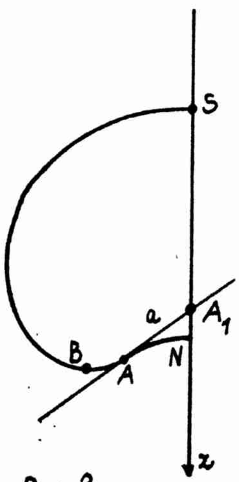

Рис. 3

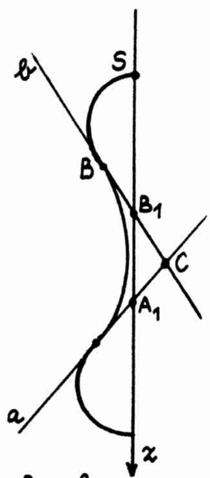

Рис. 2

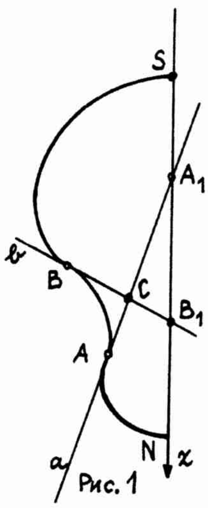

Рис. 1

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SUR LE PRODUIT SIMPLE DE DEUX DISTRIBUTIONS x)

J. JELÍNEK, Praha

Nous considérons les distributions réelles définies sur une partie ouverte G de l'espace euclidienne E^r. D'après [1] nous prenons une distribution f comme une forme linéaire continue sur $\mathcal{D}_G$. Nous désignons la valeur de cette forme dans le point $\varphi \in \mathcal{D}_G$ par $\int f(x)\,\varphi(x)\,dx$ ou $\int f\varphi$. Dans [2] le produit simple de deux distributions f, g sur une partie ouverte G de E^r est défini de la manière qui ne diffère pas essentiellement de celle-ci:

$$f(x)\, g(x) = \lim_{n\to\infty} f(x)\, g_n(x)$$

où g_n est une suite régulière convergente à g, si cette limite existe indépendante au choix de la suite régulière g_n.

La distribution $g_n = g * \rho_n$, où ρ_n est une suite régulière convergente à δ, n'y est pas définie sur l'ensemble G, mais seulement sur un ouvert $G_n \subset G$. Mais ce ne fait aucune difficulté parce que tout compact $K \subset G$ est contenu dans presque tous les G_n.

On peut facilement prouver que le produit simple ne diffère pas de celui d'après les définitions usuelles dans ces cas-ci:

1. f est de la forme $D^p\mu$ où μ est une mesure,

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x) Le texte complet sera publié dans Čech.mat.ž.

$p \geq 0$ (un système d'entiers), g est une fonction aux dérivées $D^q g$ continues pour tous les $q \leq p$, $q \geq 0$.

2. f est une fonction mesurable et bornée, g est une fonction localement intégrable.

3. f et g sont des fonctions intégrables qui sont des éléments d'espaces d'Orlič duals.

4. $x = (y, z)$, $f(x) = f_1(y)$, $g(x) = g_1(z)$. $f(x)\, g(x)$ est alors le produit direct $f_1(y) \times g_1(z)$.

Remarque 1. L'exemple cité à la fin de cette note montre que le produit simple de deux fonctions localement intégrables peut être une distribution qui n'est pas une fonction. Même le produit dans le sens usuel des fonctions en question est une fonction intégrable.

Le contenu de cette note est formé par deux caractérisations du produit simple de deux distributions (les théorèmes 1 et 2). A la fin de cette note, il y a quelques remarques sur l'associativité du produit simple.

Théorème 1. Soient f, g deux distributions définies (au moins) sur une partie ouverte G de l'espace E^r. Pour l'existence du produit simple fg sur G, il est nécessaire et suffisant qu'il y a un ouvert H dans l'espace $(E^r)_x \times (E^r)_y$, $H \supset (G)_x \times 0_y$, de sorte que $f(x)\, g(x - y)$ est une distribution qui est sur H une fonction localement intégrable par rapport à la variable y continue dans le point $y = 0$. Dans ce cas $f(x)\, g(x)$ est la section de la distribution $f(x)\, g(x - y)$ sur G dans le point $y = 0$.

Remarque 2. Le produit simple de la forme $f(x)\, g(x - y)$ est défini pour des distributions quelconques f, g ; on peut le définir de la manière classique par la formule:

$$\int f(x)\, g\,(x-y)\, \varphi\,(x, y)\, d\,x\, d\,y = $$
$$= \int f(x)\, [\int g(x-y)\, \varphi\,(x, y)\, d\,y]\, d\,x$$

car pour $\varphi(x, y) \in (\mathcal{D})_{x,y}$ on a $\int g(x-y)\, \varphi\,(x, y)\, d\,y \in$ $\in (\mathcal{D})_x$.

<u>Remarque 3</u>. L'énoncé "une distribution $F(x, y)$ (définie au moins sur H) est sur H une fonction localement intégrable par rapport à y " signifie que pour tout $\varphi \in (\mathcal{D})_x$, $\int F(x, y)\, \varphi\,(x)\, d\,x$ est une distribution qui est sur l'ensemble $\{y\ ;\ \text{support}\ \varphi \times (y) \subset H\}$ une fonction localement intégrable. De même on peut définir cet énoncé pour une fonction distributionnelle $F_y(x)$ où pour tout y pour lequel l'ensemble $\{x\ ;\ (x, y) \in H\}$ n'est pas vide, $F_y(x)$ est une distribution sur cet ensemble. L'équivalence de cette définition à celle qui est citée dans [3] est montrée dans [4]. De la façon analogique on définit la continuité de $F(x, y)$ par rapport à y et d'autres propriétés.

<u>Remarque 4</u>. Par un calcul facil on peut vérifier que, dans le cas où ρ_n est une suite régulière convergente à δ, on a pour $\varphi \in \mathcal{D}$

$$\iint f(x)\, g\,(x-y)\, \varphi\,(x)\, d\,x\ \ \rho_n(y)\, d\,y =$$
$$= \int [\, g_n(x)\, f(x)]\, \varphi\,(x)\, d\,x\ ,$$

où $g_n = g * \rho_n$ est une suite régulière convergente à g .

<u>Théorème 2</u>. Soient f, g, h des distributions sur une partie ouverte G de E^r . Pour qu'on ait $h = fg$, il est nécessaire et suffisant qu' à chaque intervale compact K dans G ($\text{int}\, K \neq \emptyset$) et à chaque voisinage du zéro $\mathcal{U}$ dans $\mathcal{D}'_K$ il existe $\lambda > 0$ de sorte que $\psi_1(x)\ ,\ \psi_2(x) \in \mathcal{D}_{\{|x| \leq \lambda\}}$, $\psi_1 \geq 0\ ,\ \psi_2 \geq 0\ ,\ \int \psi_1 = \int \psi_2 = 1 \Rightarrow h - (f * \psi_1)(g * \psi_2) \in$ $\in \mathcal{U}$ (naturellement on prend les distributions du second mem-

bre seulement sur K).

Pour démontrer le théorème 1 j'ai usé les théorèmes suivants concernant les distributions de deux variables.

Théorème 3. Soit H un ouvert dans $(E^r)_x \times (E^s)_y$, et soit $f(x, y)$ une distribution qui est sur l'ensemble H une fonction continue par rapport à la variable y . Alors il existe une fonction distributionnelle $f_y(x)$ sur H , continue par rapport à y et telle qu'on a $f_y(x) = f(x, y)$ sur H . Par ces conditions la fonction distributionnelle $f_y(x)$ est bien déterminée.

Remarque 5 . La dernière égalité signifie qu on a pour tout $\varphi \in \mathcal{D}_H$

$$\int f(x, y) \varphi (x, y) \, d x \, d y = \int [\int f_y(x) \varphi (x, y) \, d x] \, d y .$$

(Dans le second membre l'intégrale extérieure est celle d'une fonction continue dans le sens usuel).

Théorème 4. Soit $f(x, y)$ une distribution sur $P \times Q$ où $P \subset (E^r)_x$, $Q \subset (E^s)_y$ sont des intervalles compacts à l'intérieur non-vide. $f(x, y)$ soit, par rapport à y , une fonction mesurable et bornée. Il existe alors une fonction distributionnelle $f_y(x)$ $(x \in P , y \in Q)$ de manière que $f_y(x) = f(x, y)$ sur $P \times Q$, l'ensemble des distributions $\{f_y(x) ; y \in Q\}$ est borné dans $\mathcal{D}'_P$ et si pour un $\varphi \in \mathcal{D}_P$ la fonction $|\int f(x, y) \varphi (x) \, d x|$ est bornée par le nombre a sur un intervalle $Q^* \subset Q$, int $Q^* \neq \emptyset$ (dans le sens de l'égalité des fonctions et des distributions), on a le même pour la fonction $|\int f_y(x) \varphi (x) \, d x|$.

Théorème 5. Soit f une distribution sur $P \times Q$ où $P \subset (E^r)_x$, $Q \subset (E^s)_y$ sont des intervalles compacts à l'intérieur non-vide, $y_0 \in Q$. Supposons qu' à chaque $\varphi \in \mathcal{D}_P$

il y ait un intervalle $I(\varphi)$ étant un voisinage du point y_0, dans Q de sorte que sur $I(\varphi)$, $\int f(x, y)\, \varphi(x)\, dx$ est une fonction mesurable et continue dans le point y_0. Il y a alors un intervalle I étant un voisinage du point y_0 dans Q de sorte que sur $P \times I$, $f(x, y)$ est égal à une fonction distributionnelle $f_y(x)$ qui est, par rapport à la variable y, bornée et continue dans le point $y = 0$.

<u>Théorème 6</u>. Soient f, g des distributions telles qu'il existe fg sur un ouvert $G \subset E^r$. Soit μ une mesure qui est une fonction infiniment dérivable sauf le point 0 de sorte qu'il existe le produit de composition $g * \mu$ sur G. Alors il existe aussi $f.(g * \mu)$ sur G.

<u>Remarque 6</u>. La loi d'associativité n'est pas valable pour le produit simple de distributions, c'est-à-dire on n'a pas nécessairement $(fg)h = f(gh)$ même que les deux membres ont un sens, comme le montre un exemple classique sur E^1 : $f = \delta$, $g(x) = x$, $h(x) = \frac{1}{x}$ (une pseudofonction). Mais dans l'exemple cité, il n'existe pas fh. Le travail [2] contient des conditions suffisantes fondées sur les ordres des distributions f, g, h, pour la validité de la loi d'associativité. Mais il en résulte aussi l'existence fh, il pourrait donc sembler que l'existence fh seule fût une condition suffisante pour la validité de la loi d'associativité. Mais ce n'est pas vrai comme le montre l'exemple suivant.

<u>Exemple.</u> Soit f, g des fonctions sur E^1 déterminées par les formules-ci:

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{x}\,.\,(1-\log x)} & \text{s'il y a un entier } n \geq 2 \text{ pair de sorte que } x \in \left(\frac{1}{n}, \frac{1}{n-1}\right) \\ 0 & \text{pour les autres } x, \end{cases}$$

$$g(x) = \begin{cases} \dfrac{1}{\sqrt{x}\,.(1-\log x)} & \text{s'il y a un entier } n \geqq 2 \text{ pair de sorte que } x \in (\frac{1}{n+1}, \frac{1}{n}) \\ 0 & \text{pour les autres } x . \end{cases}$$

Alors, si nous prenons $\sqrt{f}$ dans le sens usuel, on a $\sqrt{f} \,.\, \sqrt{f} = f$, $\sqrt{f} \,.\, g = 0$ dans le sens usuel et aussi d'après la définition du produit simple de distributions, mais on a $f \,.\, g = \delta \,.\, \frac{1}{4} \log 2$ (δ est la mesure de Dirac) d'après la définition du produit simple de distributions. On n'a pas donc

$$(\sqrt{f} \,.\, \sqrt{f}) \,.\, g = \sqrt{f} \,.\, (\sqrt{f} \,.\, g)$$

Même pour des mesures μ, ν quelconques, le produit simple $[(\sqrt{f} * \mu) (\sqrt{f} * \nu)] \,.\, g$ a un sens; cela signifie que la supposition si forte ne suffit pas à la validité de la loi d'associativité.

Théorème 7. Soient f, g, h des distributions sur E^r, G un ouvert dans E^r. Supposons que pour chaque mesure μ au support contenu dans un voisinage du point 0 suffisament petit il existe $[(f * \mu) \,.\, g] \,.\, h$ sur G et qu'il existe gh sur G. Soit i une fonction intégrable sur E^r, infiniment dérivable sauf le point 0 de sorte qu'il existe $f_1 = f * i$. On a alors $(f_1\, g) \,.\, h = f_1 \,.\, (g\, h)$ sur G.

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О МЕТОДЕ РЕДУКЦИИ

Ю.И. ГРИБАНОВ, КАЗАНЬ

В настоящей заметке указываются условия разрешимости методом редукции [1] бесконечных систем линейных уравнений.

Пусть E -бесконечная единичная диагольная матрица, R_n - матрица, полученная из E заменой первых n единиц главной диагонали, и $P_n = E - R_n$. Обозначим через ℓ координатное пространство и через ℓ^* -дуальное к нему координатное пространство [2]. Если вектор $X \in \ell$, то при любом $n = 1, 2, \ldots$ вектор $P_n X \in \ell$ и $\| P_n X \| \leq \| X \|$. Важной характеристикой пространства ℓ является его координатное подпространство $[\ell]$, представляющее собой замыкание в метрике пространства ℓ множества всех векторов с конечным числом отличных от нуля координат. Нетрудно показать, что вектор $X \in [\ell]$ тогда и только тогда, когда $\lim_{n \to \infty} \| R_n X \| = 0$. Последовательность векторов $\{X_n\} \subset \ell$ называется слабо сходящейся в ℓ к вектору $X \in \ell$, если при любом векторе $Y = \{y_n\} \in \ell^*$

$$\lim_{n \to \infty} \sum_{k=1}^{\infty} x_k^{(n)} y_k = \sum_{k=1}^{\infty} x_k y_k .$$

Из сходимости по норме вытекает слабая сходимость. Из слабой сходимости последовательности векторов вытекает их покоординатная сходимость. Если $\ell \neq [\ell]$, то ℓ^* является собственной частью сопряженного с ℓ пространства. Поэтому в этом случае слабая сходимость в ℓ отлична от слабой сходимости в смысле теории нормированных пространств.

В рамках пространства ℓ рассмотрим бесконечную систему линейных уравнений

(I) $TX \equiv (E-A)X = H: \; x_m - \sum_{k=1}^{\infty} a_{mk} x_k = h_m \quad (m=1,2,\dots)$.

Это значит, что матричный оператор A ограничен в пространстве ℓ, известный вектор $H \in \ell$ и в расчет принимаются только принадлежащие пространству ℓ решения X этой системы уравнений. В дальнейшем будет предполагаться, что система уравнений (I) однозначно разрешима в ℓ (при любом $H \in \ell$), что в силу теоремы С. Банаха об обратном операторе эквивалентно предположению о существовании ограниченного обратного оператора T^{-1}. Наряду с системой уравнений (I) рассмотрим усеченную систему уравнений

$$(2)\quad T_n X_n \equiv (E-A_n)X_n = P_n H: \begin{cases} x_m^{(n)} - \sum_{k=1}^{n} a_{mk} x_k^{(n)} = h_m & (m=1,2,\dots,n) \\ x_m^{(n)} = 0 & (m>n) \end{cases}$$

Если при всех достаточно больших n система уравнений (2) имеет единственное решение X_n и последовательность векторов X_n сходится покоординатно к некоторому вектору $X \in \ell$, являющемуся решением системы уравнений (I), то говорят, что система уравнений (I) разрешима в ℓ методом редукции. Если кроме того $X_n \to X$ по норме пространства ℓ (соотв. слабо сходится в ℓ), то будем говорить, что метод редукции сходится по норме пространства ℓ (соотв. слабо сходится в ℓ).

В дальнейшем через X и X_n обозначаются решения систем уравнений (I) и (2) соответственно.

<u>Теорема 1</u>. Для того чтобы однозначно разрешимая в ℓ система уравнений (I) была разрешима методом редукции, сходящимся по норме пространства ℓ, необходимо и достаточно, чтобы нашлось такое число N, что определитель

(3) $|\delta_{\alpha\beta} - a_{\alpha\beta}|_{\alpha,\beta=1}^{n} \neq 0$ при любом $n \geq N$

и чтобы $\lim_{n\to\infty} \| R_n (AX_n + H)\| = 0$. Справедливы оценки $\| X - X_n \| \leq \|T^{-1}\| \, \| R_n (AX_n + H\|$ и $\| TX_n - H \| = \| R_n (AX_n + H)\|$.

Если $\|A\| < 1$, то условие (3) заведомо выполняется. Однако даже в этом случае при $\ell \neq [\ell]$ система уравнений (I) может быть неразрешимой методом редукции, сходящимся по норме пространства ℓ. В самом деле, справедлива

Теорема 2. Если $\|A\| < 1$, то система уравнений (I) разрешима методом редукции, сходящимся по норме пространства ℓ, тогда и только тогда, когда $X \in [\ell]$ или, что эквивалентно $\lim_{n\to\infty} \| R_n (AX_n + H)\| = 0$.

Таким образом, если $\ell = [\ell]$, то любая система уравнений (I) с $\|A\| < 1$ разрешима методом редукции, сходящимся по норме пространства ℓ. В частности, это справедливо для пространств ℓ_p $(p \geq 1)$. Если $\ell \neq [\ell]$, $\|A\| < 1$ и $A\{\ell\} \subseteq [\ell]$, то, как это вытекает из предыдущей теоремы, система уравнений (I) разрешима методом редукции, сходящимся по норме пространства ℓ, лишь тогда, когда известный вектор $H \in [\ell]$.

Матричный оператор A называется α-непрерывным [2], если $\lim_{n\to\infty} \| A - P_n A \| = 0$. Любой α-непрерывный оператор вполне непрерывен и область его значений расположена в $[\ell]$. Система уравнений (I) называется α-непрерывной, если A является α-непрерывным оператором.

Теорема 3. Однозначно разрешимая в ℓ α-непрерывная система уравнений разрешима методом редукчии, сходящимся по норме пространства ℓ, тогда и только тогда, когда $H \in [\ell]$.

Отсюда вытекает известный результат о вполне непрерывных системах уравнений в гильбертовом пространстве ℓ_2 ([3], гл. II, § 3 или [4], гл. XIУ, § 3). Частным случаем предыдущей теоремы является также основной результат работы [5] об ω-непрерывных системах уравнений. Если $\ell = [\ell]$, то класс вполне непрерывных операторов в ℓ совпадает с классом α-непрерывных операторов. Таким образом, если $\ell = [\ell]$, то любая однозначно разрешимая в ℓ вполне непрерывная система уравнений разрешима методом редукции, сходящимся по норме пространства ℓ .

Теорема 4. Для того чтобы однозначно разрешимая в ℓ система уравнений (I) была разрешима методом редукции, слабо сходящимся в пространстве ℓ , необходимо и достаточно, чтобы выполнялось условие (3) и $R_m A X_m \to 0$ слабо в пространстве ℓ .

Отметим ряд частных случаев этой общей теоремы, естественно дополняющей теорему I.

Теорема 5. Если система уравнений (I) однозначно разрешима в ℓ , выполняется условие (3), $\|AX_m\| \leq C < \infty$ и $\ell^* = [\ell^*]$, то эта система уравнений разрешима методом редукции, слабо сходящимся с пространстве ℓ .

Следствие. Если $\|A\| < 1$ и $\ell^* = [\ell^*]$, то система уравнений (I) разрешима методом редукции, слабо сходящимся в пространстве ℓ .

Так как $m^* = \ell_1 = [\ell_1]$, то, в частности, любая вполне регулярная система уравнений разрешима методом редукции, слабо сходящимся в пространстве m . Это является уточнением известного утверждения о разрешимости методом редукции вполне регулярной системы уравнений ([6], стр. 41).

Теорема 6. Однозначно разрешимая в ℓ α-непрерывная система уравнений разрешима методом редукции, слабо сходящимся в пространстве ℓ .

Матричный оператор A называется β-непрерывным, если $\lim_{m\to\infty} \|A - AP_m\| = 0$. Любой β-непрерывный оператор вполне непрерывен. Систему уравнений (I) будем называть β-непрерывной, если A является β-непрерывным оператором.

Теорема 7. Однозначно разрешимая в ℓ β-непрерывная система уравнений разрешима методом редукции, слабо сходящимся в пространстве ℓ. Метод редукции сходится по норме пространства ℓ в том и только том случае, когда $X \in [\ell]$ или, что эквивалентно, $\lim_{n \to \infty} \| R_n(AX_n + H)\| = 0$.

Последняя теорема уточняет одно имеющееся в [7] утверждение о β-непрерывных системах уравнений. В пространстве m любой вполне непрерывный матричный оператор является β-непрерывным оператором. Поэтому любая однозначно разрешимая в m вполне непрерывная система уравнений разрешима методом редукции, слабо сходящимся в пространстве m.

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О КРАЕВОЙ ЗАДАЧЕ РИМАНА С НЕПРЕРЫВНЫМ КОЭФФИЦИЕНТОМ

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Пусть C - замкнутая жорданова кривая, $D^{\pm}$ - внутренняя и внешняя к ней области.

В общей постановке краевая задача Римана может быть сформулирована следующим образом:

Найти кусочно аналитическую функцию $\Phi(z)$, удовлетворяющую в каком-либо смысле на C условию

(1) $\Phi^{+}(t) = G(t)\Phi^{-}(t) + g(t), \quad t \in C$,

где $G(t)$ - коэффициент задачи и $g(t)$ - ее свободный член, заданы на C , $\Phi^{\pm}(t)$ - угловые граничные значения функции $\Phi(z)$ изнутри и извне C .

В предположении, что C - кривая Ляпунова, функция $G(t)$ непрерывна и нигде на C не обращается в нуль, $g(t) \in L_p(C)$, $p > 1$, задача решалась в классе функций, представимых интегралом типа Коши с угловыми граничными значениями $\Phi^{\pm}(t) \in L_p(C)$, $p > 1$, которые почти всюду на C удовлетворяют условию (1) (см.[1],[2]).

В этой заметке мы дадим решение краевой задачи Римана с непрерывным коэффициентом для случая, когда C - гладкая кривая, у которой угол $\theta(s)$ наклона касательной к оси абсцисс, как функция длины дуги s на C , имеет модуль непрерывности $\omega(h)$ такой, что

$$(2) \qquad \int_0^c \frac{\omega(h)}{h} \ln^2 h \, dh < \infty .$$

Обозначим через $z = \psi(w)$ – функцию, однолистно и конформно отображающую круг $|w| < 1$ на область D^+.

Так же, как сделано С.Я. Альпером (см. [3], стр. 424-426) в случае, когда $\omega(h)$ удовлетворяет условию $\int_0^c \frac{\omega(h)}{h} |\ln h| dh < \infty$, можно доказать, что функция $\psi'(w)$ непрерывна в $|w| \leq 1$ и

$$(3) \quad |\psi'(e^{i\theta}) - \psi'(e^{i\theta'})| \leq \omega^*(|\theta - \theta'|), \quad 0 \leq \theta, \ \theta' \leq 2\pi,$$

где $\omega^*(h)$ такова, что

$$\int_0^c \frac{\omega^*(h)}{h} |\ln h| dh < \infty .$$

В силу (3) для любых точек $w = re^{i\theta}$ и $w' = re^{i\theta'}$, $0 \leq \theta, \ \theta' \leq 2\pi$, $0 \leq r < 1$, и $\tau = e^{i\theta}$, $0 \leq \theta \leq 2\pi$, будем иметь:

$$|\psi'(w) - \psi'(w')| \leq K\omega^*(|w - w'|)$$

и

$$(4) \qquad |\psi'(w) - \psi'(\tau)| \leq K_1 \omega_1^*(|w - \tau|),$$

где K и K_1 – некоторые постоянные, функция $\omega_1^*(h)$ удовлетворяет условию

$$(5) \qquad \int_0^c \frac{\omega_1^*(h)}{h} dh < \infty .$$

Отметим, что для доказательства второго из этих неравенств достаточно показать, что

$$|\psi''(w)| \leq \frac{\omega^*((1-r)^{\alpha})}{1-r}, \quad 0 < \alpha < 1, \quad w = re^{i\theta}, \quad 0 \leq r < 1 .$$

Используя неравенства (4) и то, что $|\psi'(w)| \geq m > 0$ в $|w| \leq 1$, нетрудно доказать, что

(6) $$\int_{|\tau|=1} \left| \frac{\psi'(\tau)}{\psi(\tau)-\psi(w)} - \frac{1}{\tau - w} \right| |d\tau| \le M, \quad |w| \le 1,$$

где постоянная M не зависит от w .

Рассмотрим интеграл типа Коши с непрерывной плотностью $f(t)$:

(7) $$F(z) = \frac{1}{2\pi i} \int_C \frac{f(t)}{t-z}\, dt .$$

В частности, когда C – окружность $|z| = 1$, то функция $F^+(z) \in H_p$ при всех $p > 0$ и, следовательно, $F^-(t) \in L_p$ при всех $p > 0$. Это вытекает из результатов В.И. Смирнова (см.[5], стр. 94 и 116) и Б.В. Хведелидзе (см.[6], сто. 19).

Покажем, что функция

$$F^+(z) = \frac{1}{2\pi i} \int_C \frac{f(t)}{t-z}\, dt, \qquad z \in D^+,$$

если кривая C удовлетворяет условию (2) принадлежит классу E_p при всех $p > 0$.

В самом деле,

(8) $$F^+(\psi(w)) = \frac{1}{2\pi i} \int_{|\tau|=1} \frac{f(\psi(\tau))}{\tau - w}\, d\tau + \frac{1}{2\pi i} \int_{|\tau|=1} f(\psi(\tau)) \left[\frac{\psi'(\tau)}{\psi(\tau)-\psi(w)} - \frac{1}{\tau - w} \right] d\tau, \quad |w| < 1 .$$

Здесь первый интеграл, как было отмечено, есть функция класса H_p при всех $p > 0$, второй – функция класса B в силу (6) и непрерывности функции $f(\psi(\tau))$. Следовательно функция $F^+(\psi(w)) \in H_p$ при всех $p > 0$. Так как функция $\psi'(w)$ непрерывна в $|w| \le 1$, то функция $F^+(z) \in E_p$ при всех $p > 0$. Ее угловые граничные значения $F^+(t)$ существуют почти всюду на C и $F^+(t) \in L_p(C)$ при всех $p > 0$.

В силу основной леммы И.И. Привалова (см.[5], стр. 190) функция $F(z)$ удовлетворяет почти всюду на C условию

(9) $$F^{+}(t)-F^{-}(t)=f(t).$$

Очевидно, что $F^{-}(t)\in L_p(C)$ при всех $p>0$.

В том случае, когда $f(t)\in L_p(C)$, $p>1$, функция $F(z)$ имеет почти всюду угловые граничные значения $F^{\pm}(t)$, причем $F^{\pm}(t)\in L_p(C)$, $p>1$, и удовлетворяют условию (9) (см.[7], стр. 273).

Следует отметить, что в классе функций, представимых интегралом типа Коши, обращающихся в нуль на бесконечности и удовлетворяющих почти всюду на C условию (9), функция (7) единственна (см.[6], стр. 66).

Приступая к изучению краевой задачи Римана, уточним ее формулировку.

Пусть C - гладкая кривая, удовлетворяющая условию (2), функция $G(t)$ непрерывна и не обращается в нуль на C , функция $g(t)\in L_p(C)$, $p>1$.

Требуется найти кусочно аналитическую функцию $\Phi(z)$, $\Phi(\infty)=0$, представимую интегралом типа Коши с угловыми граничными значениями $\Phi^{\pm}(t)$ из некоторого класса $L_p(C)$, $p>1$, и удовлетворяющую почти всюду на C условию (1).

Поскольку однородная и неоднородная задачи Римана решаются по известной схеме, то нет необходимости приводить здесь все выкладки.

Пусть $\varkappa=\operatorname{ind} G(t)=0$.

Так как функция $\ln G(t)$ непрерывна на C , то каноническая функция однородной задачи Римана

$$X(z)=e^{\Gamma(z)},$$

где $$\Gamma(z)=\frac{1}{2\pi i}\int\limits_C\frac{\ln G(t)}{t-z}\,dt.$$

Используя для $\Gamma(z)$ представление (8), нетрудно показать, что функция $X^{+}(z) \in E_p$ и $X^{\pm}(t)$ и $\frac{1}{X^{+}(t)} \in L_p(C)$ при всех $p > 0$.

Решение неоднор дной задачи Римана, как известно, представимо в виде:

(10) $$\Phi(z) = X(z)\psi(z),$$

где
$$\psi(z) = \frac{1}{2\pi i}\int_C \frac{g(t)}{X^{+}(t)}\frac{dt}{t-z}.$$

Так как $\frac{g(t)}{X^{+}(t)} \in L_{p-\varepsilon}(C)$, ε - любое малое положительное число, то угловые граничные значения $\psi^{\pm}(t) \in L_{p-\varepsilon}(C)$.

Очевидно, что функция $\Phi(z)$ имеет почти всюду на C угловые граничные значения $\Phi^{\pm}(t)$, принадлежащие классу $L_{p-\varepsilon}(C)$, ε - любое малое положительное число.

Итак, если $\varkappa = 0$, то краевая задача Римана имеет единственное решение (10).

Случай, когда $\varkappa \neq 0$ может быть так, как это обычно делается (см.[8]).

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CONCERNING UNIVERSAL CATEGORIES

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Introduction.

The present paper contains a definition of universal category and some results concerning this notion. In the § 1 we give the definition and two simple criteria which enables us to show that some frequently discussed categories are not universal. At the end of the paragraph we prove that the category of commutative semi-groups and their homomorphisms is not universal, which result is not an immediate consequence of the criteria.

The category $\mathcal{R}$ of sets with binary relations and their compatible mapping is shown to be universal in the § 2 . A proof of the fact was sketched in [2] using the constructions introduced in [1]. Here it is done in a more detailed way.

The main results of the paper are given in the § 3 . We examine some categories similar to $\mathcal{R}$ and state which of them are universal. Every transformation $\varphi : X \to X$ may be considered as a particular case of a relation on X , namely $\{(\varphi(x), x) | x \in X\}$. Another particular kind of relations form multivalued mappings $\mu : X \to X$, i.e. such relations, that for every $x \in X$ there is at least one y with $y \mu x$. Roughly speaking, the definition of $\mathcal{R}$ contains two kinds of relations: namely, relations in objects and morphisms, which are, in fact, a particular kind of compatible generalized relations. To suggest it, we write $\mathcal{R} = \mathcal{R}(r, f)$. We may replace r by f , m or r and the same for f (r means relations, m multivalued mappings, f mappings). In this way we obtain nine categories $\mathcal{R}(\alpha, \beta)$. Considering another kind of compatibility, we shall define another nine categories $\mathcal{R}^*(\alpha, \beta)$. Exact definitions of the categories $\mathcal{R}(\alpha, \beta)$, $\mathcal{R}^*(\alpha, \beta)$ are given in the beginning of § 3.

We remark that throughout the § 3 we work in a set theory without inaccessible cardinals.

§ 1.

M(a, b) denotes, as usual, the set of all the morphisms from an object a to an object b (in a given category). A class obtained by choicing a representant in every equivalence class (given by isomorphism) of A, where A is a class of objects, is called a skeleton of A. Small category is a category such that the class of all its objects is a set.

1.1. Definition. A category $\mathcal{K}$ is called universal, if any small category is isomorphic to a full subcategory of $\mathcal{K}$.

The following statement is evident:

1.2. Theorem. The dual category of a universal category is universal. If $\mathcal{K}$ is isomorphic to a full subcategory of $\mathcal{L}$, and if $\mathcal{K}$ is universal, then $\mathcal{L}$ is universal.

1.3. Remark. Since semigroups with unity elements (in particular, groups) are small categories, the following holds: Let $\mathcal{K}$ be a universal category, S^1 any semigroup with unity element. Then there exists an object a in $\mathcal{K}$ such that S^1 is isomorphic to M(a, a).

1.4. Theorem. Let $\mathcal{K}$ have the following property: There exists a class A of objects of $\mathcal{K}$ such that its skeleton is a set and such that for every $a \in \mathcal{K} \setminus A$ there exists a $b \in \mathcal{K}$ with $M(a, b) \neq \emptyset \neq M(b, a)$ and such that a is not a direct summand of b x) . Then the category $\mathcal{K}$ is not universal.

Proof. Let the category $\mathcal{K}$ be universal and let G be any group; let us denote $a(G)$ an object of $\mathcal{K}$ such that $G \approx M(a(G), a(G))$. Obviously, if G, G' are not isomorphic, the objects $a(G), a(G')$ are not isomorphic. If $a = a(G) \notin A$, there exist $b, \varphi, \psi \in \mathcal{K}$, $\varphi : a \to b$, $\psi : b \to a$, a being not a direct summand of b . Therefore $\psi \circ \varphi$ has not an inverse morphism and we have a contradiction, for $M(a, a)$ is a group. Hence $a(G) \in A$ for any group. But there is a proper class of mutually non-isomorphic groups.

1.5. Theorem . Let $\mathcal{K}$ have the following property: There exists a class A of objects of $\mathcal{K}$ such that its skeleton is a set and such that for every $b \in \mathcal{K}$ there exists an $a \in A$ such that $M(a, b) \neq \emptyset \neq M(b, a)$. Then the category $\mathcal{K}$ is not universal.

Proof. We are going to obtain the statement by proving that any object of any category has only a set of mutually non-isomorphic direct summands. Let us have an object a . Let b, b' be its non-isomorphic direct summands. Let $\alpha : b \to a$, $\beta : a \to b$, $\alpha' : b' \to a$, $\beta' : a \to b'$ be morphisms such that $\beta \circ \alpha$, $\beta' \circ \alpha'$ are isomorphisms. $\alpha \circ \beta = \alpha' \circ \beta'$ implies $(\beta \circ \alpha') \circ (\beta' \circ \alpha) = \beta \circ \alpha \circ \beta \circ \alpha$ and $(\beta' \circ \alpha) \circ (\beta \circ \alpha') = \beta' \circ \alpha' \circ \beta' \circ \alpha'$, i.e. b isomorphic with b' , in a contradiction with the assumption. The assertion concerning the set of mutually non-isomorphic direct summands is therefore an immediate consequence of the fact that $M(a, a)$ is a set.

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x) a is called a direct summand of b , if there exist morphisms $\alpha : a \to b$, $\beta : b \to a$ such that $\beta \circ \alpha$ is an isomorphism.

1.6. Corollary. If there is an object $a \in \mathcal{K}$ such that both $M(b, a)$ and $M(a, b)$ are non-void for any $b \in \mathcal{K}$, in particular, if $\mathcal{K}$ has a singleton resp. cosingleton, then $\mathcal{K}$ is not a universal category.

1.7. The criteria 1.4 and 1.5 do not give an immediate answer to the question whether the category of semigroups (in general, without a unity element) is universal. We close this paragraph by showing that at least the category of commutative semigroups and their homomorphisms is not universal. The statement is an immediate consequence of the following theorem:

Theorem. Let S be a semigroup, $a \in S$ an element such that $a^m = a^n$ for some $m \neq n$. Then there is an element $b \in S$ with $b \cdot b = b$. Consequently, a commutative semigroup either contains an element b with $b \cdot b = b$, or its endomorphism semigroup is infinite (more precisely, it contains an isomorphic image of the semigroup of natural numbers with multiplication).

Proof. Let $a^m = a^n$ with $n > m$. Hence $a^{n-m} \cdot a^m = a^m$ and consequently $a^m = a^{k(n-m)} \cdot a^m$. Let us take a k such that $k(n - m)$ m. Multiplying the both sides of the last equation by $a^{k(n-m)-m}$ we get $b \cdot b = b$ for $b = a^{k(n-m)}$. Now, let S be a commutative semigroup; let $b \cdot b \neq b$ for every $b \in S$. Hence for any a and any $m \neq n$ holds $a^m \neq a^n$. Let us denote N the semigroup of natural numbers. The mapping $\Phi : N \to E(S)$ defined by $\Phi(n)(x) = x^n$ is evidently a monomorphism.

§ 2.

2.1. Denotations. Let X, Y be sets, R, S binary relations on X, Y respectively. By a compatible (more precisely, RS-compatible) mapping $f : (X, R) \to (Y, S)$ we mean a mapping $f : X \to Y$ such that the implication $x R x' \Rightarrow f(x) S f(x')$

holds. The sets with binary relations and their compatible mappings form obviously a category, which will be denoted by $\mathcal{R}$. Let us denote $\mathcal{R}_a$ ($\mathcal{R}'_a$ resp.) the full subcategory of $\mathcal{R}$ generated by objects (X, R) with antireflexive R such that for every $x \in X$ there is a $y \in X$ with $y R x$ (with either $x R y$ or $y R x$).

Let A be a set. We denote by $A\mathcal{R}$ the category, described as follows: objects are systems $(X ; \{R_a\} , a \in A)$ where X is a set and every R_a is a binary relation on X, and the morphisms from $(X ; \{R_a\} , a \in A)$ into $(Y ; \{S_a\}, a \in A)$ are all the mappings $f : X \to Y$, which are $R_a S_a$ -compatible for every $a \in A$. Let (X, R) be a set with a binary relation. We denote $C(X, R)$ the semigroup of all the compatible mappings of (X, R) into itself. The object (X, R) is said to be rigid, if $C(X, R)$ is trivial.

2.2. Theorem. Any small category $\mathcal{K}$ is isomorphic to a full subcategory of some $A\mathcal{R}$. The set $\mathcal{K}'$ of the morphisms of $\mathcal{K}$ may be taken as the set A .

Proof. Let us denote by K the set of the objects of $\mathcal{K}$. Let us define $\Phi(a) = (\bigcup\{M(b,a) \mid b \in K\} ; \{R_\alpha\}, \alpha \in \mathcal{K}')$, where $\beta R_\alpha \gamma$ iff $\beta = \gamma \circ \alpha$; $\Phi(\beta) = \{\gamma \to \beta \circ \gamma\}$. Obviously, Φ is a 1-1 functor into $\mathcal{K}'\mathcal{R}$. Let $f : \Phi(a) \to \Phi(b)$ be a morphism in $\mathcal{K}'\mathcal{R}$. Let us denote ε the identity morphism of b . Since $\alpha = \varepsilon \circ \alpha$, we have $\alpha R_\alpha \varepsilon$ and hence $f(\alpha) R_\alpha f(\varepsilon)$, i.e. $f(\alpha) = f(\varepsilon) \circ \alpha$. Hence $f = \Phi(f(\varepsilon))$.

2.3. Lemma. Let $(X, R) \in \mathcal{R}'_a$. Then there exists a $(Y, S) \in \mathcal{R}_a$ with card $Y \geq$ card X and $C(X, R) \approx C(Y, S)$, such that the length of any cycle x) in (Y, S) is divisible by either

x) Let $(X,R) \in \mathcal{R}$. A sequence $x_1, x_2, \dots, x_n$ $(x_i \in X)$ with the property $x_i R x_{i+1}$ $(i=1,\dots,n-1)$, $x_n R x_1$ is called a cycle of the length n.

2 or 3 .

<u>Proof</u>. Let us take $Y = X \cup R \cup U$, where $U = U^1 \cup U^2$, $U^1 = \{u_1^1, u_2^1\}$, $U^2 = \{u_1^2, u_2^2, u_3^2\}$. Let us define the relation S by:

$u_1^1 \, S \, u_2^1$, $u_2^1 \, S \, u_1^1$, $u_i^2 \, S \, u_{i+1}^2$ $(i = 1, 2)$, $u_3^2 \, S \, u_1^2$;

for every $x \in X$, $i = 1,2$, $u_1^i \, S \, x$;

for every $(x, y) \in R$ $x \, S(x, y)$, $(x, y) \, S \, y$.

Evidently, the length of any cycle in (Y, S) is divisible by either 2 or 3 . Since the image of any cycle under a compatible mapping is a cycle of the same length, we get immediately $\varphi(U^i) \subset U^i$ $(i = 1,2)$ for any compatible $\varphi \in C(Y, S)$. Now, let us take an $x \in X$. Since $u_1^i \, S \, x$, we have $\varphi \, u_1^i \, S \, \varphi x$ and we easily obtain $\varphi \, u_j^i = u_j^i$ and $\varphi(X) \subset X$. Let us take $(x, y) \in R$. Since $x \, S(x, y)$ and $(x, y) \, S \, y$, we have $\varphi x \, S \, \varphi(x, y)$ and $\varphi(x, y) \, S \, \varphi y$ and we get $\varphi(x, y) = (\varphi x, \varphi y)$ by the fact that $\varphi(X) \subset X$.

Now, it is easy to see that the mapping $\Phi : C(Y, S) \to C(X, R)$ defined by $\Phi(\varphi)(x) = \varphi x$ is an isomorphism.

2.4. <u>Corollary</u>. Let m be a cardinal less than the first inaccessible one. Then there is a rigid object (X, R) in $\mathcal{R}_a$ with card $X \geq m$, such that the length of any cycle in (X, R) is divisible by either 2 or 3 .

<u>Proof</u>. By [1] there exists a rigid (X', R') in $\mathcal{R}$ with card $X \geq m$. Without a loss of generality we may suppose $m > 1$. Then $(X', R') \in \mathcal{R}'_a$, since if $x_0 \, R' \, x_0$ the constant mapping into x_0 is compatible, and, further, a point which is not in the relation with any other may be compatibly mapped everywhere. Now, we get the statement using 2.3 .

2.5. <u>Theorem</u>. In a set theory without inaccessible cardinals the category $\mathcal{R}_a$ (and hence $\mathcal{R}$, too) is universal.

Proof. By 2.2 it is sufficient to prove that, for card A accessible, the category $A\mathcal{R}$ is isomorphic to a full subcategory of $\mathcal{R}_a$. Let us take a rigid object $(B, S) \in \mathcal{R}_a$ with card B $\geq$ card A + 1 with cycles of length divisible by either 2 or 3. Let $p_1, \dots, p_4$ be mutually different primes, $p_i \neq 2, 3$, and let U_i $(i = 1, \dots, 4)$ consist of formal elements $u_i(1), \dots, u_i(p_i)$; let, finally, $U = U_1 \cup \dots \cup U_4$.

Since card A + 1 $\leq$ card B we may choose a 1-1 mapping $\alpha : A \to B$ such that there is a $b_0 \in B \setminus \alpha(A)$. Let us define $B_i = \{(a, i) \mid a \in B\}$, $S_i = \{((a, i), (a', i)) \mid (a, a') \in S\}$ $(i = 1,2)$. Let us define $\alpha_i : A \to B_i$ by $\alpha_i a = (\alpha a, i)$. Let us, for an object $\hat{X} = (X ; \{R_a\}, a \in A) \in A\mathcal{R}$ denote $Y_a = \{(x, y, a) \mid (x, y) \in R_a\}$, $Y = \bigcup\{Y_a \mid a \in A\}$ and define $\Phi(\hat{X}) = (X_1, R_1)$, where $X_1 = X \cup Y \cup U \cup B_1 \cup B_2$ and R_1 is defined as follows:

$$u_i(j)\, R_1\, u_i(j+1) \quad (i = 1, \dots, 4,\ j = 1, \dots, p_i - 1),$$

$$u_i(p_i)\, R_1\, u_i(1) \quad (i = 1, \dots, 4);$$

for every $(b, 1) \in B_1$, $u_i(1)\, R_1\, (b, 1)$ $(i = 1,2)$;

for every $(b, 2) \in B_2$, $u_i(1)\, R_1\, (b, 2)$ $(i = 3,4)$;

$$(b, i)\, R_1\, (b', i) \text{ iff } b\, S\, b' \quad (i = 1,2);$$

for every $x \in X$, $(b_0, i)\, R_1\, x$ $(i = 1,2)$;

for every $a \in A$, $(x, y) \in R_a$, $x\, R_1 (x,y,a)\, R_1\, y$, $(x,y,a) R_1 y$, $\alpha_i a\, R_1 (x,y,a)$ $(i = 1,2)$.

Now, let $\varphi : (X; \{R_a\}, a \in A) \to (X'; \{R'_a\}, a \in A)$ be a morphism. $\Phi(\varphi) : \Phi(\hat{X}) \to \Phi(\hat{X}')$ is defined as follows:

for $x \in X$ $\Phi(\varphi)x = \varphi x$, $\Phi(\varphi)(x,y,a) = (\varphi x, \varphi y, a)$,

$\Phi(\varphi)$ identical over $U \cup B_1 \cup B_2$.

Obviously, Φ forms a 1-1 functor into $\mathcal{R}_a$. Now, let $f : \Phi(\hat{X}) \to \Phi(\hat{X}')$ be a morphism. Similarly as in 2.3 we get $f(u_i(j)) = w_i(j)$, $f(B_i) \subset B_i$. By the rigidity of (B, S) immediately $f(b, i) = (b, i)$ for every $b \in B$. Since, particularly, $f(b_0, i) = (b_0, i)$, $f(\alpha_i a) = \alpha_i a$, we get $f(X) \subset X'$ and $f(Y_a) \subset Y'_a$. By $x R_1 (x, y, a)$, $(x, y, a) R_1 y$ we have $fx R'_1 f(x, y, a)$, $f(x, y, a) R'_1 fy$ and hence $f(x, y, a) = (fx, fy, a)$. Hence, finally, $f = \Phi(\varphi)$, where $\varphi : \hat{X} \to \hat{X}'$ is defined by $\varphi x = fx$.

§ 3 .

3.1. Denotations. Let X, Y, Z be sets, $A \subset Z \times Y$, $B \subset Y \times X$. The set $\{(z, x) \mid$ there exists a $y \in Y$, $(z, y) \in A$, $(y, x) \in B\}$ is denoted by $A \circ B$. The category $\mathcal{R}(r, r)$ ($\mathcal{R}^*(r, r)$ respectively) is defined as follows: Its objects are couples (X, R) , where X is a set, $R \subset X \times X$, the morphisms from (X, R) into (X', R') are triplets $(S, (X, R), (X', R'))$ such that $S \subset X' \times X$ and $S \circ R \subset R' \circ S$ ($S \circ R = R' \circ S$ respectively). The composition of morphisms is defined by the formula

$$(S', (X', R'), (X'', R'')) \circ (S, (X, R), (X', R')) = $$
$$= (S' \circ S, (X, R), (X'', R'')) .$$

We associate with every object (X, R) the adjoint morphism $a(X, R) = (R, (X, R), (X, R))$.

A multivalued compatible mapping (strongly compatible mapping, respectively) is a morphism $(S, (X, R), (X', R'))$ such that for every $x \in X$ there exists a x' with $(x', x) \in S$. If there is always exactly one such x' , we call the morphism a compatible (strongly compatible , resp.) mapping. If there is no danger of misunderstanding, we omit the words (strongly) compatible . Sometimes, we shall write simply S instead of $(S, (X, R), (X', R'))$.

By $\mathcal{R}(m, r)$ ($\mathcal{R}(f, r)$ resp.) is denoted the full subcategory of $\mathcal{R}(r, r)$ generated by the objects (X, R) such that $\mathcal{A}(X, R)$ is a multivalued mapping (a mapping, resp.).

By $\mathcal{R}(r, m)$ ($\mathcal{R}(r, f)$ resp.) is denoted the subcategory of $\mathcal{R}(r, r)$ consisting of the same objects as $\mathcal{R}(r, r)$ and of all their multivalued mappings (mappings, resp.). (It is easy to see that $\mathcal{R}(r, f)$ is isomorphic to the category from the § 2 .)

$\mathcal{R}(m, m)$ ($\mathcal{R}(f, m)$ resp.) is the full subcategory of $\mathcal{R}(r, m)$ generated by the objects (X, R) such that $\mathcal{A}(X, R)$ is a multivalued mapping (a mapping, resp.).

Finally, $\mathcal{R}(m, f)$ ($\mathcal{R}(f, f)$, resp.) is the full subcategory of $\mathcal{R}(r, f)$ generated by the objects (X, R) such that $\mathcal{A}(X, R)$ is a multivalued mapping (a mapping, resp.).

The categories $\mathcal{R}^*(\alpha, \beta)$ ($\alpha, \beta = r, m, f$) are defined analoguously.

The category $\mathcal{R}_a$ is evidently isomorphic to a full subcategory of $\mathcal{R}(m, f)$. Hence the categories $\mathcal{R}(m, f)$ and $\mathcal{R}(r, f)$ are universal ones. In present paragraph we shall prove that with exception of these two and $\mathcal{R}^*(m, f)$ and $\mathcal{R}^*(r, f)$ no category defined above is universal. On the other hand, we shall prove that both $\mathcal{R}^*(m, f)$ and $\mathcal{R}^*(r, f)$ are universal. Hence, the situation for both $\mathcal{R}(\alpha, \beta)$ and $\mathcal{R}^*(\alpha, \beta)$ ($\alpha, \beta = r, m, f$) is described by the following table (+ means: the category is universal, - : the category is not universal):

$\alpha \backslash \beta$	f	m	r
f	-	-	-
m	+	-	-
r	+	-	-

3.4. <u>Theorem</u>. The categories $\mathcal{R}^*(\alpha, \alpha)$ ($\alpha = r, m, f$) are not universal.

Proof. Let us consider the group P_3 of the permutations of a three-point set. If $\mathcal{R}^*(r, r)$ ($\mathcal{R}^*(m, m)$, $\mathcal{R}^*(f, f)$ resp.) were universal, there ought to be a relation R (multivalued mapping μ, mapping φ, resp.) on a set X such that P_3 were isomorphic to the semigroup of all the morphisms of (X, R) ((X, μ), (X, φ) resp.) into itself, i.e. to the semigroup of the sets $R' \subset X \times X$ (multivalued mappings $\mu' : X \to X$, mappings $\varphi' : X \to X$, resp.) with $R' \circ R = R \circ R'$ ($\mu' \circ \mu = \mu \circ \mu'$, $\varphi' \circ \varphi = \varphi \circ \varphi'$, resp.). Since R (μ, φ, resp.) itself is an element of the semigroup, it has to correspond to an element of P_3, which commutes with any other one. But only the unity element possesses in P_3 this property, and, on the other hand, the unity element obviously corresponds to the diagonal Δ of $X \times X$. We got a contradiction, since, except of one-point X, the semigroup of all the morphisms from (X, Δ) into (X, Δ) is not a group.

3.3. Corollary. $\mathcal{R}^*(m, r)$, $\mathcal{R}^*(f, r)$ and $\mathcal{R}^*(f, m)$ are not universal.

3.4. Lemma. Let G be a group provided by a (reflexive) partial ordering $\prec$ such that the following implication holds:

$$x, y, z \in G, \; x \prec y \Rightarrow z x \prec z y, \; x z \prec y z .$$

Let for some $g \in G$ and for every $x \in G$ $x g \prec g x$. Then $x g = g x$ for every $x \in G$.

Proof. Let $x \in G$. Since $x^{-1} g \prec g x^{-1}$, we get $g x \prec x g$ (multiplying by x from both the right and the left), and hence, assuming $x g \prec g x$, $g x = x g$.

3.5. Theorem. $\mathcal{R}(\alpha, \alpha)$ ($\alpha = r, m, f$) are not universal.

Proof. Let the semigroup of morphisms be ordered by inclusion.

Using the lemma 3.4 we may now repeat the proof of 3.2 .

3.6. Corollary. The categories $\mathcal{R}(m, r)$, $\mathcal{R}(f, r)$ and $\mathcal{R}(f, m)$ are not universal.

3.7. Theorem. $\mathcal{R}(r, m)$ and $\mathcal{R}^*(r, m)$ are not universal.

Proof. We prove, that any non-trivial group of all the morphisms of some (X, R) into itself contains a non-trivial element, commuting with every other one. Really, $(R \cup \Delta, (X, R), (X, R))$ is a multivalued mapping. If $(S, (X, R), (X, R))$ is another one, we have

$$S \circ (R \cup \Delta) = S \circ R \cup S \circ \Delta \subset (= \text{resp.})\ R \circ S \cup \Delta \circ S = (R \cup \Delta) \circ S .$$

3.8. Theorem. The category $\mathcal{R}_a$ is isomorphic to a full subcategory of $\mathcal{R}^*(m, f)$.

Proof. Let $(X, R) \in \mathcal{R}_a$. Let us denote by X_i $(i = 1, 2)$ the set $\{(x, y, i) \mid (x, y) \in R\}$, by X_3 the set $\{(x, 3) \mid x \in X\}$. Let us define a relation $\bar{R}$ on the set $\bar{X} = X \cup X_1 \cup X_2 \cup X_3$ as follows:

for every $x \in X$ $\quad x \bar{R}(x, 3)$ and $(x, 3) \bar{R} x$,

for every $(x, y) \in R$ $\quad x \bar{R}(x, y, 2)$, $y \bar{R}(x, y, 2)$,

$y \bar{R}(x, y, 1)$ and $(x, y, \ ) \bar{R} (x, y, 2)$.

Let us denote $\Phi(X, R) = (\bar{X}, \bar{R})$. Let $\varphi : (X, R) \to (Y, S)$ be a compatible mapping. The mapping $\Phi(\varphi) : \Phi(X, R) \to \Phi(Y, S)$ is defined as follows:

for $x \in X$ $\quad \Phi(\varphi)x = \varphi x$, $\Phi(\varphi)(x, 3) = (\varphi x, 3)$,

for $(x, y) \in R$, $i = 1,2$ $\quad \Phi(\varphi)(x, y, i) = (\varphi x, \varphi y, i)$.

Evidently $\Phi(\varphi) \circ \bar{R} \subset \bar{S} \circ \Phi(\varphi)$. We are going to prove the converse inclusion. Let $(a, b) \in \bar{S} \circ \Phi(\varphi)$. First, let $b = (x, y, 2)$; we have $a \bar{S} (\varphi x, \varphi y, 2)$ and therefore the element a must be equal to either $(\varphi x, \varphi y, 1)$ or φx or

φy . In any of these cases $(a, b) \in \Phi(\varphi) \circ \overline{R}$. Let $b = (x, y, 1)$; then we have $a \overline{S} (\varphi x, \varphi y, 1)$ and hence $a = \varphi y$, so that $(a, b) \in \Phi(\varphi) \circ \overline{R}$, too. Similarly, in the case of $b = x$ ($b = (x, 3)$ resp.), which leads to $a = (\varphi x, 3)$ ($a = \varphi x$ resp.). Hence, finally, $\Phi(\varphi) \circ \overline{R} = \overline{S} \circ \Phi(\varphi)$, i.e. $\Phi(\varphi)$ is a strongly compatible mapping, and Φ is an (evidently 1-1) functor into $\mathcal{R}^*(m, f)$. It remains to prove that the image of Φ is a full subcategory of $\mathcal{R}^*(m, f)$. Let $f : (\overline{X}, \overline{R}) \to (\overline{Y}, \overline{S})$ be a strongly compatible mapping. Since there are no cycles in $(\overline{X}, \overline{R})$ and $(\overline{Y}, \overline{S})$ but cycles of a type either $x, (x, 3), x, \ldots, (x, 3)$ or $(x, 3), x, (x, 3), \ldots, x$, we have $f(\{x, (x, 3)\}) \subset \{x'', (x'', 3)\}$ for every $x \in X$. Since, for every $x \in X$, there is a $y \in X$ with $y R x$, the equality $fx = (x'', 3)$ leads (by $x \overline{R} (y, x, 1)$ and $x \overline{R}(y, x, 2)$) to the equalities $f(y, x, 1) = f(y, x, 2) = x''$, in a contradiction to $f(y, x, 1) \overline{S} f(y, x, 2)$.

Hence we have $fx = x''$, $f(x, 3) = (x'', 3)$.
Now, let us turn our attention to $f(x, y, 2)$. If $f(x, y, 2)$ is equal to either x' or $(x', 3)$ or $(x', y', 1)$, we get $f(x, y, 1) = fy$, what is not possible. We have hence $f(x,y,2) = (x', y', 2)$ and $f(x, y, 1)$ is equal to either $(x', y', 1)$ or x' or y' . However, the second and the third case implies $fy = (x', 3)$ ($= (y', 3)$ resp.) in a contradiction with $fy \in Y$ proved above. Hence $f(x, y, 1) = (x', y', 1)$ and, consequently, $fy = y'$, and fx is either the x' or the y' . But $fx = y'$ implies $\overline{S} \circ f(x, y, 2) = \overline{S}(x', y', 2) = \{(x', y', 1), x', y'\}$, and $f \circ \overline{R} (x, y, 2) = f(\{(x, y, 1), x, y\}) = \{(x', y', 1), y'\}$ only. We get $fx = x'$ and we see that $f = \Phi(\varphi)$, where $\varphi : X \to Y$ is defined by $\varphi x = fx$.

3.9. <u>Corollary</u>. The categories $\mathcal{R}^*(m, f)$ and $\mathcal{R}^*(r, f)$ are universal.

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ALGEBRAIC DEPENDENCE STRUCTURES

(Preliminary communication)

Vlastimil DLAB, Praha

The present results - representing a generalization of some ideas of the papers [1] and [5] - were, together with several applications to (non-commutative) groups, lattices and modules, a subject of the author's lecture read in the Conference on General Algebra in Warsaw, September 7-11, 1964.

Let S be a given set, $\mathcal{P} S$ its powerset, $\mathcal{F} \subseteq \mathcal{P} S$ the subfamily of all its finite subsets. x and X denote always an element and a subset of S, respectively.

By a <u>relation</u> ρ <u>on</u> S we understand a subset ρ of the cartesian product $S \times \mathcal{P} S$. For a relation ρ on S, define the subfamily $\mathcal{I}_\rho \subseteq \mathcal{P} S$ of ρ-<u>independent subsets</u> by

$(\rho \to \mathcal{I}_\rho)$ $\quad I \in \mathcal{I}_\rho \longleftrightarrow \forall x (x \in I \to [x, I \setminus (x)] \notin \rho)$.

Further, define two mappings $\mathcal{D}_\rho$ and $\mathcal{D}_\rho^R$ of S into $\mathcal{P} S$ by

$(\mathcal{D}_\rho)$ $\quad X \in \mathcal{D}_\rho(x) \longleftrightarrow [x, X] \in \rho$

and

$(\mathcal{D}_\rho^R)$ $\quad X \in \mathcal{D}_\rho^R(x) \longleftrightarrow \exists I (I \subseteq X \wedge I \in \mathcal{I}_\rho \wedge x \notin I \wedge$
$\wedge [x, I] \in \rho)$.

Two relations ρ_1 and ρ_2 on S are said to be <u>associated</u> or <u>similar</u> if

$$\mathcal{I}_{\rho_1} = \mathcal{I}_{\rho_2}$$

or

$$x \notin X \to ([x, X] \in \rho_1 \longleftrightarrow [x, X] \in \rho_2),$$

respectively.

A relation ρ on S satisfying the following two conditions

(F/M) $[x, X] \in \rho \leftrightarrow \exists F (F \subseteq X \wedge F \in \mathcal{F} \wedge [x, F] \in \rho)$,

(E_r) $I \in \mathcal{I}_\rho \wedge [x_1, I] \notin \rho \wedge [x_1, I \cup (x_2)] \in \rho \rightarrow$
$\rightarrow [x_2, I \cup (x_1)] \in \rho$,

is said to be an A-dependence relation on S . It is said to be proper, or regular if, moreover,

(I) $x \in X \rightarrow [x, X] \in \rho$

or

(R) $x \notin X \wedge [x, X] \in \rho \rightarrow \exists I (I \subseteq X \wedge I \in \mathcal{I}_\rho \wedge [x, I] \in \rho)$

is satisfied, respectively.

If ρ is an A-dep. relation on S , $I \in \mathcal{I}_\rho$ and $x \notin I$, then

$$[x, X] \in \rho \leftrightarrow I \cup (x) \notin \mathcal{I}_\rho \quad .$$

For a mapping C of $\mathcal{P}$ S into $\mathcal{P}$ S , define the subfamily $\mathcal{I}_C \subseteq \mathcal{P}$ S of C-independent subsets by

($C \rightarrow \mathcal{I}_C$) $I \in \mathcal{I}_C \leftrightarrow \forall X (X \subseteq I \wedge I \subseteq C(X) \rightarrow X = I)$.

If the conditions

(φ/μ) $C(X) = \bigcup_{F \subseteq X, F \in \mathcal{F}} C(F)$,

(ε_κ) $I \in \mathcal{I}_C \wedge x_1 \in C(I \cup (x_2)) \setminus C(I) \rightarrow x_2 \in C(I \cup (x_1))$,

(ι) $X \subseteq C(X)$,

are fulfilled, then C is called an A-dependence closure operation in S . For such a closure operation

$$C(I) = \bigcup_{I \cup (x) \notin \mathcal{I}_C} I \cup (x)$$

holds for every $I \in \mathcal{I}_C$.

A subfamily $\mathcal{I}$ of $\mathcal{P}$ S satisfying the condition

(f/m) $I \in \mathcal{I} \leftrightarrow \forall F (F \subseteq I \wedge F \in \mathcal{F} \rightarrow F \in \mathcal{I})$

is said to be an A-independence net of S .

The following theorem describes the relation between any two of the following concepts of an A-dependence structure (S, ρ), (S, C) and $(S, \mathcal{J})$, where ρ, C and $\mathcal{J}$ are A-dep. relation on S, A-dep. closure operation in S and A-indep. net of S, respectively.

Theorem. To any A-dep. relation ρ on S there corresponds a well-defined A-indep. net $\mathcal{J}_\rho$ of S. On the other hand, to any A-indep. net of S there corresponds a set of (associated) A-dep. relations on S which form, under the natural operations of join and meet, a lattice $\mathbb{L}$ with infinite joins and $\mathbb{O}$. The lattice $\mathbb{L}$ splits into convex sublattices of similar relations, the greatest element of each of these sublattices being the corresponding proper relation. The correspondence in which every element of such sublattice is mapped into the corresponding greatest element is a lattice-homomorphism of $\mathbb{L}$ onto the sublattice $\mathbb{L}_p$ of all proper relations with the ideal of all regular relations. Denoting by $\mathbb{1}$, $\mathbb{O}_p$ and $\mathbb{O}$ the greatest element of $\mathbb{L}$, the least element of $\mathbb{L}_p$ and $\mathbb{L}$, respect., we have

$$\mathcal{D}_{\mathbb{1}}(x) = \mathcal{D}^R(x) \cup (\mathcal{P} S \setminus \mathcal{J}) \cup \mathcal{S}(x) ,$$

$$\mathcal{D}_{\mathbb{O}_p}(x) = \mathcal{D}^R(x) \cup \mathcal{S}(x) ,$$

$$\mathcal{D}_{\mathbb{O}}(x) = \mathcal{D}^R(x) ,$$

where $\mathcal{S}(x)$ is the subfamily of all subsets X such that $x \in X$.

As a consequence, for any A-indep. net of S, there is a uniquely determined proper regular A-dep. relation on S.

To any A-dep. closure operation C in S there corresponds a well-defined A-indep. net $\mathcal{J}_C$ of S. On the other hand, to any A-indep. net of S there corresponds a lattice of A-dep.

closure operations in S which is isomorphic to the corresponding lattice $\mathbb{L}$ of all proper A-dep. relations. The least element of this lattice is the corresponding Schmidt's "mehrstufige Austauschstrukture" (see [5]).

In what follows we consider a (fixed) A-indep. net $\mathcal{J}$ of S (with the closure operation C : $C(I) = \bigcup_{I \cup (x) \notin \mathcal{J}} I \cup (x)$).
For the purpose of establishing an invariant (rank or dimension) of certain A-dep. structures, let us introduce the following concept of a canonic subset of S . The family $\mathcal{C} \subseteq \mathcal{J}$ of all canonic subsets is defined by

$$(\mathcal{C}) \quad I \in \mathcal{C} \leftrightarrow I \in \mathcal{J} \wedge \forall X [X \in \mathcal{J} \wedge X \subseteq C(I) \wedge I \subseteq C(X) \rightarrow \rightarrow C(I) \subseteq C(X)].$$

Also, define the family $\mathcal{J}^*$ of all maximal subsets of S by

$$(\mathcal{J}^*) \qquad I \in \mathcal{J}^* \leftrightarrow I \in \mathcal{J} \wedge C(I) = S ,$$

and the family $\mathcal{L}$ of all bases of S by

$$(\mathcal{L}) \qquad \mathcal{L} = \mathcal{C} \cap \mathcal{J}^* .$$

A GA-indep. net of S is an A-indep. net $\mathcal{J}$ of S such that $\mathcal{L} \neq \emptyset$ and

$$I_1 \subseteq I_2 \wedge I_2 \in \mathcal{C} \rightarrow I_1 \in \mathcal{C} .$$

If, moreover, $\mathcal{L} = \mathcal{J}^*$, $\mathcal{J}$ is called a LA-indep. net of S .

Through the following generalization of the Steinitz's Exchange Theorem

$$X \in \mathcal{J} \wedge I \in \mathcal{C} \wedge X \subseteq C(I) \wedge I \subseteq C(X) \rightarrow$$
$$\rightarrow \forall x [x \in X \setminus I \rightarrow \exists I_0 (\emptyset \neq I_0 \subseteq I \setminus X \wedge X \setminus (x) \cup I_0 \in \mathcal{J} \wedge$$
$$\wedge I \subseteq C(X \setminus (x) \cup I_0) ,$$

one can prove the fundamental

Theorem. $X \in \mathcal{J} \wedge I \in \mathcal{C} \wedge X \subseteq C(I) \rightarrow \text{card}(X) \leq \text{card}(I)$.

Then, the implication

$$X \in \mathcal{J}^* \wedge I_1 \in \mathcal{L} \wedge I_2 \in \mathcal{L} \rightarrow \text{card}(X) \leq \text{card}(I_1) = \text{card}(I_2)$$

is a simple corollary enabling us to define the rank of any GA-dependence structure (i.e. any structure with a GA-indep.net).

The following theorem shows the relation with the results of [2], [3], [4] and [6]:

Theorem. For a given A-indep. net $\mathcal{J}$, the following conditions are equivalent:

($\mathcal{FC}$) $\quad \mathcal{J} \cap \mathcal{F} \subseteq \mathcal{C}$;

($\mathcal{C}$) $\quad \mathcal{J} = \mathcal{C}$;

($\mathcal{FN}$) $\quad I \in \mathcal{J} \cap \mathcal{F} \wedge I \cup (x) \notin \mathcal{J} \wedge I \cup (y) \notin \mathcal{J} \wedge x \neq y \rightarrow$
$\rightarrow \forall z (z \in I \rightarrow I \setminus (z) \cup (x) \cup (y) \notin \mathcal{J})$;

($\mathcal{N}$) $\quad I \in \mathcal{J} \wedge I \cup (x) \notin \mathcal{J} \wedge I \cup (y) \notin \mathcal{J} \wedge x \neq y \rightarrow$
$\rightarrow \forall z (z \in I \rightarrow I \setminus (z) \cup (x) \cup (y) \notin \mathcal{J})$;

($\mathcal{FW}$) $\quad I_1 \in \mathcal{J} \cap \mathcal{F} \wedge I_2 \in \mathcal{J} \cap \mathcal{F} \wedge \text{card}(I_1) < \text{card}(I_2) \rightarrow$
$\rightarrow \exists x (x \in I_2 \wedge x \notin I_1 \wedge I_1 \cup (x) \in \mathcal{J})$;

($\mathcal{W}$) $\quad I_1 \in \mathcal{J} \wedge I_2 \in \mathcal{J} \wedge \text{card}(I_1) < \text{card}(I_2) \rightarrow$
$\rightarrow \exists x (x \in I_2 \wedge x \notin I_1 \wedge I_1 \cup (x) \in \mathcal{J})$;

($\mathcal{FB}$) $\quad I_1 \in \mathcal{J} \cap \mathcal{F} \wedge I_2 \in \mathcal{J} \cap \mathcal{F} \wedge I_1 \subseteq C(I_2) \wedge I_2 \subseteq C(I_1) \rightarrow$
$\rightarrow \forall x [x \in I_1 \setminus I_2 \rightarrow \exists y (y \in I_2 \setminus I_1 \wedge I_1 \setminus (x) \cup (y) \in \mathcal{J})]$;

($\mathcal{B}$) $\quad I_1 \in \mathcal{J} \wedge I_2 \in \mathcal{J} \wedge I_1 \subseteq C(I_2) \wedge I_2 \subseteq C(I_1) \rightarrow$
$\rightarrow \forall x [x \in I_1 \setminus I_2 \rightarrow \exists y (y \in I_2 \setminus I_1 \wedge I_1 \setminus (x) \cup (y) \in \mathcal{J})]$.

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GENERALIZED PROXIMITY AND UNIFORM SPACES I.

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In this paper, generalized proximity and uniformity are introduced and studied. Filter is a fundamental concept of our generalizations. A semi-uniformity for a set P is a filter in $\exp(P \times P)$ intersection of which contains the diagonal Δ_P ; a proximity for a set P is given, if for each $X \subset P$ a filter in $\exp P$ is given such that its intersection contains X . Of course, these or similar generalizations and its characterizations occur in various papers e.g. by A. Appert, I. Konishi, D. Tamari, N.C. Jarutkin, W.J. Pervin (see [4]), C.H. Dowker (see [3]) (non-symmetric proximities and uniformities), S. Leader, A. Goetz, V.S. Krishnan, V.A. Efremovič and A.S. Švarc (see [2]) (characterizations of uniform spaces by means of nets). We shall prove that the categories of proximity and semi-uniform spaces and some their subcategories are S-categories over the category of sets with respect to the forgetful functors (for S-categories see [5]). Hence it is easy to characterize subobjects factor-objects, products and sums in these categories by means of theorems in [5].

Next, special properties of functors and subcategories are introduced (e.g. to be projectivity-preserving, hereditary, productive etc.). The purpose of these definitions will be seen in the part II which is in preparation. In that part we shall investigate properties of functors from the introduced categories in other ones (e.g. to preserve sub-objects, sums etc.).

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We use the terminology of [5] and now we shall introduce some symbols and terms more, used in sequel (not necessary in part I).

The category of all sets will be denoted by $\mathcal{M}$. We mark $\exp' P = \exp P - (\emptyset)$ for every set P, $\mathcal{D} M = \{x \mid \langle x,y \rangle \in M$ for some $y\}$, $\mathcal{E} M = \mathcal{D} M^{-1}$ for every <u>relation</u> M and similarly $\mathcal{D} \xi = x$, $\mathcal{E} \xi = y$ for every pair $\xi = \langle x, y \rangle$. A <u>mapping</u> is a single-valued relation unlike a <u>morphism</u> of a category of structs, e.g. of topological spaces, which is a triple $\langle f, \langle P, u \rangle, \langle Q, v \rangle\rangle$ where f is a continuous mapping of $\langle P, u \rangle$ into $\langle Q, v \rangle$ (then f = graph $\langle f, \langle P, u \rangle, \langle Q, v \rangle\rangle$).

By $\mathcal{C}(P)$ we denote the class of all nets ranging in a set P. A <u>subnet</u> of a net M is a net M/A where A is a right-cofinal subset of $\mathcal{D} M$.

We do not define the concept of <u>forgetful functor</u> but it will be clear in every situation. E.g. the forgetful functor from the category of all topological spaces into $\mathcal{M}$ is the covariant functor F for which $F \langle P, u \rangle = P$, graph $F\varphi$ = = graph φ.

We shall write $\lambda \longleftrightarrow u \longleftrightarrow \{\mathcal{U}_x \mid x \in P\}$ if u is a <u>closure</u> for a set P, if every $\mathcal{U}_x$ is the <u>neighborhood system</u> of x in $\langle P, u \rangle$ (i.e. $\mathcal{U}_x = \{U \mid U \subset P,\ x \in P - u(P - U)\}$) and if λ is the <u>convergence class</u> of $\langle P, u \rangle$(i.e. $\lambda = \{\langle M, x \rangle \mid M \in \mathcal{C}(P),\ x \in P,\ M$ is eventually in each $U \in \mathcal{U}_x\}$).

The following properties are characteristic for u, $\mathcal{U}_x$, λ:

$u\emptyset = \emptyset$, $X \subset uX$ for all $X \subset P$,
$u(X_1 \cup X_2) = uX_1 \cup uX_2$ for all $X_i \subset P$ $(i = 1,2)$;

every $\mathcal{U}_x$ is a filter in exp P intersection of which contains x ;

$\langle M, x \rangle \in \lambda$ whenever $\mathcal{E} M = (x) \subset P$,
if $\langle M, x \rangle \in \lambda$ and if M' is a subnet of M then $\langle M', x \rangle \in \lambda$, if $\langle M, x \rangle \in \mathcal{C}(P) \times P - \lambda$ then $\lambda \cap (\mathcal{C}(\mathcal{E} M') \times (x)) = \emptyset$ for some subnet M' of M .

A closure u is called topological if $uuX = uX$ for all $X \subset P$. The category of all closure spaces with continuous mappings will be denoted by C . C is an S-category over $\mathcal{M}$ with respect to the forgetful functor.

1. Some special properties of functors in S-categories

In this section, let $\mathcal{K}$ be an S-category over $\mathcal{C}$ with respect to a covariant functor T_i $(i = 1,2)$, F be a covariant functor from $\mathcal{K}_1$ in $\mathcal{K}_2$ such that $T_1 = T_2 \circ F$. The order R_A will be simply denoted by $<$.

Definition 1.1. We shall say that F is projective (more precisely projectivity-preserving), if for any nonvoid family $\{\varphi_i \mid i \in I\}$ of morphisms of $\mathcal{K}_1$ $F[\mathcal{K}_1 - \varprojlim \{\varphi_i \mid i \in I\}] = \mathcal{K}_2 - \varprojlim \{F\varphi_i \mid i \in I\}$ provided the left side exists.

We shall say that F is hereditary if for any object X of $\mathcal{K}_1$ $\langle Y', f' \rangle$ is a subobject of FX in $\mathcal{K}_2$ if and only if $FY < Y' < FY$, $T_2 f' = T_1 f$ for some subobject $\langle Y, f \rangle$ of X in $\mathcal{K}_1$.

We shall say that F is productive if for any nonvoid family $\{X_i \mid i \in I\}$ of objects of $\mathcal{K}_1$ $F[\mathcal{K}_1 - \prod \{X_i \mid i \in I\}] = \mathcal{K}_2 - \prod \{FX_i \mid i \in I\}$ provided the left side exists.

If $\mathcal{K}_1$ is a subcategory of $\mathcal{K}_2$ and if F is the identity functor from $\mathcal{K}_1$ in $\mathcal{K}_2$ then we shall say that $\mathcal{K}_1$ is projective (in $\mathcal{K}_2$), hereditary (in $\mathcal{K}_2$), productive (in $\mathcal{K}_2$) resp., provided F has the same property.

Dually <u>inductive</u> (more precisely <u>inductivity-preserving</u>), <u>cohereditary</u>, <u>coproductive</u> functors and subcategories are defined.

<u>Theorem 1.1</u>. Let F be projective. Then F is hereditary and productive. Dually for inductive functors.

Proof. Let $\langle Y, f \rangle$ be a subobject of X in $\mathcal{K}_1$. Then by theorem 5 in [5] $\langle T_1 Y, T_1 f \rangle$ is a subobject of $T_1 X$ in $\mathcal{C}$ and $Y = \mathcal{K}_1 - \underleftarrow{\mathrm{Lim}}\, f$. It follows that $\langle T_2 F Y, T_2 F f \rangle$ is a subobject of $T_2 F X$ and $F Y = \mathcal{K}_2 - \underleftarrow{\mathrm{Lim}}\, F f$. Hence $\langle FY , Ff \rangle$ is a subobject of FX in $\mathcal{K}_2$.

Let $\langle Y', f' \rangle$ be a subobject of FX in $\mathcal{K}_2$. Then $\langle T_2 Y', T_2 f' \rangle$ is a subobject of T_2 FX in $\mathcal{C}$ and $Y' = \mathcal{K}_2 - \underleftarrow{\mathrm{Lim}}\, f'$. There exists $\langle Y, f \rangle$ such that $Y = \mathcal{K}_1 - \underleftarrow{\mathrm{Lim}}\, f$, $\mathcal{E} f = X$, $T_1 f = T_2 f'$. $\langle Y, f \rangle$ is a subobject of X in $\mathcal{K}_1$ and by the first part of our proof $\langle FY, Ff \rangle$ is a subobject of FX in $\mathcal{K}_2$. By remark 2 in [5] $FY < Y' < FY$. The assertion about products follows at once from a special case of theorem 4 in [5].

<u>Theorem 1.2</u>. Assume that $X = \mathcal{K}_1 - \underleftarrow{\mathrm{Lim}}\, \{ \varphi_i \mid i \in I \}$. Then $FX = \mathcal{K}_2 - \underleftarrow{\mathrm{Lim}}\, \{ F \varphi_i \mid i \in I \}$ if and only if there are morphisms ψ_i in $\mathcal{K}_1$ such that $T_1 \psi_i = T_1 \varphi_i$, $\mathcal{E} \psi_i = \mathcal{E} \varphi_i$ for all $i \in I$ and that $\mathcal{D} \psi_i = \mathcal{D} \psi_j \in F^{-1} [\mathcal{K}_2 - \underleftarrow{\mathrm{Lim}}\, \{ F \varphi_i \mid i \in I \}]$ for all $\langle i, j \rangle \in I \times I$.

Dually for inductive limits.

Proof. The necessity is obvious. We shall prove the sufficiency. Evidently $FX < \mathcal{K}_2 - \underleftarrow{\mathrm{Lim}}\{F\varphi_i \mid i \in I\}$. By the assumption there is an object $Y \in F^{-1}[\mathcal{K}_2 - \underleftarrow{\mathrm{Lim}}\{F\varphi_i \mid i \in I\}]$ and morphisms $\psi_i \in T_1^{-1}[T_1\varphi_i] \cap \mathrm{Hom}_{\mathcal{X}_1}(Y, E\varphi_i)$ for each $i \in I$. Hence $Y < X$ and consequently $\mathcal{K}_2 - \underleftarrow{\mathrm{Lim}}\{F\varphi_i \mid i \in I\} < FX$.

Corollary. Let $\mathcal{K}_1$ be a full subcategory of $\mathcal{K}_2$.

(a) $\mathcal{K}_1$ is projective in $\mathcal{K}_2$ if and only if for any nonvoid family $\{\varphi_i \mid i \in I\}$ of morphisms of $\mathcal{K}_1$ $\mathcal{K}_2 - \underleftarrow{\mathrm{Lim}}\{\varphi_i \mid i \in I\}$ is an object of $\mathcal{K}_1$ provided it exists.

(b) $\mathcal{K}_1$ is projective in $\mathcal{K}_2$ if each object X of $\mathcal{K}_2$ has its upper modification $\langle Y, \varphi \rangle$ in $\mathcal{K}_1$ such that $T_2\varphi = i_{T_2X}$.

Dually for inductive subcategories.

2. Proximity spaces

Theorem 2.1 Let P be a set. Consider the following conditions for $p \subset \exp P \times \exp P$, $\wp \subset \mathcal{C}(P) \times \exp' P$, $\mathcal{N} \subset \exp P \times \exp\exp P$:

(1) $\langle X, \emptyset \rangle \in p \cup p^{-1}$ for no $X \subset P$;

(2) $\langle X, Y \rangle \in p$ whenever $X \cup Y \subset P$, $X \cap Y \neq \emptyset$;

(3) if $Y_1 \cup Y_2 \subset P$ then $\langle X, Y_1 \cup Y_2 \rangle \in p$ if and only if either $\langle X, Y_1 \rangle \in p$ or $\langle X, Y_2 \rangle \in p$;

(α) $\langle M, X \rangle \in \wp$ whenever $E M = (x) \subset X \subset P$;

(β) if $\langle M, X \rangle \in \wp$ and if M' is a subnet of M then $\langle M', X \rangle \in \wp$

(γ) if $\langle M, X \rangle \in \mathcal{C}(P) \times \exp P - \wp$ then $\wp \cap (\mathcal{C}(E M') \times (X)) = \emptyset$ for some subnet M' of M ;

(a) $\mathcal{N}$ is a single-valued relation $\{\mathcal{N}_X \mid X \in \exp P\}$;

(b) $\bigcap \mathcal{N}_X \supset X$ for each $X \subset P$, $\mathcal{N}_\emptyset = \exp P$;

(c) $\mathcal{N}_X$ is a filter in $\exp P$ for each $X \in \exp P$.

Then:

$R_1 = \{\langle p, \wp\rangle \mid p = \{\langle X, Y\rangle \mid Y \subset P$ and $\wp \cap (\mathcal{C}(Y) \times (X)) \neq \emptyset\}\} = \{\langle p, \wp\rangle / \wp = \{\langle M, X\rangle \mid M \in \mathcal{C}(P)\ \langle X, \varepsilon M'\rangle \in p$ for each subnet M' of $M\}$ is a one-to-one relation for the class of all relations p on $\exp P$ satisfying (1), (2), (3) and the class of all relations $\wp \subset \mathcal{C}(P) \times \exp' P$ satisfying (α), (β), (γ);

$R_2 = \{\langle p, \mathcal{N}\rangle \mid p = \{\langle X, Y\rangle \mid Y \subset P$ and $P - Y \notin \mathcal{N}_X\} = \{\langle p, \mathcal{N}\rangle \mid \mathcal{N}_X = \{Y \mid Y \subset P$ and $\langle X, P - Y\rangle \notin p\}$ for each $X \subset P\}$

is a one-to-one relation for the class of all relations p on $\exp P$ satisfying (1), (2), (3) and the class of all relations $\mathcal{N} \subset \exp P \times \exp \exp P$ satisfying (a), (b), (c);

$R_3 = \{\langle \wp, \mathcal{N}\rangle \mid \wp = \{\langle M, X\rangle \mid M \in \mathcal{C}(P)$, M is eventually in each $Y \in \mathcal{N}_X\}\} = \{\langle \wp, \mathcal{N}\rangle \mid \mathcal{N}_X = \{Y \mid Y \subset P$ and $\wp \cap (\mathcal{C}(P - Y) \times (X)) = \emptyset\}$ for each $X \subset P\}$

is a one-to-one relation for the class of all relations $\wp \subset \mathcal{C}(P) \times \exp' P$ satisfying (α), (β), (γ) and the class of all relations $\mathcal{N} \subset \exp P \times \exp \exp P$ satisfying (a), (b), (c).

We write $\wp \longleftrightarrow p \longleftrightarrow \mathcal{N}$ provided $\wp$, p, $\mathcal{N}$ fulfil the above conditions and $\langle p, \wp\rangle \in R_1$, $\langle p, \mathcal{N}\rangle \in R_2$ (then $\langle \wp, \mathcal{N}\rangle \in R_3$, because $R_3 = R_2 \circ R_1^{-1}$).

Definition 2.1. Let $\wp \longleftrightarrow p \longleftrightarrow \{\mathcal{N}_X \mid X \in \exp P\}$. Then p is called a proximity for P and the pair $\langle P, p\rangle$ is called a proximity space. The relation $\wp$ is the convergence class of $\langle P, p\rangle$ and every $\mathcal{N}_X$ is the neighborhood system of X in $\langle P, p\rangle$.

Definition 2.2. A proximity p, a proximity space $\langle P, p\rangle$ resp., is called monotone if one of the following equivalent conditions is fulfilled ($\wp \longleftrightarrow p \longleftrightarrow \{\mathcal{N}_X \mid X \in P\}$) :

1) $\langle X, Y\rangle \in p$, $P \supset Z \supset X$ implies $\langle Z, Y\rangle \in p$;

2) $\langle M, X\rangle \in \wp$, $P \supset Z \supset X$ implies $\langle M, Z\rangle \in \wp$;

3) $P \supset X \supset Y$ implies $\mathcal{N}_X \subset \mathcal{N}_Y$.

Definition 2.3. Let r be a relation in $\exp' P$. Then $p = \{\langle X, Y\rangle \mid$ either $X \cap Y \neq \emptyset$ or for each finite cover $\mathcal{O}$ of Y there is a $Z \in \mathcal{O}$ such that $\langle X, Z'\rangle \in r$ whenever $P \supset Z' \supset Z\} \cap (\exp' P \times \exp' P)$ is a proximity which is called generated by r .

We shall say that a relation $\sigma \subset \mathcal{C}(P) \times \exp' P$ generates a convergence class $\wp$ if $P \subset \mathcal{C}(P) \times \exp' P$ and $\wp$ is the smallest relation greater than σ, satisfying the conditions (α), (β), (γ) of theorem 2.1.

($\wp = \{\langle M, X\rangle \mid M \in \mathcal{C}(P)$ and $\sigma' \cap (\mathcal{C}(\mathcal{E} M') \times (X)) \neq \emptyset$ for each subnet M' of $M\}$ where $\sigma' = \{\langle M, X\rangle \mid M \in \mathcal{C}(P)$, $X \subset P$ and either $\mathcal{E} M = (x) \subset X$ or M is a subnet of a net N for which $\langle N, X\rangle \in \sigma\}$).

Remark 2.1. (a) If r from definition 2.3 fulfils the condition (2) of theorem 2.1 then p is the greatest proximity smaller than r . The generating relations occurring in this paper always satisfy (1),(2) and the part "if" of (3),(α),(β) resp. (i.e. $\sigma = \sigma'$).

(b) In generating $\mathcal{N}$ we restrict ourselves on the well-known concepts of sub-bases or bases of filters.

(c) We shall write usually $X\, p\, Y$ instead of $\langle X, Y\rangle \in p$ and X non p Y instead of $\langle X, Y\rangle \notin p$, $X \cup Y \subset P$.

Definition 2.4. Let f be a mapping of a proximity space $\langle P, p\rangle$ into another one $\langle Q, q\rangle$ and let $\wp \leftrightarrow p \leftrightarrow \{\mathcal{U}_X \mid X \subset P\}$, $\sigma \leftrightarrow q \leftrightarrow \{\mathcal{V}_X \mid X \subset P\}$.

(A) We say that f is upper proximally continuous if one of the following equivalent conditions is fulfilled:

(a) if $X \, p \, Y$ then $f[X] \, q \, f[Y]$;

(b) if $\langle M, X \rangle \in \wp$ then $\langle f \circ M, f[X] \rangle \in \delta$;

(c) if $X \subset P$, $V \in \mathcal{V}_{f[X]}$ then $f^{-1}[V] \in \mathcal{U}_X$.

(B) We say that f is <u>lower proximally continuous</u> if one of the following equivalent conditions is fulfilled:

(a) if $X \cup Y \subset Q$, $f^{-1}[X] \, p \, f^{-1}[Y]$ then $X \, q \, Y$;

(b) if $X \subset Q$, $\langle M, f^{-1}[X] \rangle \in \wp$ then $\langle f \circ M, X \rangle \in \delta$;

(c) if $V \in \mathcal{V}_X$ then $f^{-1}[V] \in \mathcal{U}_{f^{-1}[X]}$.

(C) We say that f is <u>proximally continuous</u> if f is both upper and lower proximally continuous.

<u>Remark 2.2</u>. Evidently, the class of all proximity spaces with upper proximally continuous mappings, lower proximally continuous mappings, proximally continuous mappings resp., forms a category $\mathcal{P}^U$, $\mathcal{P}^L$, $\mathcal{P}$ resp.

<u>Theorem 2.2</u>. Let f be a mapping of a proximity space $\langle P, p \rangle$ into another one $\langle Q, q \rangle$.

(a) Let $f[P] = Q$ or let q be monotone. Then if f is upper proximally continuous it is also lower proximally continuous and hence proximally continuous.

(b) Let f be one-to-one or let p be monotone. Then if f is lower proximally continuous it is also upper proximally continuous and hence proximally continuous.

<u>Example 2.1.</u> Let $P = (a, b, c)$, $Q = (\alpha, \beta)$,

$p = \{\langle A, B \rangle \mid A \cap B \neq \emptyset$ or $A \neq (a, b)\} \cap (\exp' P \times \exp' P)$

$q = \{\langle X, Y \rangle \mid X \cap Y \neq \emptyset$ or $\alpha \notin X\} \cap (\exp' Q \times \exp' Q)$.

If we put $f a = f b = \alpha$, $f c = \beta$, $g \alpha = c$, $g \beta = b$ then f is a lower proximally continuous mapping of the proximity space $\langle P, p \rangle$ onto the monotone proximity space $\langle Q, q \rangle$ which is not upper proximally continuous and g is an upper proximally continuous one-to-one mapping of the monotone proximity space $\langle Q, q \rangle$ into the proximity space $\langle P, p \rangle$ which is not lower proximally continuous.

Definition 2.5. We say that a proximity p is finer than a proximity q or that a proximity q is coarser than a proximity p (sign $p < q$) if p, q are proximities for the same set P and the identity mapping $\Delta_P : \langle P, p \rangle \to \langle P, q \rangle$ is proximally continuous (i.e. $U\mathfrak{D}p = U\mathfrak{D}q$ and $p \subset q$).

Theorem 2.3. The set of all proximities for a set P is complete in the order $<$. Let $A \neq \emptyset$ and for each $\alpha \in A$ p_α be a proximity for a set P and $\wp_\alpha \longleftrightarrow p_\alpha \longleftrightarrow \{\mathcal{U}_X^\alpha \mid X \subset P\}$. Let

$$\sigma_1 \longleftrightarrow q_1 = \sup\{p_\alpha \mid \alpha \in A\} \longleftrightarrow \mathcal{V}_X^1 \mid X \subset P\},$$
$$\sigma_2 \longleftrightarrow q_2 = \inf\{p_\alpha \mid \alpha \in A\} \longleftrightarrow \mathcal{V}_X^2 \mid X \subset P\}.$$

Then

1) $q_1 = U\{p_\alpha \mid \alpha \in A\}$;
2) $\mathcal{V}_X^1 = \bigcap\{\mathcal{U}_X^\alpha \mid \alpha \in A\}$ for each $X \subset P$;
3) $\bigcup\{\wp_\alpha \mid \alpha \in A\}$ generates σ_1 ;
4) $\bigcap\{p_\alpha \mid \alpha \in A\}$ generates q_2 ;
5) $\bigcup\{\mathcal{U}_X^\alpha \mid \alpha \in A\}$ is a subbase of $\mathcal{V}_X^2$ for each $X \subset P$;
6) $\sigma_2 = \bigcap\{\wp_\alpha \mid \alpha \in A\}$.

If $\{p_\alpha \mid \alpha \in A\}$ is left-directed then $q_2 = \bigcap\{p_\alpha \mid \alpha \in A\}$ and $\mathcal{V}_X^2 = \bigcup\{\mathcal{U}_X^\alpha \mid \alpha \in A\}$ for all $X \subset P$.

Theorem 2.4. The categories $\mathcal{P}^U$, $\mathcal{P}^L$ are S-categories over $\mathcal{M}$ with respect to the forgetful functors.

Proof. The proof of the conditions (1),(2),(5) of definition 1 in [5] is easy (notice that $\{\langle X, Y \rangle \mid X \cup Y \subset P,\ X \cap Y \neq \emptyset\}$, $\exp' P \times \exp' P$ resp. is the finest, the coarsest resp., proximity for a set P). (4) was proved in theorem 2.3 . It remains to prove (3). Let $\langle f, \langle P, p \rangle, \langle Q, q \rangle\rangle$ be a morphism of $\mathcal{P}^U$, $\mathcal{P}^L$ resp. Let R be a set and $\varphi : P \to R$, $\psi : R \to Q$ mappings with the composition $\psi \circ \varphi = f$. We want to define a proximity r for R such that φ, ψ are upper proximally continuous, lower proximally continuous resp. It is sufficient to put

$r = \{\langle X, Y\rangle \mid X \cup Y \subset R,\ \psi[X]\ q\ \psi[Y]\}$ in the first case and

$r = \{\langle X, Y\rangle \mid X \cup Y \subset R$ and either $X \cap Y \neq \emptyset$ or $\varphi^{-1}[X]\ p\ \varphi^{-1}[Y]\}$ in the second case.

Remark 2.3. The category $\mathcal{P}$ fulfils all the conditions of definition 1 in [5] except (3) as follows from the following proposition.

Let $\langle P, p\rangle$ be a non-monotone proximity space. Then there is a proximity q for P and a proximally continuous mapping $f : \langle P, p\rangle \rightarrow \langle P, q\rangle$ such that the mappings

$$\varphi = f : \langle P, p\rangle \rightarrow \langle f[P], r\rangle,$$

$$\psi = \Delta_{f[P]} : \langle f[P], r\rangle \rightarrow \langle P, q\rangle$$

are proximally continuous for no proximity r for $f[P]$.

We shall give a short proof of this proposition. There are subsets K, M, N of P such that $M\ p\ N$, K non $p\ N$, $K \supset M$. Let $f = \Delta_{P-(K-M)} \cup (K - M) \times (m)$ where $m \in M$. Then f is a proximally continuous mapping of $\langle P, p\rangle$ into $\langle P, q\rangle$ where $q = \{\langle X, Y\rangle \mid X \cup Y \subset P$ and either $X \cap Y \neq \emptyset$ or $f^{-1}[X]\ p\ f^{-1}[Y]$ or $f[A] = X$, $A\ p\ f^{-1}[Y]$ for some $A \subset P\}$. Now, let r be a proximity for $f[P]$. The proximal continuity of φ implies $M\ r\ N$ and the proximal continuity of ψ implies M non $r\ N$.

Remark 2.4. Let us denote by $\mathcal{P}_M$ the full subcategory of $\mathcal{P}$ generated by all monotone proximity spaces. It follows from theorem 2.2 that $\mathcal{P}_M$ is a full subcategory both of $\mathcal{P}^U$ and $\mathcal{P}^L$.

Lemma 2.1. For each proximity p_o there exists a coarsest monotone proximity p_1 finer than p_o and a finest monotone proximity p_2 coarser than p_o. If $\wp_i \leftrightarrow p_i \leftrightarrow \{\mathcal{U}_X^i \mid X \subset P\}$, $(i = 0,1,2)$ then

1) $\{\langle X, Y\rangle \mid Z\, p_o\, Y$ whenever $Z \supset X\}$ generates p_1;

2) $\bigcup\{\mathcal{U}_Y^o \mid P \supset Y \supset X\}$ is a subbase of $\mathcal{U}_X^1$ for each $X \subset P$;

3) $\wp_1 = \{\langle M, X\rangle \mid X \in \exp' P$ and $\langle M, Y\rangle \in \wp_o$ whenever $X \subset Y \subset P\}$;

4) $p_2 = \{\langle X, Y\rangle \mid P \supset X$ and $Z\, p_o\, Y$ for some $Z \subset X\}$;

5) $\mathcal{U}_X^2 = \bigcap\{\mathcal{U}_Y^o \mid Y \subset X\}$ for each $X \subset P$;

6) $\{\langle M, X\rangle \mid P \supset X$ and $\langle M, Z\rangle \in \wp_o$ for some $Z \subset X\}$ generates $\wp_2$.

<u>Theorem 2.5</u>. Each object $\langle P, p_o\rangle$ of $\mathcal{P}^L$ has its lower modification $\langle\langle P, p_1\rangle, \langle\Delta_P, \langle P, p_1\rangle, \langle P, p_o\rangle\rangle\rangle$ in $\mathcal{P}_M$ and each object $\langle P, p_o\rangle$ of $\mathcal{P}^U$ has its upper modification $\langle\langle P, p_2\rangle, \langle\Delta_P, \langle P, p_o\rangle, \langle P, p_2\rangle\rangle\rangle$ in $\mathcal{P}_M$. Hence each object of $\mathcal{P}$ has its upper and lower modifications in $\mathcal{P}_M$.

<u>Corollary 1</u>. $\mathcal{P}_M$ is an S-category over $\mathcal{M}$ with respect to the forgetful functor.

Proof. See theorem 1 in [5].

<u>Corollary 2</u>. $\mathcal{P}_M$ is projective in $\mathcal{P}^U$ and inductive in $\mathcal{P}^L$.

Proof. See corollary (b) of theorem 1.2.

<u>Remark 2.5</u>. It follows from theorems 2.2, 2.4 and the foregoing corollary that

$$\mathcal{P}^U - \underrightarrow{\mathrm{Lim}}\,\{\varphi_i \mid i \in I\} = \mathcal{P}_M - \underrightarrow{\mathrm{Lim}}\,\{\varphi_i \mid i \in I\}$$

if $\{\varphi_i \mid i \in I\}$ is a nonvoid family of epimorphisms of $\mathcal{P}_M$ and that

$$\mathcal{P}^L - \underleftarrow{\mathrm{Lim}}\,\{\varphi_i \mid i \in I\} = \mathcal{P}_M - \underleftarrow{\mathrm{Lim}}\,\{\varphi_i \mid i \in I\}$$

if $\{\varphi_i \mid i \in I\}$ is a nonvoid family of monomorphisms of $\mathcal{P}_M$. Hence $\mathcal{P}_M^{\cdot}$ is cohereditary in $\mathcal{P}^U$ and hereditary in $\mathcal{P}^L$

Lemma 2.2. Let $\mathcal{K}$ be an S-category over a category $\mathcal{C}$ with respect to a covariant functor T , $\mathcal{K}'$ be a subcategory of $\mathcal{K}$ and an S-category over $\mathcal{C}$ with respect to $T/\mathcal{K}'$ and let $\mathcal{C}$ have inversion property . Suppose that an object X of $\mathcal{K}$ has an upper modification in $\mathcal{K}'$. If there is an object Y' of $\mathcal{K}'$ such that $\langle X, Y' \rangle \in R_{TX}$ (for R_A see definition 1 in [5]) then there is a smallest object Y of $\mathcal{K}'$ greater than X in the order R_{TX} (then $T\varphi = i_{TX}$ for some $\varphi \in \mathrm{Hom}_{\mathcal{K}}(X, Y)$) and $\langle Y, \varphi \rangle$ is an upper modification of X in $\mathcal{K}'$.

Proof. Let $\langle Z, \psi \rangle$ be an upper modification of X in $\mathcal{K}'$. Evidently $\varphi = \chi \circ \psi$ for some $\chi \in \mathrm{Hom}_{\mathcal{K}'}(Z, Y')$. Hence ψ is a monomorphism. It is easy to see that ψ is also an epimorphism and hence a bimorphism. Indeed, otherwise $\psi' \circ \psi = \psi'' \circ \psi$ for some different morphisms ψ', ψ'' of $\mathcal{K}'$ with $\mathcal{E}\psi' = \mathcal{E}\psi''$ and this contradicts our assumption that $\langle Z, \psi \rangle$ is an upper modification. As $T\psi$ is invertible, there are isomorphisms χ', ψ' in $\mathcal{K}'$ such that $T\psi' = T\psi$, $\psi' \circ \chi' = i_Z$. It follows from the equalities

$$T(\chi \circ \psi') = T\chi \circ T\psi' = T\chi \circ T\psi = T(\chi \circ \psi) = T\varphi = i_{TX} ,$$

$$T(\chi' \circ \psi) = T\chi' \circ T\psi = T\chi' \circ T\psi' = i_{TX}$$

that $\mathcal{D}\psi'$ is an object of $\mathcal{K}'$ greater than X and smaller than Y' . Now, it is sufficient to put $Y = \mathcal{D}\psi'$.

Theorem 2.6. (a) The class of all objects of $\mathcal{P}^L$ having the upper modifications in $\mathcal{P}_M$ is precisely the class of objects of $\mathcal{P}_M$. (b) Let us put for a moment

$\tau = \{\langle p, q\rangle \mid q$ is the coarsest monotone proximity finer than $p\}$ and $\mu_Q = \{\langle p, q\rangle \mid \langle\langle Q, q\rangle, \Delta_Q\rangle$ is a subobject of $\langle \bigcup \mathcal{D}\, p, p\rangle$ in $\mathcal{P}^U\}$. A proximity space $\langle P, p\rangle$, as an object of $\mathcal{P}^U$, has its lower modification in $\mathcal{P}_M$ if and only if $\tau \mu_Q p = \mu_Q \tau p$ for all $Q \subset P$ (i.e. if $Q \subset P$, q is the coarsest monotone proximity finer than $p \cap (\exp Q \times \exp Q)$ then $K\, p\, N$ provided $M\, q\, N$, $P \supset K \supset M$).

Proof. (a) Let $\langle P, p\rangle$ be a non-monotone proximity space. There are subsets K, M, N of P such that $M\, p\, N$, K non $p\, N$, $K \supset M$. Put $Q = (a,b,c)$, $q = \exp' Q \times \exp' Q - (\langle (a),(b)\rangle)$, $f = (K \times (a)) \cup (N \times (b)) \cup ((P - (K \cup N)) \times (c))$. Then $\langle Q, q\rangle$ is a monotone proximity space, f is a lower proximally continuous mapping $\langle P, p\rangle \longrightarrow \langle Q, q\rangle$ but f is not lower proximally continuous of $\langle P, p'\rangle$ into $\langle Q, q\rangle$ where p' is the finest monotone proximity coarser than p. Hence $\langle P, p\rangle$ has no upper modification in $\mathcal{P}_M$ (see the foregoing lemma).

(b) Our assertion follows from the characterization of subobjects expressed in theorem 5 of [5] and from the fact that a mapping f of a monotone proximity space $\langle Q', q'\rangle$ into $\langle P, p\rangle$ is upper proximally continuous if and only if the monotone proximity of $\mathcal{P}^U - \underrightarrow{\mathrm{Lim}} \langle f, \langle Q', q'\rangle, \langle f[Q'], \mu_{f[Q']}\, p\rangle\rangle$ is finer than $\mu_{f[Q']}\, p$ (see theorem 2 of [5] and remark 2.4.).

Example 2.2. Let P be a set, $P \supset X_0 \neq \emptyset$, card $(P - X_0) \geqq 2$, $p = \{\langle X, Y\rangle \mid X \cup Y \subset P$, either $X \cap Y \neq \emptyset$ or $X = X_0$, $Y \neq \emptyset\}$. $\langle P, p\rangle$ is a non-monotone proximity space fulfilling the condition of theorem 2.6(b).

Remark 2.6. It was said in the introduction that we can construct products, subobjects etc. in $\mathcal{P}^U$, $\mathcal{P}^L$, $\mathcal{P}_M$ resp., from those in $\mathcal{M}$. It follows from theorems in [5] that for this construction it is sufficient to know characterizations of sup, inf in R_A and of objects in $\mathcal{P}^U$, $\mathcal{P}^L$, $\mathcal{P}_M$ resp., projectively (inductively) generated by one morphism. Characterizations of sup, inf are described in theorem 2.3 ; characterizations of generated objects are left to the reader.

3. Semi-uniform spaces

Theorem 3.1. Let P be a set. Consider the following conditions for $\mathcal{U} \subset \exp(P \times P)$, $\mathcal{C} \subset \mathcal{C}(P \times P)$:

(a) $\mathcal{U}$ is a filter in $\exp(P \times P)$;

(b) $\bigcap \mathcal{U} \supset \Delta_P$;

(α) $M \in \mathcal{C}$ whenever $\mathcal{E} M = (\langle x, x \rangle \cdot) \subset P \times P$;

(β) if $M \in \mathcal{C}$ and if M' is a subnet of M then $M' \in \mathcal{C}$;

(γ) if $M \in \mathcal{C}(P \times P) - \mathcal{C}$ then $\mathcal{C} \cap \mathcal{C}(\mathcal{E} M') = \emptyset$ for some subnet M' of M .

Then $R = \{\langle \mathcal{U}, \mathcal{C} \rangle \mid \mathcal{C} = \{M \mid M \in \mathcal{C}(P \times P)$, M is eventually in each $U \in \mathcal{U}\}\} = \{\langle \mathcal{U}, \mathcal{C} \rangle \mid \mathcal{U} = \{U \mid U \subset P \times P$, each $M \in \mathcal{C}$ is eventually in $U\}\}$ is a one-to-one relation, for the class of all $\mathcal{U} \subset \exp(P \times P)$ satisfying (a),(b) and the class of all $\mathcal{C} \subset \mathcal{C}(P \times P)$ satisfying (α),(β),(γ). We write $\mathcal{U} \longleftrightarrow \mathcal{C}$ provided $\mathcal{U}$, $\mathcal{C}$ fulfil the above conditions and $\langle \mathcal{U}, \mathcal{C} \rangle \in R$.

Definition 3.1. Let $\mathcal{U} \longleftrightarrow \mathcal{C}$, $P = \mathcal{D} \bigcup \mathcal{U}$. Then we call $\mathcal{U}$ a semi-uniformity for P , $\langle P, \mathcal{U} \rangle$ a semi-uniform space and $\mathcal{C}$ the convergence class of $\langle P, \mathcal{U} \rangle$.

Definition 3.2. Let $\mathcal{U} \longleftrightarrow \mathcal{C}$. The semi-uniformity $\mathcal{U}$, the semi-uniform space $\langle P, \mathcal{U} \rangle$ resp. is called symmetric if one of the following equivalent conditions is fulfilled:

1) if $U \in \mathcal{U}$ then $U^{-1} \in \mathcal{U}$;

2) if $M \in \mathcal{C}$ then $\{\langle a, \langle \mathcal{E} M_a, \mathcal{D} M_a \rangle\rangle \mid a \in \mathcal{D} M\} \in \mathcal{C}$;

The semi-uniformity $\mathcal{U}$, the semi-uniform space $\langle P, \mathcal{U} \rangle$ resp., is called uniformity, uniform space resp., if one of the following equivalent conditions is fulfilled:

1) each $U \in \mathcal{U}$ contains $V \circ V$ for some $V \in \mathcal{U}$;

2) if $M \in \mathcal{C}$, $N \in \mathcal{C}$, $\mathcal{D} M = \mathcal{D} N$, $\mathcal{E} M_a = \mathcal{D} N_a$ for all $a \in \mathcal{D} M$, then $\{\langle a, \langle \mathcal{D} M_a, \mathcal{E} N_a \rangle\rangle \mid a \in \mathcal{D} M\} \in \mathcal{C}$.

Remark 3.1. We shall use this notation:
if $M \in \mathcal{C}(P \times P)$ then $\alpha M = \{\langle a, \mathcal{D} M_a \rangle \mid a \in \mathcal{D} M\}$, $\beta M = \{\langle a, \mathcal{E} M_a \rangle \mid a \in \mathcal{D} M\}$. Hence we can assign in one-to-one way to each $\mathcal{C} \subset \mathcal{C}(P \times P)$ the relation $\rho_{\mathcal{C}} = \{\langle \alpha M, \beta M \rangle \mid M \in \mathcal{C}\}$ on $\mathcal{C}(P)$. If $\mathcal{U} \leftrightarrow \mathcal{C}$ then $\rho_{\mathcal{C}}$ is reflexive. By definition 3.2 $\mathcal{U}$ is symmetric if and only if $\rho_{\mathcal{C}}$ is symmetric, $\mathcal{U}$ is a uniformity if and only if $\rho_{\mathcal{C}}$ is transitive. So $\mathcal{U}$ is a symmetric uniformity if and only if $\rho_{\mathcal{C}}$ is an equivalence.

Remark 3.2. Similarly as in definition 2.3 we shall say that $\mathcal{D} \subset \mathcal{C}(P \times P)$ generates a convergence class $\mathcal{C}$ if $\mathcal{C} \subset \mathcal{C}(P \times P)$ is the smallest class containing $\mathcal{D}$ and satisfying the conditions (α), (β), (γ) of theorem 3.1. ($\mathcal{C} = \{M \mid M \in \mathcal{C}(P \times P)$, $\mathcal{D}' \cap \mathcal{C}(\mathcal{E} M') \neq \emptyset$ for each subnet M' of $M\}$, where $\mathcal{D}' = \mathcal{D} \cup \{M \mid M \in \mathcal{C}(P \times P)$, $\mathcal{E} M = (\langle x, x \rangle)$ for some $x\}$. As a rule $\mathcal{D}' = \mathcal{D}$.)

Definition 3.3. Let f be a mapping of a semi-uniform space $\langle P, \mathcal{U} \rangle$ into another one $\langle Q, \mathcal{V} \rangle$ and let $\mathcal{U} \leftrightarrow \mathcal{C}$, $\mathcal{V} \leftrightarrow \mathcal{D}$. We say that f is uniformly continuous if $(f \times f)^{-1}[V] \in \mathcal{U}$ for each $V \in \mathcal{V}$ or equivalently if $(f \times f) \circ M \in \mathcal{D}$ for each $M \in \mathcal{C}$.

Remark 3.3. Evidently, the semi-uniform spaces with uniformly continuous mappings form a category $\mathcal{U}$. We denote by $\mathcal{U}_S$, $\mathcal{U}_U$, $\mathcal{U}_{SU}$ resp., the full subcategory of $\mathcal{U}$ generated by symmetric semi-uniform spaces, uniform spaces, symmetric uniform spaces resp.

Definition 3.4. We say that a semi-uniformity $\mathcal{U}$ is finer than another one $\mathcal{V}$ or that $\mathcal{V}$ is coarser than $\mathcal{U}$ (sign $\mathcal{U} < \mathcal{V}$) if $\mathcal{U}$, $\mathcal{V}$ are semi-uniformities for the same set P and if the identity mapping $\Delta_P : \langle P, \mathcal{U} \rangle \to \langle P, \mathcal{V} \rangle$ is uniformly continuous (i.e. $\mathcal{D} \cup \mathcal{U} = \mathcal{D} \cup \mathcal{V}$, $\mathcal{U} \supset \mathcal{V}$).

Theorem 3.2. The set of all semi-uniformities for a set P is complete in the order $<$. Let for each $\alpha \in A$ $(A \neq \emptyset)$ $\mathcal{U}_\alpha$ be a semi-uniformity for P and $\mathcal{U}_\alpha \leftrightarrow \mathcal{C}_\alpha$. Suppose that $\mathcal{V}_1 = \sup \{ \mathcal{U}_\alpha \mid \alpha \in A \} \leftrightarrow \mathcal{D}_1$, $\mathcal{V}_2 = \inf \{ \mathcal{U}_\alpha \mid \alpha \in A \} \leftrightarrow$ $\leftrightarrow \mathcal{D}_2$.

Then

1) $\mathcal{V}_1 = \bigcap \{ \mathcal{U}_\alpha \mid \alpha \in A \}$;

2) $\bigcup \{ \mathcal{C}_\alpha \mid \alpha \in A \}$ generates $\mathcal{D}_1$;

3) $\bigcup \{ \mathcal{U}_\alpha \mid \alpha \in A \}$ is a subbase of $\mathcal{V}_2$;

4) $\mathcal{D}_2 = \bigcap \{ \mathcal{C}_\alpha \mid \alpha \in A \}$.

If $\{ \mathcal{U}_\alpha \mid \alpha \in A \}$ is left-directed then $\mathcal{V}_2 = \bigcup \{ \mathcal{U}_\alpha \mid \alpha \in A \}$.

Theorem 3.3. $\mathcal{U}$ is an S-category over $\mathcal{M}$ with respect to the forgetful functor.

Proof. We shall prove only the condition (3) of definition 1 in [5]. ((4) was proved in theorem 3.2 and the remaining conditions are trivially fulfilled; notice that $(P \times P)$, $\{ U \mid P \times P \supset U \supset \Delta_P \}$ resp., is the coarsest, the finest resp., semi-uniformity for P.) Let f be a uniformly continuous mapping of $\langle P, \mathcal{U} \rangle$ into $\langle Q, \mathcal{V} \rangle$, $f = \psi \circ \varphi$, $\mathcal{D}\psi = R$. If we put $\mathcal{W} = \{ W \mid R \times R \supset W \supset (\psi \times \psi)^{-1} [V]$ for some $V \in \mathcal{V} \}$, then the mappings $\varphi : \langle P, \mathcal{U} \rangle \to \langle R, \mathcal{W} \rangle$,

$\psi : \langle R, \mathcal{W} \rangle \to \langle Q, \mathcal{V} \rangle$ are uniformly continuous.

Lemma 3.1. For every semi-uniformity $\mathcal{U}_o$ there exists a coarsest symmetric semi-uniformity $\mathcal{U}_1$ finer than $\mathcal{U}_o$, a finest symmetric semi-uniformity $\mathcal{U}_2$ coarser than $\mathcal{U}_o$, a finest uniformity $\mathcal{U}_3$ coarser than $\mathcal{U}_o$ and a finest symmetric uniformity $\mathcal{U}_4$ coarser than $\mathcal{U}_o$. If $\mathcal{U}_i \longleftrightarrow \mathcal{C}_i$ for $i \in (0,1,2,3,4)$ then

1) $\{U \mid U \in \mathcal{U}_o$ or $U^{-1} \in \mathcal{U}_o\}$ is a subbase for $\mathcal{U}_1$; $\{M \mid M \in \mathcal{C}_o$ and $\{\langle a, \langle \mathcal{E} M_a, \mathcal{D} M_a \rangle\rangle \mid a \in \mathcal{D} M\} \in \mathcal{C}_o\} = \mathcal{C}_1$;

2) $\{U \mid U \in \mathcal{U}_o$ and $U^{-1} \in \mathcal{U}_o\} = \mathcal{U}_2$; $\{M \mid M \in \mathcal{C}_o$ or $\{\langle a, (M_a)^{-1} \rangle \mid a \in \mathcal{D} M\} \in \mathcal{C}_o\}$ generates $\mathcal{C}_2$;

3) $\{U \mid$ there is a sequence $\{U_n\}$ in $\mathcal{U}_o$ such that $U_o \subset U$ and $U_{n+1} \circ U_{n+1} \subset U_n$ for each $n\} = \mathcal{U}_3$;

4) $\{U \mid$ there is a sequence $\{U_n\}$ in $\mathcal{U}_o$ such that $U_o \subset U$ and $U_{n+1} \circ U_{n+1} \subset U_n$, $U_n = U_n^{-1}$ for each $n\} = \mathcal{U}_4$.

Theorem 3.4. Each object $\langle P, \mathcal{U}_o \rangle$ of $\mathfrak{U}$ has its lower modification $\langle\langle P, \mathcal{U}_1 \rangle, \langle \Delta_P, \langle P, \mathcal{U}_1 \rangle, \langle P, \mathcal{U}_o \rangle\rangle\rangle$ in $\mathfrak{U}_S$, its upper modification $\langle\langle P, \mathcal{U}_2 \rangle, \langle \Delta_P, \langle P, \mathcal{U}_o \rangle, \langle P, \mathcal{U}_2 \rangle\rangle\rangle$ in $\mathfrak{U}_S$, its upper modification $\langle\langle P, \mathcal{U}_3 \rangle, \langle \Delta_P, \langle P, \mathcal{U}_o \rangle, \langle P, \mathcal{U}_3 \rangle\rangle\rangle$ in $\mathfrak{U}_U$, its upper modification $\langle\langle P, \mathcal{U}_4 \rangle, \langle \Delta_P, \langle P, \mathcal{U}_o \rangle, \langle P, \mathcal{U}_4 \rangle\rangle\rangle$ in $\mathfrak{U}_{SU}$ and each object $\langle P, \mathcal{U}_o \rangle$ of $\mathfrak{U}_U$ has its lower modification $\langle\langle P, \mathcal{U}_1 \rangle \langle \Delta_P, \langle P, \mathcal{U}_1 \rangle, \langle P, \mathcal{U}_o \rangle\rangle\rangle$ in $\mathfrak{U}_{SU}$.

Proof follows from the foregoing lemma and for the upper modifications from the fact that if $f : \langle P, \mathcal{U} \rangle \to \langle Q, \mathcal{W} \rangle$ is a uniformly continuous mapping then the semi-uniformity $\{V \mid P \times P \supset V \supset (f \times f)^{-1}[W]$ for some $W \in \mathcal{W}\}$ is coarser than $\mathcal{U}$ and it is symmetric semi-uniformity, uniformity, symmetric uniformity resp., provided $\mathcal{W}$ has the same property.

The last assertion is the consequence of the fact that $\mathcal{U}_1$ is a uniformity provided $\mathcal{U}_0$ is a uniformity.

Corollary. $\mathcal{U}_s$, $\mathcal{U}_u$, $\mathcal{U}_{su}$ are S-categories over $\mathcal{M}$ with respect to the forgetful functors.

Proof. See theorem 1 in [5].

Corollary. $\mathcal{U}_s$ is projective and inductive in $\mathcal{U}$. $\mathcal{U}_u$ is projective in $\mathcal{U}$. $\mathcal{U}_{su}$ is projective in $\mathcal{U}$, $\mathcal{U}_s$, $\mathcal{U}_u$ and inductive in $\mathcal{U}_u$.

Proof. See corollary (b) of theorem 1.2.

Remark 3.4. The upper modification of an object ξ of $\mathcal{U}$ in $\mathcal{U}_{su}$ is the upper modification in $\mathcal{U}_u$ of the upper modification of ξ in $\mathcal{U}_s$.

Example 3.1. a) Let $\mathcal{U} = \{U \mid P \times P \supset U \supset (X \times X) \cup \cup (Y \times Y)\}$ where $X \cup Y = P$, $X \cap Y \neq \emptyset$, $X \neq P$, $Y \neq P$. The symmetric semi-uniformity $\mathcal{U}$ has neither the coarsest symmetric uniformity finer than $\mathcal{U}$ nor the coarsest uniformity finer than $\mathcal{U}$. Indeed, $\sup \{\mathcal{V} \mid \mathcal{V} < \mathcal{U}$, $\mathcal{V}$ is a symmetric uniformity $\} = \mathcal{U}$ because

$\mathcal{U} = \{U \mid P \times P \supset U \supset (X \times X) \cup \Delta_P\} \cap \{U \mid P \times P \supset U \supset (Y \times Y) \cup \cup \Delta_P\}$.

(b) The finest symmetric semi-uniformity coarser than the uniformity $\{U \mid P \times P \supset U \supset (P \times (a)) \cup \Delta_P\}$, where a is an element of at least three-point set P , is not a uniformity. Hence, the upper modification of an object ξ of $\mathcal{U}$ in $\mathcal{U}_{su}$ need not be the upper modification in $\mathcal{U}_s$ of the upper modification of ξ in $\mathcal{U}_u$.

Theorem 3.5. Let $\langle f, \langle P, \mathcal{U}_0 \rangle , \langle Q, \mathcal{V}_0 \rangle\rangle = f'$ be a morphism of $\mathcal{U}$. Put $\langle P, \mathcal{U}_1 \rangle = \mathcal{U} - \underleftarrow{\mathrm{Lim}}\, f'$, $\langle Q, \mathcal{V}_1 \rangle = \mathcal{U} - \underrightarrow{\mathrm{Lim}}\, f'$, $\mathcal{U}_i \longleftrightarrow \mathcal{C}_i$, $\mathcal{V}_i \longleftrightarrow \mathcal{D}_i$ $(i = 0,1)$. Then

1) $\mathcal{U}_1 = \{U \mid P \times P \supset U \supset (f \times f)^{-1}[V]$ for some $V \in \mathcal{V}_0\}$;

2) $\mathcal{C}_1 = \{M \mid M \in \mathcal{C}(P \times P),\ (f \times f) \circ M \in \mathcal{D}_0\}$;

3) $\mathcal{V}_1 = \{V \mid V \supset \Delta_Q,\ (f \times f)^{-1}[V] \in \mathcal{U}_0\} = \{V \mid Q \times Q \supset \supset V \supset \Delta_Q \cup (f \times f)[U]$ for some $U \in \mathcal{U}_0\}$;

4) $\{M \mid M \in \mathcal{C}(Q \times Q),\ (f \times f) \circ N = M$ for some $M \in \mathcal{C}_0\}$ generates $\mathcal{D}_1$.

Corollary. $\mathcal{U}_U$ is coproductive in $\mathcal{U}$ and $\mathcal{U}_{SU}$ is coproductive in $\mathcal{U}$, $\mathcal{U}_S$.

Proof. $\mathcal{V}_1$ in theorem 3.5 is a uniformity provided $\mathcal{U}_0$ is a uniformity and f is one-to-one. Our corollary now follows from the following obvious statement:
if $\{\mathcal{V}_\alpha \mid \alpha \in A\}$ is a nonvoid set of uniformities for a set P with the property
$\langle \alpha, \alpha' \rangle \in A \times A - \Delta_A$, $V_\alpha \in \mathcal{V}_\alpha$, $V_{\alpha'} \in \mathcal{V}_{\alpha'}$, implies $\{x \mid \text{card } V_\alpha^{-1}[x] > 1\} \cap \{x \mid \text{card } V_{\alpha'}^{-1}[x] > 1\} = \emptyset$ then $\sup \{\mathcal{V}_\alpha \mid \alpha \in A\}$ is a uniformity.

Example 3.2. Let $P = (a,b,c,d)$, $Q = (\alpha, \beta, \gamma)$, $U_0 = (a, b) \times (a, b) \cup (c, d) \times (c, d)$, $V_0 = (\alpha, \beta) \times \times (\alpha, \beta) \cup (\beta, \gamma) \times (\beta, \gamma)$.
$\mathcal{U} = \{U \mid P \times P \supset U \supset U_0\}$ is a symmetric uniformity for P , $\mathcal{V} = \{V \mid Q \times Q \supset V \supset V_0\}$ is a symmetric semi-uniformity for Q which is not a uniformity, $\langle Q, \mathcal{V} \rangle = \mathcal{U} - \underrightarrow{\text{Lim}}\, f$ where $f = \langle (\langle a, \alpha \rangle, \langle b, \beta \rangle, \langle c, \beta \rangle, \langle d, \gamma \rangle), \langle P, \mathcal{U} \rangle, \langle Q, \mathcal{V} \rangle \rangle$.
It follows that $\mathcal{U}_{SU}$ is not cohereditary in $\mathcal{U}$, $\mathcal{U}_S$ and hence $\mathcal{U}_U$ is not cohereditary in $\mathcal{U}$.

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ON THE PROXIMITY GENERATED BY ENTIRE FUNCTIONS

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We examine the proximity structure of the complex plane generated by the set of all entire functions. It is shown that this structure coincides with the finest proximity compatible with the usual topology of the plane.

In § 1, some fundamental concepts concerning proximity spaces are recalled, and the problems under consideration are formulated in terms of projectively generating mappings (however, this formulation is not used in what follows). In § 2, the problems in question are stated by means of current elementary topological concepts, In § 3, main theorems are stated, as well as some auxiliary propositions. Finally, § 4 contains the proofs.

§ 1 .

The simplest concepts of the theory of topological and uniform spaces are assumed to be known; however, for convenience, we recall certain concepts concerning proximity spaces (the theory of these spaces is due mainly to Yu.M. Smirnov [see, e.g. 5] ; a short survey of main concepts and results, as well as a list of references, is contained, e.g., in [1].

If M is a set, then a binary relation ϑ on the collection of all subsets of M is called a proximity structure (or simply a proximity) on M if, for any subsets X, Y, Z of M,

(1) $X \vartheta Y \Longleftrightarrow Y \vartheta X$,

(2) $(X \cup Y) \vartheta Z \Longleftrightarrow (X \vartheta Z$ or $Y \vartheta Z)$,

(3) $X \cap Y \neq \emptyset \Longrightarrow X \vartheta Y$,

(4) $X \vartheta Y \Longrightarrow X \neq \emptyset$,

(5) if X non ϑ Y , then there exist $X_1 \subset M$, $Y_1 \subset M$ such that $X_1 \cup Y_1 = M$ and neither $X \vartheta Y_1$ nor $Y \vartheta X_1$.

The pair (M, ϑ) is called a proximity space. If $X \vartheta Y$, we shall say that X and Y are near (under ϑ); if not, they are said to be distant (under ϑ). A mapping f of a proximity space (M, ϑ) into another one, (M_1, ϑ_1) , is called proximally continuous if $X \vartheta Y$ implies $f(X) \vartheta_1 f(Y)$. If ϑ , ϑ_1 are proximities on the same set M and $X \vartheta Y \Longrightarrow X \vartheta_1 Y$, then we shall say that ϑ_1 is coarser than ϑ or that ϑ is finer than ϑ_1 . It is well known that if ϑ is a proximity on M , then the formula $x \in \overline{X} \Longleftrightarrow (x) \vartheta X$ defines a completely regular topology on M ; we shall say that this topology is induced by the proximity ϑ .

We recall two simple instances of proximity spaces. If (M, ρ) is a metric space, then let ϑ be determined as follows: $X \vartheta Y$ if and only if for every $\varepsilon > 0$ there are points $x \in X$, $y \in Y$ with $\rho(x, y) < \varepsilon$; we shall say that ϑ is induced by the metric ρ . If M is a normal topological space, put $X \vartheta Y \Longleftrightarrow \overline{X} \cap \overline{Y} \neq \emptyset$; then ϑ is the finest proximity inducing the prescribed topology of M .

We shall now introduce the concept of projective generation (for topological, uniform, and proximity spaces). It will not be used in what follows; however, its introduction may help to show the connection of problems considered here with certain notions of a quite general character.

Let us say, for convenience, t-space, u-space, p-space instead of topological (uniform, proximity) space. The terms t-, u-, p-continuous mapping, t-, u-, p-structure will be used in an analoguous way. Finally, the letter c will be used as a "variable" to be replaced by t or u or p .

Now let X be a set; for any $a \in A$, let f_a be a mapping of X into a c-space X_a . It is easy to show that there exists a coarsest c-structure on X under which every f_a is c-continuous; we shall say that this structure is projectively generated by the mappings f_a . An important special case is obtained if f_a are mappings into the real line $\mathbb{R}$ or the complex plane $\mathbb{C}$ endowed with the usual structure (recall that the proximity and uniform structure of $\mathbb{R}$ or $\mathbb{C}$ are defined as follows: X and Y are near if $\inf_{x \in X, y \in Y} |x - y| = 0$; $\mathcal{G}$ is a uniform covering if there is a number $\varepsilon > 0$ such that for every point x there exists a $G \in \mathcal{G}$ with $|x - y| < \varepsilon \Rightarrow y \in G$).

It appears that the characterization of projectively generated proximity and uniform structures is not quite trivial even for some rather simple and natural sets of generators f_a . We shall consider the following two problems here.

Let H denote the set of all entire functions on $\mathbb{C}$ (i. e. of those functions $\varphi : \mathbb{C} \to \mathbb{C}$ which are holomorphic at every point $x \in \mathbb{C}$).

(I) To characterize the proximity on $\mathbb{C}$ projectively generated by H ; in particular, to decide whether it coincides with the finest proximity compatible with the usual topology of $\mathbb{C}$ (i.e. with the proximity under which X and Y are distant if and only if $\overline{X} \cap \overline{Y} = \emptyset$).

(II) To characterize the uniformity generated by H ; to decide whether it coincides with the finest uniformity compatible with the usual topology of $\mathbb{C}$.

The problem (I) is answered in the present note whereas problem (II) remains unsolved.

Clearly, there arise similar questions if we consider, instead of H , the class of all holomorphic mappings $f : \mathbb{C} \to E$ where E is locally convex topological complex linear space.

Finally, the above-mentioned problems are closely connected with the theory of Λ -structures introduced in [1] by one of the present authors. However, we shall not go into these questions here.

§ 2 .

Let δ and ν denote, respectively, the proximity and the uniformity generated by the set H of all entire functions.

Clearly, the structures δ and ν may be described as follows:

Two sets $X \subset \mathbb{C}$ and $Y \subset \mathbb{C}$ are near (under δ) if and only if, for any entire functions $f_1, \dots, f_n$ and any $\varepsilon > 0$, there exist points $x \in X$, $y \in Y$ with $|f_k(x) - f_k(y)| < \varepsilon$ for $k = 1, \dots, n$.

A collection $\mathcal{G}$ of subsets of $\mathbb{C}$ is a uniform covering of the space $(\mathbb{C}, \nu)$ if and only if there exist entire functions $f_1, \dots, f_n$ and a number $\varepsilon > 0$ with the following property: for every $x \in \mathbb{C}$ there is a set $G \in \mathcal{G}$ such that $y \in G$ whenever $|f_k(x) - f_n(y)| < \varepsilon$ for $k = 1, \dots, n$.

Problems (I) and (II) may now be reformulated as follows.

<u>Problem 1</u>. Let $X \subset \mathbb{C}$, $Y \subset \mathbb{C}$ be disjoint closed sets. To decide whether there exists a natural n (which may depend on X and Y) such that, for appropriate entire functions $f_1, \dots, f_n$, $\max_{1 \le k \le n} |f_k(x) - f_k(y)| \geq 1$ for any

$x \in X$, $y \in Y$.

Problems 2 and 2'. Let $\mathcal{G}$ be an open covering (for Problem 2': a finite open covering) of $\mathbb{C}$. To decide whether there exists a natural n (which may depend on $\mathcal{G}$) such that, for appropriate entire functions $f_1, \dots, f_n$, the following holds: if $x \in \mathbb{C}$, $y \in \mathbb{C}$ and $\max_{1 \le k \le n} |f_k(x) - f_k(y)| < 1$, then there is a set $G \in \mathcal{G}$ with $x \in G$, $y \in G$.

It is well known that, for any two distant sets X and Y in a proximity space M, there exists a proximally continuous function h which separates X and Y in the sense that $h(z) = 0$ for $z \in X$, $h(z) = 1$ for $z \in Y$. It is also clear that, in the case of the space $\mathbb{C}$, h cannot be an entire function, in general (not even for a far weaker condition requiring that $x \in X \Longrightarrow |h(x)| < \varepsilon$, $y \in Y \Longrightarrow |h(y) - 1| < \varepsilon$).

Therefore, the following question seems to be natural:

Problem 3. Let $X \subset \mathbb{C}$, $Y \subset \mathbb{C}$ be disjoint closed sets, and let $G \subset \mathbb{C}$, $H \subset \mathbb{C}$ be disjoint open non-void. To decide whether there exists a natural number n (which may depend on X, Y, G, H) such that there exist entire functions $f_1, \dots, f_n$ with the folllowing property: for any $x \in X$, $y \in Y$, there is a number $k = 1, \dots, n$ with $f_k(x) \in G$, $f_k(y) \in H$.

To illustrate this problem, we are going to show that, for certain sets X, Y, G, H, $n = 2$ is not sufficient.

Example. Let $\{\alpha_k\}$ be an increasing sequence of positive numbers, $\alpha_k \to \infty$. Denote by T_k the set of all $x \in \mathbb{C}$ such that $|x| = \alpha_k$; let X and Y denote, respectively, the union of all T_k with k odd, and with k even. Let

$G \subset \mathbb{C}$, $H \subset \mathbb{C}$ be disjoint bounded open non-void.

Suppose that there are entire functions f_1, f_2 such that, for any $x \in X$, $y \in Y$, either $f_1(x) \in G$, $f_1(y) \in H$ or $f_2(x) \in G$, $f_2(y) \in H$. We may suppose that f_1 , f_2 are not constant. For $k = 1,3,5,\dots$, let A_k and B_k denote the set of those $x \in T_k$ for which $f_1(x) \in G$, respectively, $f_2(x) \in G$. Clearly, $T_k \subset A_k \cup B_k$, $k = 1,3,5,\dots$. There exists an odd k_0 such that $B_k - A_k \neq \emptyset$; for otherwise $f_1(T_k) \subset G$ for $k = 1,3,5,\dots$ which is a contradiction since G is bounded. Choose $x_0 \in B_{k_0} - A_{k_0}$. For any $y \in Y$, we obtain $f_1(x_0)$ non $\in G$, hence $f_2(x_0) \in G$, $f_2(y) \in H$; thus $f_2(y) \subset H$ which is a contradiction since H is bounded.

§ 3 .

We may now state the main propositions. Observe that Theorem 1 solves Problem 3 . As an immediate consequence, we obtain Theorem 2, which solves Problem 1; Theorem 3 (which solves Problem 2') also follows from Theorem 1. However, the solutions are not definitive; we do not know whether a smaller number of functions is sufficient.

__Theorem 1__. Let $X \subset \mathbb{C}$, $Y \subset \mathbb{C}$ be disjoint closed and let $G \subset \mathbb{C}$, $H \subset \mathbb{C}$ be disjoint open non-void. Then there exist entire functions $f_1,\dots, f_9$ such that, for any $x \in X$, $y \in Y$, $f_k(x) \in G$, $f_k(y) \in H$ for some $k = 1,\dots, 9$.

__Remark__. The example above shows that we cannot replace 9 with 2 in this assertion. On the other hand, we do not know whether 9 can be replaced by some $k = 3,\dots, 8$.

__Theorem 2__. Let $X \subset \mathbb{C}$, $Y \subset \mathbb{C}$ be disjoint closed. Then there exist entire functions $f_1,\dots, f_9$ such that, for any $x \in X$, $y \in Y$, $\max_{k=1,\dots,9} |f_k(x) - f_k(y)| \geq 1$.

__Remark__. We do not know whether 9 can be replaced by some $k = 1, \dots, 8$. - If we denote by μ the metric proximity on $\mathbb{C}$ (i.e. the proximity under which X and Y are near if and only if $\inf_{x \in X, y \in Y} |x - y| = 0$), then Theorem 2 asserts that, for any $X \subset \mathbb{C}$, $Y \subset \mathbb{C}$ with $\overline{X} \cap \overline{Y} = \emptyset$, there exists a holomorphic mapping $f : \mathbb{C} \to (\mathbb{C}, \mu)^9$ such that $f(X)$ and $f(Y)$ are distant.

Theorem 3. Let $\mathcal{G}$ be an open cover of $\mathbb{C}$; let $\mathcal{G}$ consist of p sets. Then there exist entire functions $f_1, \dots$ $\dots, f_{9p}$ such that the following holds: if $x \in \mathbb{C}$, $y \in \mathbb{C}$ and $\max_{k=1,\dots,9p} |f_k(x) - f_k(y)| < 1$ then there is a set $G \in \mathcal{G}$ with $x \in G$, $y \in G$.

Remark. In contradistinction to the preceding theorems, the number of functions given in this theorem depends on $\mathcal{G}$. We do not know whether this dependence is substantial or whether there is a number q with the following property: for any finite open cover $\mathcal{G}$ of $\mathbb{C}$, there exists a holomorphic mapping of $\mathbb{C}$ into $\mathbb{C}^q$ such that, with an appropriate $\varepsilon > 0$, $\|f(x) - f(y)\| < \varepsilon$ implies the existence of a set $G \in \mathcal{G}$ such that $x \in G$, $y \in G$.

The proof of Theorem 1 leans on two propositions from the theory of functions of a complex variable. The first of them is well known theorem of M.V. Keldysh [2] ; its proof is omitted. The second proposition is an easy consequence of the first (and a special case of some general theorems due to M.V. Keldysh and M.A. Lavrentiev [3] ; see also [4]).

Proposition A . If $E \subset \mathbb{C}$ is compact and $\mathbb{C} - E$ is connected, then for any complex-valued function f continuous on E and holomorphic on Int E , and any $\varepsilon > 0$, there exists a polynomial g such that $|f(z) - g(z)| < \varepsilon$ for $z \in E$.

Proposition B. Let $E \subset \mathbb{C}$ be closed an suppose that there exist compact sets $B_k \subset \mathbb{C}$, $k = 1,2,\dots,$ such that

(1) $\bigcup B_k = \mathbb{C}$,

(2) for any $k = 1,2,\dots,$ B_k is contained in $\operatorname{Int} B_{k+1}$ and does not intersect $\overline{E - B_k}$,

(3) the complement of $E \cap B_1$ as well as of every $B_k \cup (E \cap B_{k+1})$, $k = 1,2,\dots,$ is connected.

Then, for any complex-valued f continuous on E and holomorphic on $\operatorname{Int} E$ and any monotone positive function ϑ on reals $t \geq 0$, there exists an entire function g such that $|g(z) - f(z)| < \vartheta(|z|)$ for every $z \in E$.

In the proof of Theorem 1, the following assertion will be used.

Proposition C. If $D \subset \mathbb{C}$ and $S_i \subset \mathbb{C}$, $i = 1,\dots, n$, are convex compact sets, and every two S_i, S_j, $i \neq j$, are disjoint, then $\mathbb{C} - D - \bigcup_{i=1}^{n} S_i$ is connected.

This proposition (in an essentially more general form) is well known. Its proof is omitted.

§ 4.

Proof of Proposition B. Let ε_k denote the greatest lower bound of $\vartheta(|z|)$ for $z \in B_k$. Then $\varepsilon_k \geq \varepsilon_{k+1} > 0$ for $k = 1,2,\dots$. Let $\delta_k > 0$ be such that $\sum_{i=k}^{\infty} \delta_i < \varepsilon_k$, $k = 1,2,\dots$. Put $B_0 = \emptyset$; let $g_0(z) = 0$ for every $z \in \mathbb{C}$.

By Proposition A, there exists a polynomial g_1 such that

$$|f(z) - g_1(z)| < \delta_1 \quad \text{for } z \in E \cap B_1 .$$

Now suppose that for a certain $m = 1,2,\dots$ there are already chosen certain polynomials $g_1, g_2, \dots, g_m$ such that

$$(*)\begin{cases} |f(z) - g_n(z)| < \delta_n & \text{for } z \in (E \cap B_n) - B_{n-1} \text{ and } 1 \leq n \leq m, \\ |g_n(z) - g_{n-1}(z)| < \delta_n & \text{for } z \in B_{n-1} \text{ and } 1 \leq n \leq m . \end{cases}$$

As a matter of fact, this has been done for $m = 1$. We shall now construct a polynomial g_{m+1} for which (*) holds with $m + 1$ instead of m.

Put $\varphi(z) = 0$ for $z \in B_m$, $\varphi(z) = f(z) - g_m(z)$ for $z \in (E \cap B_{m+1}) - B_m$. Then φ is continuous on $B_m \cup (E \cap B_{m+1})$ and holomorphic in its interior. Since the complement of $B_m \cup (E \cap B_{m+1})$ is connected, there exists, by Proposition A, a polynomial h such that $|h(z)| < \delta_{m+1}$ for $z \in B_m$, $|f(z) - g_m(z) - h(z)| < \delta_{m+1}$ for $z \in (E \cap B_{m+1}) - B_m$.

Now put $g_{m+1} = g_m + h$. Then, clearly, (*) holds with $m + 1$ instead of m. By induction, we obtain a sequence of polynomials $g_1, g_2 \dots$ satisfying the inequalities (*). Put $g(z) = \lim_{k \to \infty} g_k(z)$. It is easy to see that this sequence converges locally uniformly in $\mathbb{C}$; hence g is an entire function.

Clearly, if $z \in (E \cap B_k) - B_{k-1}$, $k = 1,2,\dots,$ then $|f(z) - g(z)| < \sum_{i=k}^{\infty} \delta_i < \varepsilon_k$, and therefore $|f(z) - g(z)| < \vartheta(|z|)$.

Proof of Theorem 1. For any $\delta > 0$ let $\mathcal{S}(\delta)$ denote the collection of all squares with sides of length δ and vertices of the form $p\delta + iq\delta$ where p, q are integers. For $n = 1,2,\dots$ let D_n denote the set of those $z \in \mathbb{C}$ for which $-n \leq \mathcal{R}(z) \leq n$, $-n \leq \mathcal{J}(z) \leq n$. Choose positive numbers δ_n, $n = 0,1,2,\dots$ in such a way that δ_0^{-1} is an integer greater than 2, and

(1) for each $n = 0,1,2,\dots$, $\delta_n = q_n \delta_{n+1}$ where q_n is an integer greater than 1, and

(2) if $n = 1,2,3,\dots$, $x \in X \cap D_n$, $y \in Y \cap D_n$, then $\max(|\mathcal{R}(x-y)|, |\mathcal{J}(x-y)|) > 4\delta_{n-1}$.

Put $D_0 = \emptyset$ and, for $n = 1,2,\dots,$ denote by $\mathcal{K}_n$ the collection of those squares $S \in \mathcal{S}(\delta_n)$ which are contained in $D_n - \operatorname{Int} D_{n-1}$. Put $\mathcal{K} = \bigcup \mathcal{K}_n$. Then $\mathcal{K}$ is a locally finite collection of compact sets and the following condition is fulfilled: if $S_i \in \mathcal{K}$, $i = 1, \dots, 4$, and $\bigcup_{i=1}^{4} S_i$ is connected, then either $X \cap \bigcup_{i=1}^{4} S_i = \emptyset$ or $Y \cap \bigcup_{i=1}^{4} S_i = \emptyset$.

We shall now construct three collections $\mathcal{K}^{(0)}, \mathcal{K}^{(1)}, \mathcal{K}^{(2)}$ of rectangles in the following way: $\mathcal{K}^{(0)}$ and $\mathcal{K}^{(2)}$ consist of squares with sides parallel to the axes; a square belongs to $\mathcal{K}^{(0)}$ if and only if, for some n , the length of its side is equal to $\frac{1}{4}\delta_n$, and its center x is a vertex of some $S_1 \in \mathcal{K}_n$, but of no square $S_2 \in \mathcal{K}_{n+1}$; a square belongs to $\mathcal{K}^{(2)}$ if and only if, for some n , the length of its side is equal to $\delta_n - \frac{1}{8}\delta_{n+1}$ and its centre coincides with the centre of some $S \in \mathcal{K}_n$; finally, it may be shown that the closure of $\mathbb{C} - \bigcup \mathcal{K}^{(0)} - \bigcup \mathcal{K}^{(2)}$ may be expressed as the union of a disjoint collection of rectangles, and this collection is taken as $\mathcal{K}^{(1)}$.

Obviously, the collection $\mathcal{K}^* = \mathcal{K}^{(0)} \cup \mathcal{K}^{(1)} \cup \mathcal{K}^{(2)}$ has the following properties: (1) $\bigcup \mathcal{K}^* = \mathbb{C}$, (2) $\mathcal{K}^*$ is locally finite, (3) each $\mathcal{K}^{(j)}$ is a disjoint collection, (4) every $T \in \mathcal{K}^*$ is a compact convex set, (5) every $T \in \mathcal{K}^*$ is contained in the star (with respect to $\mathcal{K}$) of some $x \in \mathbb{C}$.

For $j = 0,1,2$, denote by $\mathcal{X}^{(j)}$ and $\mathcal{Y}^{(j)}$ the collection of those $T \in \mathcal{K}^{(j)}$ which intersect the set X (respectively, Y); let $X^{(j)}$ denote the union of all $T \in \mathcal{X}^{(j)}$, and similarly for $Y^{(j)}$; put $X^* = X^{(0)} \cup X^{(1)} \cup X^{(2)}$, $Y^* = Y^{(0)} \cup Y^{(1)} \cup Y^{(2)}$. Then $X \subset X^*$, $Y \subset Y^*$, $X^* \cap Y^* = \emptyset$, and $X^{(j)}$, $Y^{(j)}$ are closed. Choose points $a \in G$, $b \in H$; let $\varepsilon > 0$ be such that $|x - a| < \varepsilon$ implies $x \in G$, $|y - b| < \varepsilon$ implies $y \in H$; put $\varepsilon' = |b - a|^{-1}\varepsilon$.

To conclude the proof, it is now sufficient to find, for any given i, $j = 0,1,2$, and entire function $g = g_{ij}$ such that $|g(z)| < \varepsilon'$ for $z \in X^{(i)}$, $|g(z) - 1| < \varepsilon'$ for $z \in Y^{(j)}$. If such functions are constructed, then putting $f_{3i+j+1}(z) = a + (b - a)\, g_{ij}(z)$ we obtain functions $f_1, \dots, f_9$ with properties described in the theorem.

Now let i, j be given. Put $E = X^{(i)} \cup Y^{(j)}$ and denote by B_k the union of D_k and all those $T \in \mathcal{X}^{(i)} \cup \mathcal{Y}^{(j)}$ which intersect D_k. Then Proposition C implies that the assumptions from proposition B are fulfilled. Put $\vartheta(t) = \varepsilon'$ for $0 \leq t$, $f(z) = 0$ for $z \in X^{(i)}$, $f(z) = 1$ for $z \in Y^{(j)}$. By Proposition B, there exists an entire function g such that $|g(z) - f(z)| < \varepsilon'$ for every $z \in E$, hence $|g(z)| < \varepsilon'$ for $z \in X^{(i)}$, $|g(z) - 1| < \varepsilon'$ for $z \in Y^{(j)}$.

Proof of Theorem 3. Let $\mathcal{G}$ consist of sets $G_1, \dots, G_p$. Choose open sets V_i such that $\overline{V}_i \subset G_i$, $\bigcup_1^p V_i = \mathbb{C}$. By Theorem 2, there exist, for any $i = 1, \dots, p$, entire functions $f_{i,1}, \dots, f_{i,9}$ such that $\max_{k=1,\dots,9} |f_{i,k}(x) - f_{i,k}(y)| \geq 1$ whenever $x \in \overline{V}_i$, $y \in \mathbb{C} - G_i$. Consider the functions $f_{1,1}, \dots, f_{1,9}, \dots, f_{p,9}$. If $x \in \mathbb{C}$, $y \in \mathbb{C}$ and $|f_{i,j}(x) - f_{i,j}(y)| < 1$ for all $i = 1, \dots, p$, $j = 1, \dots, 9$, then, for some i, $x \in \overline{V}_i$ and therefore y does not belong to $\mathbb{C} - G_i$, hence $y \in G_i$. This concludes the proof.

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