Werk

Label: Article

Jahr: 1994

PURL: https://resolver.sub.uni-goettingen.de/purl?313123012_0119|log43

Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

119 (1994) MATHEMATICA BOHEMICA No. 2, 203–211

COMPARISON THEOREMS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

JOZEF DŽURINA, Košice

(Received September 24, 1992)

Summary. In this paper the oscillatory and asymptotic properties of the solutions of the functional differential equation

$$L_n u(t) + p(t) f(u[g(t)]) = 0$$

are compared with those of the functional differential equation

$$\alpha_n u(t) + q(t) h(u[w(t)]) = 0.$$

Keywords: Property (A), canonical form

AMS classification: Primary 34C10

We consider the n-th order ($n \geqslant 2$) functional differential equation

$$\text{(1)} \qquad \left(\frac{1}{r_{n-1}(t)} \dots \left(\frac{1}{r_1(t)} u'(t)\right)' \dots\right)' + p(t) f(u[g(t)]) = 0,$$

where $r_i, g, p \in C([t_0, \infty))$, $f \in C(\mathbf{R})$, $p(t) > 0$, $r_i(t) > 0$, $i = 1, 2, \dots, n-1$, $xf(x) > 0$ for $x \neq 0$ and $g(t) \to \infty$ as $t \to \infty$.

We introduce the notation

$$\text{(2)} \qquad L_0 u(t) = u(t), \quad L_i u(t) = \frac{1}{r_i(t)} (L_{i-1} u(t))', \quad L_n u(t) = (L_{n-1} u(t))', \quad i = 1, 2, \dots, n-1.$$

The author would like to thank the referee for some useful comments.

Then equation (1) can be rewritten as

$$L_n u(t) + p(t) f(u[g(t)]) = 0.$$

The domain $D(L_n)$ of L_n is defined to be the set of all functions $u\colon [T_u, \infty) \to \mathbb{R}$ such that $L_i u(t)$, $0 \leqslant i \leqslant n$ exist and are continuous on $[T_u, \infty)$. By a proper solution of equation (1) we mean a function $u(t) \in D(L_n)$ which satisfies (1) for all sufficiently large t and $\sup\{|u(t)|\colon t \geqslant T\} > 0$ for every $T > T_u$. We make the standing hypothesis that equation (1) does possess proper solutions. A proper solution of (1) is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory. Equation (1) is said to be oscillatory if all its proper solutions are oscillatory.

We say that the operator L_n is in canonical form if

$$\int^{\infty} r_i(s)\,\mathrm{d}s = \infty \quad \text{for} \quad 1 \leqslant i \leqslant n-1. \tag{3}$$

It is well known that any differential operator of the form (2) can always be represented in a canonical form in an essentially unique way (see Trench [10]). In the sequel we will suppose L_n is in canonical form.

Lemma 1. *Let* (3) *hold. If* $u(t)$ *is a nonoscillatory solution of* (1), *then there exists a* t_1 *and an integer* ℓ, $0 \leqslant \ell \leqslant n-1$ *such that* $\ell \not\equiv n \pmod 2$ *and*

$$\begin{aligned} u(t) L_i u(t) &> 0 \quad \text{on} \quad [t_1, \infty), \quad 1 \leqslant i \leqslant \ell, \\ (-1)^{i-\ell} u(t) L_i u(t) &> 0 \quad \text{on} \quad [t_1, \infty), \quad \ell + 1 \leqslant i \leqslant n. \end{aligned} \tag{4}$$

Lemma 1 generalizes a well known lemma of Kiguradze [6] and can be proved similarly.

A function $u(t)$ satisfying (4) is said to be of degree ℓ. The set of all nonoscillatory solutions of degree ℓ of (1) is denoted by $\mathcal{N}_\ell$. If we denote by $\mathcal{N}$ the set of all nonoscillatory solutions of (1), then

$$\begin{aligned} \mathcal{N} &= \mathcal{N}_0 \cup \mathcal{N}_2 \cup \cdots \cup \mathcal{N}_{n-1}, \quad \text{for } n \text{ odd}, \\ \mathcal{N} &= \mathcal{N}_1 \cup \mathcal{N}_3 \cup \cdots \cup \mathcal{N}_{n-1}, \quad \text{for } n \text{ even}. \end{aligned}$$

Following Kusano and Naito [7] we are interested in the situation when $\mathcal{N} = \mathcal{N}_0$, especially when every nonoscillatory solution $u(t)$ of (1) satisfies

$$\lim_{t \to \infty} u(t) = 0. \tag{5}$$

Definition 1. Let L_n be in canonical form. Equation (1) is said to have property (A) if for n even (1) is oscillatory (i.e. $\mathcal{N} = \emptyset$) and for n odd every nonoscillatory solution $u(t)$ of (1) satisfies (5).

Remark 1. Let us denote $M_0 = 1$ and

$$M_i(t) = \int_{t_0}^{t} r_i(s_i) \int_{t_0}^{s_i} \cdots \int_{t_0}^{s_2} r_1(s_1)\,\mathrm{d}s_1 \ldots \mathrm{d}s_i, \quad i = 1, \ldots, n-1.$$

Then a stronger asymptotic result can be established, namely, according to Theorem 1 in [1] if $u(t)$ satisfies (5) then also

$$\lim_{t\to\infty} M_i(t) L_i u(t) = 0, \quad i = 0, 1, \ldots, n-1.$$

The following lemma is elementary but quite useful in the sequel.

Lemma 2. *Let $p(t) \geqslant 0$, $q(t) \geqslant 0$. Suppose that a function $a(t)$ is positive, continuous and nondecreasing on $[t_0, \infty)$. If*

$$\int_t^\infty p(s)\,\mathrm{d}s \geqslant \int_t^\infty q(s)\,\mathrm{d}s, \quad t \geqslant t_0,$$

then

$$\int_t^\infty p(s)a(s)\,\mathrm{d}s \geqslant \int_t^\infty q(s)a(s)\,\mathrm{d}s, \quad t \geqslant t_0.$$

In the literature many comparison results have been established to the effect that if a differential equation with a deviating argument has property (A) then so does another related equation with larger deviating argument. Attempts in this direction have been undertaken e.g. by Kusano and Naito [7], Mahfoud [9] and Erbe [3] and [4].

The aim of this paper is to present comparison results in the opposite direction, that is we wish to derive property (A) of an equation with a deviating argument from the corresponding property of another equation with larger deviating argument. Therefore, let functions $g(t)$ and $w(t)$ be subject to the conditions

$$g, w \in C^1, \quad g'(t) > 0, \quad w'(t) > 0, \quad w(t) \geqslant g(t). \tag{6}$$

To be able to build our comparison technique we use the following differential operator which was introduced by Kusano and Naito [7].

$$\alpha_n = \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{r_{n-1}[\tau(t)]\tau'(t)} \frac{\mathrm{d}}{\mathrm{d}t} \cdots \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{r_1[\tau(t)]\tau'(t)} \frac{\mathrm{d}}{\mathrm{d}t},$$

where $\tau(t) = g(w^{-1}(t))$ and $w^{-1}(t)$ is the inverse function to $w(t)$.

We compare oscillatory and asymptotic properties of solutions of equation (1) with those of the equation

$$\alpha_n u(t) + q(t)h\left(u[w(t)]\right) = 0, \tag{7}$$

where $q \in C([t_0, \infty))$, $h \in C(\mathbb{R})$, $q(t) > 0$, $xh(x) > 0$ for $x \neq 0$ and $w(t)$ satisfies (6).

Note that the function $\tau(t)$ expressed in terms of arguments $g(t)$ and $w(t)$ of equations (1) and (7) is the main tool for comparing (1) and (2).

Theorem 1. *Suppose that* (3) *and* (6) *hold. Let* $h(x)$ *be nondecreasing. Further assume that*

$$f(x)\operatorname{sgn} x \geqslant h(x)\operatorname{sgn} x \qquad \text{for } x \neq 0, \tag{8}$$

$$\int_t^\infty p(s)\,\mathrm{d}s \geqslant \int_t^\infty q(s)\,\mathrm{d}s \qquad \text{for } t \geqslant t_0. \tag{9}$$

Then equation (1) *has property* (A) *if so does equation* (7).

Proof. Let $u(t)$ be a nonoscillatory solution of (1). We may assume that $u(t)$ is positive (for $u(t) < 0$ we can use a similar argument). Then there exists an integer $\ell \in \{0, 1, \ldots, n-1\}$ such that $n + \ell$ is odd, and a $t_1 \geqslant t_0$ associated with $u(t)$ by Lemma 1. Assume $\ell \geqslant 1$. Integrating (1) and using $\tau(t) \leqslant t$ we obtain

$$L_{n-1}u\big(\tau(t)\big) \geqslant \int_t^\infty p(s)f\big(u[g(s)]\big)\,\mathrm{d}s \tag{10}$$

for $t \geqslant t_2$ ($\geqslant t_1$) provided t_2 is sufficiently large. First, note that $u(t)$ is nondecreasing as $\ell \geqslant 1$. Combining (10) with (8) one gets

$$L_{n-1}u\big(\tau(t)\big) \geqslant \int_t^\infty p(s)h\big(u[g(s)]\big)\,\mathrm{d}s, \quad t \geqslant t_2.$$

Since the composite function $h\big(u[g(t)]\big)$ is nondecreasing, according to Lemma 2 we obtain

$$L_{n-1}u\big(\tau(t)\big) \geqslant \int_t^\infty q(s)h\big(u[g(s)]\big)\,\mathrm{d}s, \quad t \geqslant t_2. \tag{11}$$

We multiply (11) by $r_{n-1}\big(\tau(t)\big)\tau'(t)$ and integrate the resulting inequality over $[t, \infty)$. Repeating this procedure, we arrive at

$$\begin{aligned} L_\ell u\big(\tau(t)\big) \geqslant \int_t^\infty & r_{\ell+1}[\tau(s_{\ell+1})]\tau'(s_{\ell+1}) \\ & \times \int_{s_{\ell+1}}^\infty \cdots \int_{s_{n-1}}^\infty q(s_n)h\left(u[g(s_n)]\right)\mathrm{d}s_n \ldots \mathrm{d}s_{\ell+1}\,. \end{aligned} \tag{12}$$

We multiply (12) by $r_\ell(\tau(t))\tau'(t)$ and integrate over $[t_2, t]$. Continuing in this manner we obtain

$$
\begin{aligned}
u[\tau(t)] \geqslant c + \int_{t_2}^{t} r_1[\tau(s_1)]\tau'(s_1) \int_{t_2}^{s_1} \cdots \int_{t_2}^{s_{\ell-1}} r_\ell[\tau(s_\ell)]\tau'(s_\ell) \\
\times \int_{s_\ell}^{\infty} \cdots \int_{s_{n-1}}^{\infty} q(s_n)h\big(u[g(s_n)]\big)\,\mathrm{d}s_n \ldots \mathrm{d}s_1, \quad t \geqslant t_2,
\end{aligned} \tag{13}
$$

where $c = u[\tau(t_2)] > 0$. Denote the right hand side of (13) by $z(t)$. By repeated differentiation of $z(t)$ one can verify that $z(t)$ is a function of degree ℓ and, on the other hand,

$$
\alpha_n z(t) + q(t)h\big(u[g(t)]\big) = 0. \tag{14}
$$

As $u[\tau(t)] \geqslant z(t)$ and $\tau\big(w(t)\big) = g(t)$ we see that

$$
u\big(g(t)\big) = u\big(\tau[w(t)]\big) \geqslant z\big(w(t)\big)
$$

for all large t, say $t \geqslant t_3$. Combining this fact with (14) we see that $z(t)$ is a solution of the differential inequality

$$
\big\{\alpha_n z(t) + q(t)h\big(z[w(t)]\big)\big\} \operatorname{sgn} z[w(t)] \leqslant 0, \quad t \geqslant t_3.
$$

Then by Kusano and Naito (see [7]) equation (7) has also an eventually positive solution $x(t)$ satisfying

$$
\lim_{t\to\infty} x(t) \geqslant c > 0,
$$

which contradicts the hypotheses.

Now, let $\ell = 0$ (note that this is possible only when n is odd). To obtain a contradiction assume that $c_0 = \lim_{t\to\infty} u(t) > 0$. Integrating (1), in view of (8) we have

$$
\begin{aligned}
L_0 u\big(\tau(t)\big) \geqslant c_0 + \int_{t}^{\infty} r_1[\tau(s_1)]\tau'(s_1) \\
\times \int_{s_1}^{\infty} \cdots \int_{s_{n-1}}^{\infty} p(s_n)h\left(u[g(s_n)]\right)\mathrm{d}s_n \ldots \mathrm{d}s_1
\end{aligned} \tag{15}
$$

for all large t. Since $u(t)$ is decreasing ($\ell = 0$) one gets

$$
\frac{3}{2}c_0 \geqslant u(t) \geqslant c_0, \quad t \geqslant t_2,
$$

where t_2 is large enough. This fact together with (15) and (9) implies

$$c_0 \geqslant \frac{c_0}{2} + \int_t^\infty r_1[\tau(s_1)]\tau'(s_1) \times \int_{s_1}^\infty \cdots \int_{s_{n-1}}^\infty q(s_n)h(c_0)\,\mathrm{d}s_n \ldots \mathrm{d}s_1 \tag{16}$$

for all large t, say $t \geqslant t_3$. Again, let us denote the right hand side of (16) by $z(t)$. Then

$$\alpha_n z(t) + q(t)h(c_0) = 0, \quad t \geqslant t_3. \tag{17}$$

Noting that $z(w(t)) \leqslant c_0$, (17) yields

$$\{\alpha_n z(t) + q(t)h(z[w(t)])\} \operatorname{sgn} z[w(t)] \leqslant 0.$$

Again according to Kusano and Naito [7] equation (7) has also an eventually positive solution $x(t)$ with the property

$$x(t) \geqslant \frac{c_0}{2} > 0.$$

This contradicts the assumptions and the proof is complete. □

Example 1. Consider an even order delay equation

$$\left(\frac{1}{r(t)}y^{(m)}\right)^{(m)} + p(t)f(y[g(t)]) = 0, \quad m \geqslant 1, \tag{18}$$

where functions r, p, f and g are the same as in Theorem 1 and $g(t) \leqslant t$. Then by Theorem 1 equation (18) is oscillatory if the ordinary equation without delay

$$\left(\frac{1}{r(g(t))g'(t)}y^{(m)}\right)^{(m)} + p(t)f(y(t)) = 0, \quad m \geqslant 1$$

is oscillatory.

Corollary 1. *Let* (3) *hold. Let* $r_i(t)$; $1 \leqslant i \leqslant n-1$ *be nonincreasing. Assume that* f *is nondecreasing. Suppose that* $g \in C^1$ *and* $g'(t) > 0$. *Then for any constant* M *equation* (1) *has property* (A) *if and only if so does the equation*

$$L_n u(t) + p(t)f(u[g(t) + M]) = 0. \tag{19}$$

Proof. Assume that $M > 0$. The "only if" part follows from Theorem 1 of Kusano and Naito [7]. Now suppose that equation (19) has property (A). We put $w(t) = g(t) + M$. Then

$$\tau(t) = g\left(w^{-1}(t)\right) = t - M \leqslant t.$$

By Theorem 1 of Kusano and Naito [7] the equation

$$\alpha_n u(t) + p(t) f\big(u[g(t) + M]\big) = 0 \tag{20}$$

has property (A) as $r_i(\tau(t))\tau'(t) \geqslant r_i(t)$. The "if" part now follows from Theorem 1 applied to equations (20) and (1).

Now let $M < 0$. We put $\tilde{g}(t) = g(t) + M$. Since $-M > 0$, by the first part of the proof of this theorem we see that the equation

$$L_n u(t) + p(t) f\big(u[\tilde{g}(t)]\big) = 0$$

has property (A) if and only if so does the equation

$$L_n u(t) + p(t) f\big(u[\tilde{g}(t) + (-M)]\big) = 0,$$

which we wanted to verify. □

Example 2. Consider an n-order linear delay differential equation

$$\left(\frac{1}{r(t)} \cdots \left(\frac{1}{r(t)} u'(t)\right)' \cdots\right)' + p(t)u(g(t)) = 0, \tag{21}$$

where r, p and g are subject to the same conditions as in (1). Let us denote $b(t) = r(g(t))g'(t)$ and put $w(t) = t$. By Theorem 1 equation (21) has property (A) if the differential equation without delay

$$\left(\frac{1}{b(t)} \cdots \left(\frac{1}{b(t)} u'(t)\right)' \cdots\right)' + p(t)u(t) = 0$$

has property (A), which by Theorem 5 in [2] occurs if

$$\liminf_{t\to\infty} \left(\int_{t_0}^{t} b(s)\,\mathrm{d}s\right)^{n-1} \left(\int_{t}^{\infty} p(s)\,\mathrm{d}s\right) > \frac{M_1}{n-1}, \tag{22}$$

where M_1 is the maximum of all local maxima of the polynomial

$$P_n(k) = -k(k-1)\ldots(k-n+1).$$

If we set $g(s) = x$ then (22) reduces to

$$\liminf_{t\to\infty} \left(\int_{t_0^*}^{g(t)} r(x)\,\mathrm{d}x \right)^{n-1} \left(\int_t^{\infty} p(s)\,\mathrm{d}s \right) > \frac{M_1}{n-1},$$

which is a weaker sufficient condition for equation (21) to have property (A) than Kusano and Naito have required in [7].

For n even, as we see from Definition 1, property (A) of equation (1) reduces to oscillation of (1). In [8] Kusano and Naito discussed the oscillatory character of a special case of (1), namely the even order linear differential equation

$$L_n u(t) + p(t)u(t) = 0, \tag{23}$$

by comparing (23) with a set of second order differential equations (see Theorem B in [8]). We adapt their method together with Theorem 1 and make use of the results obtained to extend the results of Kusano and Naito [8] and Trench [11].

Let $1 \leqslant i \leqslant n-1$ and $t, s \in [t_0, \infty)$. We define

$$I_0 = 1$$

$$I_i(t, s; r_i, \ldots, r_1) = \int_s^t r_i(x) I_{i-1}(x, s; r_{i-1}, \ldots, r_1)\,\mathrm{d}x\,.$$

Let us denote $b_i(t) = r_i(g(t))g'(t)$ for $i = 1, 2, \ldots, n-1$. For simplicity of notation we put

$$J_i(t, s) = I_i(t, s; b_1, \ldots, b_i),$$
$$K_i(t, s) = I_i(t, s; b_{n-1}, \ldots, b_{n-i}).$$

Theorem 2. *Suppose that $n \geqslant 4$ is even. Assume that all the conditions of Theorem 1 are satisfied with $h(x) = x$ and $w(t) = t$. Define for $i = 1, 3, \ldots, n-3$*

$$a_i(t) = b_{i+1}(t) \int_t^{\infty} J_{i-1}(s, t) K_{n-i-2}(s, t) q(s)\,\mathrm{d}s, \tag{24}$$

$$a_{n-1}(t) = b_{n-2}(t) \int_t^{\infty} J_{n-3}(s, t) q(s)\,\mathrm{d}s\,. \tag{25}$$

Then equation (1) *is oscillatory if the second order equations*

$$\left(\frac{1}{b_i(t)} y'(t) \right)' + a_i(t) y(t) = 0, \quad i = 1, 3, \ldots, n-1 \tag{26}$$

are oscillatory.

Proof. Applying Theorem B in [8] we conclude that equation (7) is oscillatory, and hence (1) is oscillatory by Theorem 1. □

Corollary 2. *Let all the conditions of Theorem* 1 *hold with* $h(x) = x$ *and* $w(t) = t$. *If*

$$\liminf_{t\to\infty} \left(\int_{t_0}^{g(t)} r_i(s)\,\mathrm{d}s \right) \left(\int_t^{\infty} a_i(s)\,\mathrm{d}s \right) > \frac{1}{4}, \quad i = 1, 3, \ldots, n-1, \tag{27}$$

where $a_i(t)$, $i = 1, 3, \ldots, n-1$ *are defined as in* (24) *and* (25), *then equation* (1) *is oscillatory.*

Proof. We make a change of variables in the first integral in (27) by using the substitution $s = g(x)$ and obtain

$$\liminf_{t\to\infty} \left(\int_{t_0^*}^{t} b_i(x)\,\mathrm{d}x \right) \left(\int_t^{\infty} a_i(s)\,\mathrm{d}s \right) > \frac{1}{4}, \quad i = 1, 3, \ldots, n-1. \tag{28}$$

It is known (see [5]) that (28) is sufficient for all solutions of (26) to be oscillatory. Hence, Corollary 2 follows from Theorem 2. □

References

[1] *J. Džurina*: Oscillation and asymptotic properties of n-th order differential equations. Czech. Math. J. *42* (1992), 11–14.

[2] *J. Džurina*: Comparison theorems for nonlinear ODEs. Math. Slovaca *42* (1992), 299–315.

[3] *L. Erbe*: Oscillation criteria for second order nonlinear delay equations. Canad. Math. Bull. *16* (1973), 49–56.

[4] *L. Erbe*: Oscillation and asymptotic behavior of solutions of third order differential delay equations. SIAM J. Math. Anal. *7* (1976), 491–500.

[5] *E. Hille*: Non-oscillation theorems. Trans. Amer. Math. Soc. *64* (1948), 234–258.

[6] *I. T. Kiguradze*: On the oscillation of solutions of the equation $\mathrm{d}^m u/\mathrm{d}t^m + a(t)|u|^n \times \operatorname{sign} u = 0$. Mat. Sb. *65* (1964), 172–187. (In Russian.)

[7] *T. Kusano and M. Naito*: Comparison theorems for functional differential equations with deviating arguments. J. Math. Soc. Japan *3* (1981), 509–532.

[8] *T. Kusano and M. Naito*: Oscillation criteria for general linear ordinary differential equations. Pacific J. Math. *92* (1981), 345–355.

[9] *W. E. Mahfoud*: Comparison theorems for delay differential equations. Pacific J. Math. *83* (1979), 187–197.

[10] *W. F. Trench*: Canonical forms and principal systems for general disconjugate equations. Trans. Amer. Math. Soc. *189* (1974), 319–327.

[11] *W. F. Trench*: Oscillation properties of perturbed disconjugate equations. Proc. Amer. Math. Soc. *52* (1975), 147–155.

Author's address: *Jozef Džurina*, Department of Mathematical Analysis, Šafárik University, Jesenná 5, 041 54 Košice, Slovakia, e-mail: dzurina@turing.upjs.sk.