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ON PROJECTIVE INTERVALS IN A MODULAR LATTICE

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Summary. In this paper a combinatorial result concerning pairs of projective intervals of a modular lattice will be established.

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1. PRELIMINARIES

The recent papers dealing with combinatorial questions concerning partially ordered sets are rather frequent (cf., e.g., [2], [3], [4]).

Let L be a modular lattice. We denote by $\mathscr{D}$ the collection of all systems $D = (a_1, a_2, a_3, u, v)$ of distinct elements of L such that

$$u = a_1 \wedge a_2 = a_1 \wedge a_3 = a_2 \wedge a_3, \qquad v = a_1 \vee a_2 = a_1 \vee a_3 = a_2 \vee a_3.$$

An interval $[a_1, a_2]$ of L will be said to be an m-interval if there is $D \in \mathscr{D}$ such that (under the above notation), $[a_1, a_2]$ is projective to $[u, a_1]$.

Let $\alpha = [b_1, b_2]$ and $\beta = [c_1, c_2]$ be distinct projective intervals of L. Assume that α is nontrivial (i.e. $b_1 \neq b_2$); then β is nontrivial as well.

There exists a least positive integer n such that for some $\alpha_0, \alpha_1, \ldots, \alpha_n$ in L the following conditions are satisfied:

(i) $\alpha_0 = \alpha$ and $\alpha_n = \beta$;

(ii) for each $i \in \{1, 2, \ldots, n\}$, the interval α_i is transposed to the interval α_{i-1}. We denote $\mu(\alpha, \beta) = n$.

Let $S(\alpha)$ be the collection of all systems $(y_0, y_1, y_2, \ldots, y_m)$ with $b_1 = y_0 < y_1 < y_2 < \ldots < y_m = b_2$. The collection $S(\beta)$ is defined analogously. For each $i \in \{1, 2, \ldots, m\}$ let $k(i)$ be a positive integer.

A system of distinct intervals

(1)
$$(\beta_{ij})(i = 1, 2, \ldots, m; j = 1, 2, \ldots, k(i))$$

will be said to be a p-system for the intervals α and β if the following conditions are satisfied:

(i) there are $Y = (y_0, y_1, \ldots, y_n) \in S(\alpha)$ and $Z = (z_1, z_2, \ldots, z_n) \in S(\beta)$ such that for each $i \in \{1, 2, \ldots, m\}$ we have $\beta_{i1} = [y_{i-1}, y_1]$ and $\beta_{i,k_i} = [z_{i-1}, z_i]$;

(ii) for each $i \in \{1, 2, \ldots, m\}$ and each $j \in \{1, 2, \ldots, k(i)\}$ the interval $\beta_{i,j-1}$ is transposed to $\beta_{i,j}$. The collection of all p-systems for α and β will be denoted by $p(\alpha, \beta)$. For $A \in P(\alpha, \beta)$ (where A is as in (1)) let A_0 be the of all $\beta_{ij} \in A$ such that β_{ij} fails to be an m-interval. We put

$$\nu(A) = \text{card } A_0,$$
$$\nu_0(\alpha, \beta) = \min\{\nu(A): A \in P(\alpha, \beta)\}.$$

In this note it will be proved that we always have

(2)
$$\nu_0(\alpha, \beta) \leqslant 3$$

and this estimate cannot be sharpened in general.

The estimate (2) is a consequence of the following result:

(A) *Let $\alpha = [b_1, b_2]$ and $\beta = [c_1, c_2]$ be nontrivial intervals of a modular lattice L. Assume that α is projective to β. Then there exist elements x_0, x_1, ..., x_m, y_0, y_1, ..., y_m in L such that the following conditions are satisfied:*

(i) $b_1 = x_0 < x_1 < \ldots < x_m = b_2$, $c_1 = y_0 < y_1 < \ldots < y_m = c_2$ *and for each $i \in \{1, 2, \ldots, m\}$ the interval $[x_{i-1}, x_i]$ is projective to $[y_{i-1}, y_i]$;*

(ii) *there is $i(1) \in \{1, 2, \ldots, m\}$ such that $[x_{i-1}, x_i]$ is an m-interval for each $i \in \{1, 2, \ldots, m\} \setminus \{i(1)\}$, and either $[x_{i(1)-1}, x_{i(1)}]$ is an m-interval, or there is an interval $[t_1, t_2] \subseteq L$ such that $[x_{i(1)-1}, x_{i(1)}]$ is transposed to $[t_1, t_2]$ and $[t_1, t_2]$ is transposed to $[y_{i(1)-1}, y_{i(1)}]$.*

THE PROOF OF (A)

We will apply the notation from Section 1. Again, let α and β be distinct nontrivial intervals of a modular lattice L. Assume that α and β are projective. A p-system A for α and β will be said to be reduced if (under the notation as above), whenever $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, k(i) - 1\}$, then $\beta_{i,j-1}$ fails to be transposed to $\beta_{i,j+1}$.

The following lemma is easy to verify.

2.1. Lemma. *Let $A \in P(\alpha, \beta)$. Then there exists $A' \in P(\alpha, \beta)$ such that $A' \subseteq A$ and A' is reduced.*

Let $[c_1, c_2]$ and $[d_1, d_2]$ be transposed intervals of L; then we have either

(i) $\quad c_2 \wedge d_1 = c_1, \qquad c_2 \vee d_1 = d_2,$

or

(ii) $\quad d_2 \wedge c_1 = d_1, \qquad d_2 \vee c_1 = c_2.$

If (i) is valid, then we write $[c_1, c_2] \nearrow [d_1, d_2]$; the validity of (ii) will be recorded by writing $[c_1, c_2] \searrow [d_1, d_2]$.

2.2. Lemma. *Let $A \in P(\alpha, \beta)$ and assume that A is reduced. Let A be as in (1). If $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, k(i) - 1\}$, $\alpha_{i,j-1} \nearrow \alpha_{i,j}$, then $\alpha_{i,j} \searrow \alpha_{i,j+1}$ (and dually).*

The proof is trivial.

Let $A \in P(\alpha, \beta)$ be as in (1). Let $i \in \{1, 2, \ldots, m\}$, $z_{i1} \in L$, $x_{i-1,1} < z_{i1} < x_{i1}$. We define elements $z_{i2}, z_{i3}, \ldots, z_{i,k(i)}$ by induction as follows: if $z_{i,j-1}$ ($j \in \{2, \ldots, k(i)-1\}$) is already defined and if $\alpha_{i,j-1} \nearrow \alpha_{i,j}$, then we put $z_{ij} = z_{i,j-1} \vee d_1$, where d_1 is the least element of $\alpha_{i,j}$; on the other hand, if $\alpha_{i,j-1} \searrow \alpha_{i,j}$, then we set $z_{ij} = z_{i,j-1} \wedge d_2$, where d_2 is the largest element of α_{ij}.

Consider the system A' which we obtain from the system A if the i-th row $(\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,i(k)})$ of A is replaced by the rows

$$\alpha'_{i,1}, \quad \alpha'_{i,2}, \quad \ldots, \quad \alpha'_{i,i(k)}$$
$$\alpha''_{i,1}, \quad \alpha''_{i,2}, \quad \ldots, \quad \alpha''_{i,n},$$

where

$$\alpha'_{i,j} = \{t \in \alpha_{ij} : t \leqslant z_{i,j}\}, \qquad \alpha''_{i,j} = \{t \in \alpha_{ij} : t \geqslant z_{i,j}\}.$$

Then we obviously have:

2.3. Lemma. *A' is a p-system for the intervals α and β.*

The system A' will be said to be generated by the system A and by the element z_{i1}.

Let $y, z \in L$, $b_1 < y < b_2$, $c_1 < z < c_2$. Suppose that $[b_1, y]$ is projective to $[c_1, z]$ and that $[y, b_2]$ is projective to $[z, c_2]$.

2.4. Lemma. *Let $A \in p([b_1, y], [c_1, z])$. (We apply the same notation as in (1) with the distinction that we now have y and z instead of b_2 and c_2.) Let $\beta_{m+1,i}$ ($i = 1, 2, \ldots, k(m+1)$) be intervals of L such that $\beta_{m+1,1} = [y, b_2]$, $\beta_{m+1,k(m+1)} = [z, c_2]$*

and for each $i \in \{2, 3, \ldots, k(m+1)\}$ the interval $\beta_{m+1, i-1}$ is transposed to $\beta_{m+1, i}$. Let A' be the system

$$(\beta_{ij}\,(i = 1, 2, \ldots, m+1;\, j = 1, 2, \ldots, k(i))).$$

Then $A' \in p(\alpha, \beta)$.

Proof. This is an immediate consequence of the definition of $p(\alpha, \beta)$. The assertion dual to 2.4. is also valid. □

2.5. Lemma. Let x, y and z be elements of a modular lattice L. Assume that the relations

$$[x \wedge y, x] \nearrow [y, x \vee y] \quad \text{and} \quad [y, x \vee y] \searrow [y \wedge z, z]$$

are valid. Then the sublattice L_1 of L generated by the elements x, y and z is a homomorphic image of the lattice on Fig. 1.
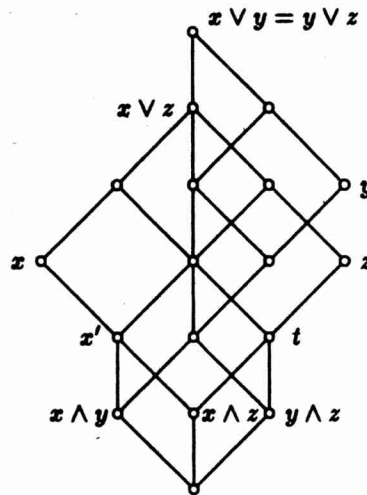

Fig. 1

Proof. If we consider the free modular lattice with three free generators (cf. e.g. [1], Chap. III, Theorem 8) x, y and z, and if we take into account that in our case we have $x \vee y = y \vee z$, then we obtain the assertion of the lemma. □

Theorem. Let α and β be nontrivial distinct intervals of a modular lattice L. Assume that α is projective to β. Then there is $A \in P(\alpha, \beta)$ such that (under the notation as in (1) the following condition is satisfied: there is $i(1) \in \{1, 2, \ldots, m\}$

such that, whenever $i \in \{1, 2, \ldots, m\} \backslash \{i(1)\}$ and $j \in \{1, 2, \ldots, k(i)\}$, then β_{ij} is an m-interval; next, either $\beta_{i(1),1}$ is an m-interval, or $k(i(1)) \leqslant 3$.

P r o o f.   Under the notation as in Section 1, let $\mu(\alpha, \beta) = n$. We have $n \geqslant 1$. If $n = 1$, then the assertion obviously holds (it suffices to consider the system (α_0, α_1)).

Suppose that $n \geqslant 2$ and let us apply induction with respect to n. First we consider the system

$$(\alpha_k) \qquad (k = 0, 1, 2, \ldots, n)$$

which obviously belongs to $p(\alpha, \beta)$. Without loss of generality we may assume that this system is reduced. Next, we can suppose that $\alpha_0 \nearrow \alpha_1 \searrow \alpha_2$ is valid (in the case $\alpha_0 \searrow \alpha_1 \nearrow \alpha_2$ we apply a dual procedure).

Let x, y and z be the greatest element of α_0, the least element of α_1 and the greatest element of α_2, respectively. (Cf. Fig. 1.) Then

$$\alpha_0 = [x \wedge y, x], \quad \alpha_1 = [y, x \vee y], \quad \alpha_2 = [x \wedge z, z].$$

At the same time, $x \vee y = y \vee z$. Put $x' = (x \wedge y) \vee (x \wedge z)$. We have obviously

$$x \wedge y \leqslant x' \leqslant x.$$

From $x \wedge y < x$ we infer that either $x \wedge y < x'$ or $x' < x$.

Let us distinguish the following cases.

(a) Let $x \wedge y = x'$. Then $\alpha = \alpha_0 = [x', x]$. In view of Fig. 1, α_0 is an m-interval; therefore $\alpha_1, \alpha_2, \ldots, \alpha_n$ are m-intervals as well. Now it suffices to put $a = (\alpha_i)$ $(i = 0, 1, 2, \ldots, n)$.

(b) Let $x' = x$. Then $\alpha = \alpha_0 = [x \wedge y, x]$. Next, $\alpha_2 = [y \wedge z, t]$, where $t = (x \wedge z) \vee (y \wedge z)$. Denote $\alpha_1' = [x \wedge y \wedge z, x \wedge z]$. We have (cf. Fig. 1)

$$\alpha_0 \searrow \alpha_1' \nearrow \alpha_2.$$

Thus the system A' consisting of the intervals

$$\alpha_0, \ \alpha_1', \ \alpha_2, \ \alpha_3, \ \ldots, \ \alpha_n$$

belongs to $P(\alpha, \beta)$. Since $\alpha_2 \nearrow d_3$, according to 2.2 the system A' fails to be reduced. Thus in view of 2.1 there exists a system

$$\beta_0, \ \beta_1, \ \ldots, \ \beta_l$$

which belongs to $P(\alpha, \beta)$ such that $l < n$. Therefore by the induction hypothesis, the assertion of the theorem is valid for α and β.

(c) Let $x \wedge y < x' < x$. Let A_1 be the system

$$(\alpha_i) \qquad (i = 0, 1, 2, \ldots, n)$$

and let A_2 be the system generated by A_1 and the element x'. Then (under the notation as in Lemma 2.3) the system A_2 consists of intervals

$$\alpha_0', \ \alpha_1', \ \ldots, \ \alpha_n',$$
$$\alpha_o'', \ \alpha_1'', \ \ldots, \ \alpha_n'',$$

where

$$\alpha_0' = [x \wedge y, x'], \quad \alpha_n' = [y \wedge z, t],$$
$$\alpha_0'' = [x', x], \qquad \alpha_n'' = [t, z].$$

Since α_0'' is an m-interval, all α_i' ($i = 1, 2, \ldots, n$) must be m-intervals. Next, by the same argument as in (b) we can verify that there exists a system A_3 consisting of intervals

$$\beta_0, \ \beta_1, \ \ldots, \ \beta_l$$

with $1 < n$ such that $A_3 \in p([x', x], [t, z])$. Hence by the induction hypothesis, the assertion of the theorem is valid for the intervals $[x', x]$ and $[t, z]$. Now it suffices to apply Lemma 2.3. □

Theorem (A) in Section 1 is obviously a consequence of (in fact, equivalent to) Theorem 2.6.

2.7. Example. Let L be as in Fig. 1 Consider the intervals $\alpha = [x \wedge y, x']$ and $\beta = [y \wedge z, t]$. It is easy to verify that $\mu_0(\alpha, \beta) = 3$. Hence the estimate (2) cannot be sharpened in general.

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