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ABOUT A GENERALIZATION OF TRANSVERSALS

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Summary. The aim of this paper is to generalize several basic results from transversal theory, primarily the theorem of Edmonds and Fulkerson.

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1. INTRODUCTION

There are two fundamental results concerning both transversals and matroids. The first was proved by Rado [14], who established a necessary and sufficient condition for a finite family of sets to possess a transversal which is independent in a given matroid. Perfect [13] extended this theorem to partial transversals. The second result, proved by Edmonds and Fulkerson [2] (and independently also by Mirsky and Perfect [12]), says that the partial transversals of a finite family of sets form a matroid.

There are plenty of generalizations of these two results. A comprehensive survey of this field is in [11], [12] and [16], for later results see e. g. [6], [7], [17]. In this paper we introduce $\mathscr{M}$-polytransversals, which are in fact characteristic vectors of some special (matroid relative) systems of representatives. We show that $\mathscr{M}$-polytransversals of a finite family of sets form an integral polymatroid. Using this fact we can extend the Rado–Perfect theorem and also the result of Ford and Fulkerson [3] about common transversals of two families of sets. Our results generalize the classical theorems known for transversals and also some recent results of [7] and [6].

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2. Preliminaries

We expect the reader to be familiar with the theory of matroids. All terminology related to matroids and polymatroids is essentially the same as that of Welsh [16].

By $\mathbb{Z}_+$ ($\mathbb{R}_+$) we denote the set of nonnegative integral (real) numbers and the symbol $\mathbb{Z}_+^S$ ($\mathbb{R}_+^S$) denotes the space of integer (real) valued nonnegative vectors with coordinates indexed by a finite set S. For each $\mathbf{u} \in \mathbb{R}_+^S$ and $s \in S$ denote the sth coordinate of $\mathbf{u}$ by $\mathbf{u}(s)$. For $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}_+^S$ we write $\mathbf{u} \leqslant \mathbf{v}$ iff $\mathbf{u}(s) \leqslant \mathbf{v}(s)$ for any $s \in S$. For $\mathbf{u} \in \mathbb{R}_+^S$ and $X \subseteq S$ define $\mathbf{u}(X) = \sum_{s \in X} \mathbf{u}(s)$, and call the quantity $|\mathbf{u}| = \mathbf{u}(S) = \sum_{s \in S} \mathbf{u}(s)$ the *modulus* $|\mathbf{u}|$ of $\mathbf{u}$.

A *polymatroid* $\mathbf{P}$ *on* S is a pair (S, ϱ) where S, are the *ground set*, is a nonempty finite set an ϱ, the *ground set rank function*, is a function $\varrho\colon 2^S \to \mathbb{R}_+$, such that ϱ is *nondecreasing* (i.e., $\varrho(X) \leqslant \varrho(Y)$ for any $X \subseteq Y \subseteq S$), *submodular* (i.e., $\varrho(X) + \varrho(Y) \geqslant \varrho(X \cup Y) + \varrho(X \cap Y)$ for any X, $Y \subseteq S$) and $\varrho(\emptyset) = 0$. The vectors $\mathbf{u} \in \mathbb{R}_+^S$ such that $\mathbf{u}(X) \leqslant \varrho(X)$ for all $X \subseteq S$ are the *independent vectors of* $\mathbf{P}$. For each vector $\mathbf{a} \in \mathbb{R}_+^S$, the *vector rank* $r(\mathbf{a})$ of $\mathbf{a}$ is given by

$$r(\mathbf{a}) = \min_{X \subseteq S} \big(\mathbf{a}(X) + \varrho(S \setminus X)\big) \tag{1}$$

or equivalently, $r(\mathbf{a}) = \max(|\mathbf{u}|;\ \mathbf{u} \leqslant \mathbf{a},\ \mathbf{u}$ is independent in $\mathbf{P})$.

A polymatroid $\mathbf{P} = (S, \varrho)$ is *integral* if ϱ is integral. Moreover, if $\varrho(\{s\}) = 0, 1$ for any $s \in S$ then $\mathbf{P}$ is a polymatroid of a matroid on S with rank function ϱ. The following theorem is one of the basic results of matroid theory (see [1], [10]).

Theorem 1. *Let $\mathbf{P}_1 = (S, \varrho_1)$ and $\mathbf{P}_2 = (S, \varrho_2)$ be two polymatroids on S and let $k \in \mathbb{R}_+$. Then there exists a vector $\mathbf{u}$ of $\mathbb{R}_+^S$ independent in both $\mathbf{P}_1$ and $\mathbf{P}_2$ and with modulus at least k iff for any $X \subseteq S$,*

$$\varrho_1(X) + \varrho_2(S \setminus X) \geqslant k.$$

Furthermore, if $\mathbf{P}_1$ and $\mathbf{P}_2$ are both integral we may insist that the independent vector $\mathbf{u}$ be integer valued.

Throughout this paper S and T denote finite sets, $\mathscr{A}$ denotes the family $(A_t\colon t \in T)$ of subsets of S and $\mathscr{M}$ denotes the family $(M_s\colon s \in S)$ of matroids on T. For any $J \subseteq T$ and $s \in S$, denote

$$A(s, J) = \{t \in J; s \in A_t\} \quad (\subseteq T).$$

A family $(x_t\colon t \in J)$ $(J \subseteq T)$ of elements of S is called a *partial system of representatives* (in abbreviation *partial SR*) of $\mathscr{A}$ if $x_t \in A_t$ for any $t \in J$. $|J|$

($|T \setminus J|$) is called the *length (defect)* of the partial SR $(x_t : t \in J)$ of $\mathscr{A}$. A partial SR $(x_t : t \in J)$ of $\mathscr{A}$ will be called a *partial $\mathscr{M}$-system of representatives (partial $\mathscr{M}$-SR)* of $\mathscr{A}$ if the set $\{t \in J : x_t = s\}$ is independent in M_s for any $s \in S$.

If $(x_t : t \in J)$ is a partial $\mathscr{M}$-SR of $\mathscr{A}$, then the vector $\mathbf{u} \in \mathbb{Z}_+^S$ satisfying $\mathbf{u}(s) = |\{t \in J : x_t = s\}|$ for any $s \in S$ is called the *partial $\mathscr{M}$-polytransversal* of $\mathscr{A}$. We will call $|J|$ ($|T \setminus J|$) the *length (defect)* of the partial $\mathscr{M}$-polytransversal $\mathbf{u}$. Clearly $\sum_{s \in S} \mathbf{u}(s) = |J|$.

As usual, the partial SR, partial $\mathscr{M}$-SR and partial $\mathscr{M}$-polytransversal of $\mathscr{A}$ with defect 0 are called the *system of representatives, $\mathscr{M}$-system of representatives* and *$\mathscr{M}$-polytransversal of $\mathscr{A}$*, respectively.

If $\mathscr{M}$ is a family of uniform matroids of rank 1 then the partial $\mathscr{M}$-polytransversals of $\mathscr{A}$ are the characteristic vectors of the classical partial transversals of $\mathscr{A}$. We dealt with $\mathscr{M}$-SR also in [7] and proved the following variant of Hall's theorem [4] for $\mathscr{M}$-SR.

Lemma 1. *Let $\mathscr{A} = (A_t : t \in T)$ be a finite family of subsets of a finite set S and let $\mathscr{M}$ be a family $(M_s : s \in S)$ of matroids on T with rank functions ϱ_s, respectively. Then the maximal length of a partial $\mathscr{M}$-system of representatives of $\mathscr{A}$ (thus also the maximal length of a partial $\mathscr{M}$-polytransversal of $\mathscr{A}$) is equal to*

$$\min_{J \subseteq T} \left(\sum_{s \in S} \varrho_s(A(s, J)) + |T \setminus J| \right).$$

It is straightforward to check the following lemma (see [9]).

Lemma 2. *Let M' be a matroid on a finite set T with rank function ϱ' and let $\mathscr{B} = (B_s : s \in S)$ be a finite family of subsets of T. Then the function $\varrho : 2^S \to \mathbb{R}_+$ satisfying*

$$\varrho(X) = \varrho'(\cup\{B_s ; s \in X\}) \tag{2}$$

for any $X \subseteq S$ is the ground set rank function of an integral polymatroid $\mathbf{P}$ on S.

3. Properties of $\mathscr{M}$-polytransversals

Primarily we can extend the theorem of Edmonds and Fulkerson [2] to $\mathscr{M}$-polytransversals.

Theorem 2. *Let $\mathscr{A} = (A_t : t \in T)$ be a finite family of subsets of a finite set S and let $\mathscr{M}$ be a family $(M_s : s \in S)$ of matroids on T with rank functions ϱ_s, respectively. Then the partial $\mathscr{M}$-polytransversals of $\mathscr{A}$ are the integer valued independent vectors of the integral polymatroid $\mathsf{P} = (S, \varrho)$ such that for any $X \subseteq S$,*

$$\varrho(X) = \min_{J \subseteq T} \Big(\sum_{s \in X} \varrho_s(A(s,J)) + |T \setminus J| \Big). \tag{3}$$

Proof. Let ϱ be the function defined by (3). Then, by Lemma 1, $\varrho(X)$ denotes the maximal length of a partial $\mathscr{M}$-polytransversal of the family $\mathscr{A}_X = (A_t \cap X : t \in T)$ of subsets of X.

Take the family $\mathscr{B} = (B_s : s \in S)$ of subsets of T such that $B_s = A(s,T)$ for any $s \in S$. Let M_s' be the restriction of M_s to $A(s,T)$ $(s \in S)$ and M' the union of all M_s', $s \in S$. Let ϱ' be the rank of M'.

It is easy to check that there exists a one-to-one correspondence between the $\mathscr{M}$-SR of $\mathscr{A}_X$ and the subsets of $\cup\{B_s;\ s \in X\}$ which are independent in M'. Then, by Lemmas 1 and 2, (2) and (3) determine the same function, i.e. $\mathsf{P} = (S, \varrho)$ is an integral polymatroid and any $\mathscr{M}$-polytransversal of $\mathscr{A}$ is independent in P.

On the other hand, let $\mathbf{u} \in \mathbb{Z}_+^S$ be independent in P, i.e. $\mathbf{u}(X) \leqslant \varrho(X)$ for any $X \subseteq S$. Denote by $M_s^{\mathbf{u}}$ the truncation of M_s at $\mathbf{u}(s)$, i.e. the rank $\varrho_s^{\mathbf{u}}$ of $M_s^{\mathbf{u}}$ satisfies

$$\varrho_s^{\mathbf{u}}(J) = \min\{\varrho_s(J), \mathbf{u}(s)\} \quad (s \in S, J \subseteq T).$$

Denote by $\mathscr{M}^{\mathbf{u}}$ the family of matroids $(M_s^{\mathbf{u}} : s \in S)$ on T. We assert that

$$\mathbf{u}(S) \leqslant \min_{J \subseteq T} \Big(\sum_{s \in S} \varrho_s^{\mathbf{u}}(A(s,J)) + |T \setminus J| \Big). \tag{4}$$

Indeed, if this is not the case, take $K \subseteq T$ such that

$$\mathbf{u}(S) > \sum_{s \in S} \varrho_s^{\mathbf{u}}(A(s,K)) + |T \setminus K| = \sum_{s \in S} \Big(\min\{\varrho_s(A(s,K)), \mathbf{u}(s)\} \Big) + |T \setminus K|,$$

and let $Y = \{s \in S; \varrho_s(A(s,K)) \leqslant \mathbf{u}(s)\}$. Then

$$\mathbf{u}(S) > \sum_{s \in Y} \varrho_s(A(s,K)) + \mathbf{u}(S \setminus Y) + |T \setminus K| \geqslant \mathbf{u}(S \setminus Y) + \varrho(Y).$$

Therefore $\mathbf{u}(Y) > \varrho(Y)$ – a contradiction. Thus (4) holds.

Let $\mathbf{v} \in \mathbb{Z}_+^S$ be a partial $\mathcal{M}^{\mathbf{u}}$-polytransversal of $\mathcal{A}$ with the maximal length. Then, by Lemma 1 and (4), $\mathbf{u}(S) \leqslant \mathbf{v}(S)$. But, by definition of $M_s^{\mathbf{u}}$, $\mathbf{u}(s) \geqslant \mathbf{v}(s)$ for any $s \in S$. Thus $\mathbf{u} = \mathbf{v}$ and $\mathbf{u}$ is a partial $\mathcal{M}^{\mathbf{u}}$-polytransversal (and also a partial $\mathcal{M}$-polytransversal) of $\mathcal{A}$. Thus the partial $\mathcal{M}$-polytransversals of $\mathcal{A}$ are the integer valued independent vectors of the integral polymatroid $\mathbf{P} = (S, \varrho)$, which concludes the proof. □

The polymatroid $\mathbf{P} = (S, \varrho)$ from Theorem 2 will be called the *polymatroid of partial $\mathcal{M}$-polytransversals of $\mathcal{A}$*.

Theorem 2 has interesting consequences. Primarily, we can extend the theorems of Rado and Perfect.

Corollary 1. *Let $\mathcal{A} = (A_t : t \in T)$ be a finite family of subsets of a finite set S and let $\mathcal{M}$ be a family $(M_s : s \in S)$ of matroids on T with rank functions ϱ_s, respectively. Let $\mathbf{P}_1 = (S, \varrho_1)$ be an integral polymatroid on S with vector rank r_1 and $d \in \mathbb{Z}_+$, $d \leqslant |T|$. Then $\mathcal{A}$ has a partial $\mathcal{M}$-polytransversal of $\mathcal{A}$ with defect d which is independent in $\mathbf{P}_1$ if and only if for all $J \subseteq T$,*

$$r_1\big(\varrho_s(A(s, J)) : s \in S\big) \geqslant |J| - d$$

(note that $\big(\varrho_s(A(s, J)) : s \in S\big)$ denotes a vector in $\mathbb{Z}_+^S$).

Proof. Let $\mathbf{P} = (S, \varrho)$ be the (integral) polymatroid of partial $\mathcal{M}$-polytransversals of $\mathcal{A}$. Then Theorems 1 and 2 imply that $\mathcal{A}$ has the required property if and only if

$$\begin{aligned} |T| - d &\leqslant \min_{X \subseteq S} \big(\varrho(X) + \varrho_1(S \setminus X)\big) \\ &= \min_{X \subseteq S} \min_{J \subseteq T} \Big(\sum_{s \in X} \varrho_s(A(s, J)) + |T \setminus J| + \varrho_1(S \setminus X) \Big). \end{aligned}$$

Thus, by (1),

$$|T| - d \leqslant \min_{J \subseteq T} \big(r_1\big(\varrho_s(A(s, J)) : s \in S\big) + |T \setminus J|\big),$$

concluding the proof. □

Ford and Fulkerson's theorem [3] gives a condition for two families of sets to have a common transversal. We extend this result.

Corollary 2. *For $j = 1, 2$, let $\mathcal{A}^{(j)} = (A_t^{(j)} : t \in T^{(j)})$ be a finite family of subsets of a finite set S and let $\mathcal{M}^{(j)}$ be a family $(M_s^{(j)} : s \in S)$ of matroids on $T^{(j)}$ with* *rank functions* $\varrho_s^{(j)}$*, respectively. Then there exists* $\mathbf{u} \in \mathbb{Z}_+^S$, $|\mathbf{u}| \geqslant k$ *(*$k \in \mathbb{Z}_+$*), such that* $\mathbf{u}$ *is a partial* $\mathscr{M}^{(j)}$*-polytransversal of* $\mathscr{A}^{(j)}$ *for both* $j = 1, 2$*, if and only if for any* $J \subseteq T^{(1)}$, $K \subseteq T^{(2)}$,

$$\sum_{s\in S} \left(\min \{ \varrho_s^{(1)}(A^{(1)}(s,J)), \varrho_s^{(2)}(A^{(2)}(s,K)) \} \right)$$
$$\geqslant |J| + |K| - |T^{(1)}| - |T^{(2)}| + k.$$

Proof. follows immediately from Theorems 1 and 2. □

$\mathscr{M}$-polytransversals and $\mathscr{M}$-SR generalize several known notions from transversal theory.. For instance, if $\mathscr{M}$ is a system of uniform matroids of rank k then we get in fact the k-transversals from [15] and [16]. A little more complicated "k-transversals" were introduced in [6], but they can be also described by a special class of $\mathscr{M}$-polytransversals. In [7] we dealt with another generalization of transversals, the so called "$\mathscr{M}$-transversals". Note that from Theorem 2 some of the results from [7] can be obtained, too.

As pointed out in [9] (see also [5], [8], [10]), any integral polymatroid on S can be represented by the construction of Lemma 2. Then it follows from the proof of Theorem 2 that any integral polymatroid on S can be represented as a polymatroid of $\mathscr{M}$-polytransversals of a family of sets $\mathscr{A}$. This contrasts with the known fact that transversal matroids from a proper subclass of matroids.

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