A NOTE ON INTEGRATION OF RATIONAL FUNCTIONS

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Summary. Let P and Q be polynomials in one variable with complex coefficients and let n be a natural number. Suppose that Q is not constant and has only simple roots. Then there is a rational function φ with $\varphi' = P/Q^{n+1}$ if and only if the Wronskian of the functions $Q', (Q^2)', \ldots$ $\ldots, (Q^n)', P$ is divisible by Q.

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0. Introduction. Let f be a rational function of one variable. If we ask how to recognize whether f has a rational primitive, we may get various "reasonable" answers. Let us observe, first of all, that every such f can be expressed as P/Q^m, where P and Q are polynomials, Q is not identically zero and has no multiple roots (which will be assumed throughout this introduction) and m is a natural number. We may even require P and Q to have coefficients in the smallest field Θ containing the coefficients of the polynomials whose ratio is f. (It is possible to obtain P and Q by so called rational operations.) Then we can find polynomials A and B with coefficients in Θ such that $P/Q^m = (A/Q^{m-1})' + B/Q$. (We may proceed, e.g., as in the proof of Lemma 21.) It is obvious that f has a rational primitive if and only if B is divisible by Q. This argument in some sense solves our problem.

Let us now compare the described procedure with the assertion (iv) on p. 19 of Hardy's book [1]:

P/Q^2 has a rational primitive if and only if $PQ'' - P'Q'$ is divisible by Q.

This assertion gives a very simple answer to the mentioned problem, if $m = 2$. For the case $m = 3$ it is not difficult to prove the following:

P/Q^3 has a rational primitive if and only if $P(3Q''^2 - Q'Q''') - 3P'Q'Q'' + P''Q'^2$ is divisible by Q.

This being so, it will not surprise the reader that for every positive integer n we can find expressions $V_0, \ldots, V_n$ such that P/Q^{n+1} has a rational primitive if and only if $PV_0 + P'V_1 + \ldots + P^{(n)}V_n$ is divisible by Q; V_j is the sum of terms of the form

$$c(Q')^{j_1}(Q'')^{j_2}\cdots(Q^{(n+1)})^{j_{n+1}},$$

where c is an integer and $j_1, \ldots, j_{n+1}$ are nonnegative integers with $j_1 + \ldots + j_{n+1} = n$ and $j + j_1 + 2j_2 + \ldots + (n+1) j_{n+1} = 2n$ (so that $j + j_2 + \ldots + nj_{n+1} = n$). We get these expressions, if we take in Theorem 22 for F the mapping Φ defined in 7.

If we choose there $F = \Lambda$, where Λ is as in 14, we see that P/Q^{n+1} has a rational primitive if and only if the Wronskian of the functions $Q', (Q^2)', \ldots, (Q^n)', P$ is divisible by Q. This result is remarkable for its simplicity, but it is in some sense unpractical. The mentioned Wronskian has namely the form $PW_0 + P'W_1 + \ldots \ldots + P^{(n)}W_n$, where W_j are determinants whose direct computation is considerably more difficult than the computation of the expressions V_j, if $n > 1$. However, it follows from 13 and 14 that

$$W_j = V_j(Q')^{\binom{n}{2}} \prod_{k=1}^{n-1} k!\,.$$

1. Notation. Let $\mathfrak{P}$ be the set of all polynomials in one variable with coefficients in a given field of numbers. Throughout this note Q is a given element of $\mathfrak{P}$. For $f, g \in \mathfrak{P}$ the symbol $f \circ g$ means the corresponding composite function (i.e. $(f \circ g)(x) = f(g(x))$). For any positive integers i, k let a_{ik}, b_{ik} be polynomials defined as follows: If $k \leqq i$, let $a_{ik} = k!\binom{i}{k} Q^{i-k}$; if $k > i$, let $a_{ik} = 0$. Further let $b_{1k} = Q^{(k)}$; $b_{i1} = 0$, $b_{i,k+1} = b'_{ik} + Q'b_{i-1,k}$ $(i = 2, 3, \ldots, k = 1, 2, \ldots)$. Obviously $a_{kk} = k!$, $b_{ik} = 0$ for $k < i$, $b_{kk} = (Q')^k$.

2. Lemma. *Let* $K \in \mathfrak{P}$. *Then* $(K \circ Q)^{(k)} = \sum_{j=1}^{k} (K^{(j)} \circ Q)\, b_{jk}$ $(k = 1, 2, \ldots)$.

Proof. This is obvious, if $k = 1$. If the assertion holds for some k, then $(K \circ Q)^{(k+1)} = \sum_{j=1}^{k} (K^{(j+1)} \circ Q)\, Q' b_{jk} + \sum_{j=1}^{k} (K^{(j)} \circ Q)\, b'_{jk} = (K' \circ Q)\, b'_{1k} + \sum_{i=2}^{k} (K^{(i)} \circ Q)(b_{i-1,k}Q' + b'_{ik}) + (K^{(k+1)} \circ Q)\, Q' b_{kk} = \sum_{j=1}^{k+1} (K^{(j)} \circ Q)\, b_{j,k+1}$.

3. Conventions, notation. In what follows n is a nonnegative integer. For each $y \in \mathfrak{P}$ let $\varrho(y) = (y, y', \ldots, y^{(n)})$. For $i = 1, 2, \ldots$ let $b_i = (b_{i1}, \ldots, b_{i,n+1})$.

Let $\mathfrak{F} = \mathfrak{F}_n$ be the set of all mappings F of $\mathfrak{P}$ to $\mathfrak{P}$ for which there are $S_0, \ldots, S_n \in \mathfrak{P}$ such that

$$F(y) = \sum_{j=0}^{n} y^{(j)} S_j \quad (y \in \mathfrak{P})\,. \tag{1}$$

Remark. It is easy to see that the polynomials S_j are uniquely determined by F. (We may, e.g., apply the relations

$$F(y_i) = \sum_{j=0}^{i-1} y_i^{(j)} S_j + i!\, S_i \quad (i = 0, \ldots, n)\,,$$

where $y_i(x) = x^i$.) Further it is clear that $F(y)$ is the scalar product $\varrho(y)\, S$, where $S = (S_0, \ldots, S_n)$.

4. Lemma. *Let i be a natural number. Then*

$$\varrho((Q^i)') = \sum_{j=1}^{i} a_{ij} b_j\,.$$

Proof. Set $K(x) = x^i$. Clearly $K^{(j)} \circ Q = a_{ij}$ for each $j > 0$. Let k be a natural number. By 2 we have $(Q^i)^{(k)} = \sum_{j=1}^{k} a_{ij} b_{jk}$. Since $a_{ij} = 0$ for $j > i$ and $b_{jk} = 0$ for $j > k$, we have also $(Q^i)^{(k)} = \sum_{j=1}^{i} a_{ij} b_{jk}$. Now we observe that $\varrho((Q^i)') = ((Q^i)', \ldots, (Q^i)^{(n+1)})$.

5. Lemma. *Let L be a linear subspace of $\mathfrak{P}$. Suppose that the following holds:*

(2) *For each $y \in L$ and each $z \in \mathfrak{P}$ we have $yz \in L$.*

(3) *If $z \in \mathfrak{P}$ and $zQ' \in L$, then $z \in L$.*

Let F be given by (1) *and let $F((Q^i)') \in L$ for $i = 1, \ldots, n$. Then*

(4) $F((Q^{n+1})') - (n+1)!\,(Q')^{n+1} S_n \in L$.

If, moreover,

(5) $F((Q^{n+1})') \in L$ *or* $S_n \in L$,

then $S_j \in L$ for $j = 0, \ldots, n$.

Proof. Set $S = (S_0, \ldots, S_n)$. By 4 we have $F((Q^i)') = \varrho((Q^i)')\, S = \sum_{j=1}^{i-1} a_{ij}(b_j S) + i!\,(b_i S)$ $(i = 1, \ldots, n+1)$. We see that $b_1 S \in L$; by (2) we have $b_2 S \in L, \ldots, b_n S \in L$ and $F((Q^{n+1})') - (n+1)!\,(b_{n+1} S) \in L$. Clearly

$$b_i S = (Q')^i S_{i-1} + \sum_{j=i+1}^{n+1} b_{ij} S_{j-1} .$$

Choosing $i = n + 1$ we get (4). Now it follows from (3) and (5) that $S_n \in L$, $S_{n-1} \in L, \ldots, S_0 \in L$.

6. Lemma. *Let L be as in 5. Let $\alpha_j, \beta_j \in \mathfrak{P}$ $(j = 0, \ldots, n)$, $G(y) = \sum_{j=0}^{n} y^{(j)} \alpha_j$, $H(y) = \sum_{j=0}^{n} y^{(j)} \beta_j$ $(y \in \mathfrak{P})$. Let $G((Q^i)') \in L$, $H((Q^i)') \in L$ for $i = 1, \ldots, n$. Then $(\alpha_n H - \beta_n G)(y) \in L$ for each $y \in \mathfrak{P}$.*

Proof. Set $F = \alpha_n H - \beta_n G$. Then we have (1) with $S_n = 0$. Clearly $F((Q^i)') \in L$ for $i = 1, \ldots, n$ and, by 5, $F(y) \in L$ for each $y \in \mathfrak{P}$.

7. Notation. Let $v = (0, \ldots, 0)$ $(n + 1$ terms). For each $(n+1) \times (n+1)$-matrix Z with rows $z_0, \ldots, z_n$ let Z^* be the matrix with rows $v, z_0, \ldots, z_{n-1}$. For each $f \in \mathfrak{P}$ let $E(f)$ be the matrix with entries e_{ik}, where $e_{ik} = 0$ for $k < i$ and $e_{ik} = \binom{k}{i} f^{(k-i)}$ for $k \geqq i$ $(i, k = 0, \ldots, n)$. Let I be the $(n+1) \times (n+1)$ identity matrix and let w be its last row. Further let

$$M = n\, E(Q') - (E(Q))^* + QI^* .$$

Let $m_0, \ldots, m_n$ be the rows of M. For each $y \in \mathfrak{P}$ let $\Phi(y)$ be the determinant with rows $m_0, \ldots, m_{n-1}, \varrho(y)$.

8. Lemma. *Let $f, g \in \mathfrak{P}$. Then $\varrho(fg) = \varrho(f)\, E(g)$, $\varrho(f'g) = \varrho(f)\,(E(g))^* + f^{(n+1)} g w$; in particular, $\varrho(f') = \varrho(f)\, I^* + f^{(n+1)} w$.*

(The easy proof is omitted.)

9. Lemma. *M is an upper triangular matrix with diagonal entries* $(n-k)\,Q'$ $(k = 0, \ldots, n)$; *in particular*, $m_n = v$.

Proof. Let $H = E(Q) - QI$. Then $H = (h_{ik})$ is an upper triangular matrix with $h_{kk} = 0$ $(k = 0, \ldots, n)$ and $h_{k-1,k} = kQ'$ $(k = 1, \ldots, n)$. Obviously $M = n\,E(Q') - H^*$ from which our assertion follows at once.

10. Lemma. *Let* $f \in \mathfrak{P}$. *Then* $\Phi(nfQ' - f'Q) = -Q\,\Phi(f')$.

Proof. By 8 we have $\varrho(nfQ' - f'Q) = n\,\varrho(f)\,E(Q') - \varrho(f)\,(E(Q))^* - f^{(n+1)}Qw = \varrho(f)\,M - Q(\varrho(f)\,I^* + f^{(n+1)}w) = \varrho(f)\,M - Q\,\varrho(f')$. Since, by 9, we have $m_n = v$, $\varrho(f)\,M$ is a linear combination of the rows $m_0, \ldots, m_{n-1}$. This easily implies our assertion.

11. Lemma. *We have* $\Phi((Q^i)') = 0$ *for* $i = 1, \ldots, n$. *If we define* V_j *by*

$$\Phi(y) = \sum_{j=0}^{n} y^{(j)} V_j \quad (y \in \mathfrak{P}), \tag{6}$$

then

$$V_n = n!\,(Q')^n. \tag{7}$$

Proof. We may suppose that $n > 0$. If we choose $f = 1$ in 10, we get $\Phi(Q') = 0$. Now, if $i < n$ and $\Phi((Q^i)') = 0$, we set $f = Q^i$ in 10 and we get $\Phi((Q^{i+1})') = [(i+1)/(n-i)]\,\Phi(nQ^iQ' - iQ^{i-1}Q'Q) = 0$. It is obvious that V_n is a triangular determinant with diagonal entries $(n-k)\,Q'$ $(k = 0, \ldots, n-1)$. This completes the proof.

12. Convention. In sections 13 and 14 we define mappings Ψ and Λ. The reader can prove easily that theorems 13 and 14 hold, if $n = 0$ or $Q' = 0$. (If $n = 0$, then $\Phi(y) = \Psi(y) = \Lambda(y) = y$; if $Q' = 0$ and $n > 0$, then $\Phi(y) = \Psi(y) = \Lambda(y) = 0$ $(y \in \mathfrak{P})$.) Therefore in the corresponding proofs we will suppose that $n > 0$ and that Q is not constant. Then (3) holds with $L = \{0\}$.

13. Theorem. *For each* $y \in \mathfrak{P}$ *let* $\Psi(y)$ *be the determinant with rows* $b_1, \ldots, b_n$, $\varrho(y)$ (see 3). *Let* T_j *be defined by*

$$\Psi(y) = \sum_{j=0}^{n} y^{(j)} T_j. \tag{8}$$

Then

$$\Psi((Q^i)') = 0 \quad \textit{for} \quad i = 1, \ldots, n \tag{9}$$

and

$$n!\,\Psi = (Q')^{\binom{n}{2}}\,\Phi. \tag{10}$$

Proof. The relation (9) follows from 4. It is easy to see that T_n is a triangular determinant with diagonal entries $Q', \ldots, (Q')^n$. Let (6) hold. By 11 and 6 with $L = \{0\}$ we have $V_n \Psi = T_n \Phi$. This combined with (7) yields (10).

14. Theorem. *For each $y \in \mathfrak{P}$ let $\Lambda(y)$ be the Wronskian of the functions Q', $(Q^2)', \ldots, (Q^n)', y$. Let Ψ be as in 13. Then $\Lambda = \Psi \prod_{k=1}^{n} k!$.*

Proof. Let A, B, C be matrices with entries $a_{ik}, b_{ik}, (Q^i)^{(k)}$ $(i, k = 1, \ldots, n)$. By 4, where we take $n - 1$ instead of n, we have $C = AB$. Let us define W_j by $\Lambda(y) = \sum_{j=0}^{n} y^{(j)} W_j$ and let (8) hold. Then $\det A = \prod_{k=1}^{n} k!$, $\det B = T_n$ and $W_n = \det C = \det A \det B$. Clearly $\Lambda((Q^i)') = 0$ for $i = 1, \ldots, n$. By (9) and 6 with $L = \{0\}$ we have $T_n \Lambda = W_n \Psi$ which easily implies our assertion.

15. Conventions, notation. In what follows we suppose that Q is a polynomial that is not identically zero and has no multiple roots (so that it is relatively prime to Q'). If $f, g \in \mathfrak{P}$, then the relation $f \equiv g$ means that $f - g = hQ$ for some $h \in \mathfrak{P}$. Let $\mathfrak{V} = \mathfrak{V}_n$ be the set of all mappings $F \in \mathfrak{F}_n$ such that $F((Q^i)') \equiv 0$ $(i = 1, \ldots, n)$. Let $\mathfrak{W} = \mathfrak{W}_n$ be the set of all mappings $F \in \mathfrak{V}_n$ for which $F((Q^{n+1})')$ is relatively prime to Q.

16. Lemma. *Let $F \in \mathfrak{V}$ and let (1) hold. Then $F \in \mathfrak{W}$ if and only if S_n is relatively prime to Q.*

Proof. We set $L = \{y \in \mathfrak{P};\ y \equiv 0\}$ in 5 and apply (4).

17. Theorem. *The mappings Φ, Ψ and Λ are elements of $\mathfrak{W}$.*

Proof. By (7) and 16 we have $\Phi \in \mathfrak{W}$. Now we apply 13 and 14.

18. Lemma. *Let $F \in \mathfrak{V}, f \in \mathfrak{P}$. Then $F(nfQ' - f'Q) \equiv 0$.*

Proof. Let (1) and (6) hold and let $L = \{y \in \mathfrak{P};\ y \equiv 0\}$. Set $y = nfQ' - f'Q$. By 6 and 10 we have $V_n F(y) \equiv S_n \Phi(y) \equiv 0$ and, by 11, V_n is relatively prime to Q. Thus $F(y) \equiv 0$.

19. Lemma. *Let $n > 0$, $F \in \mathfrak{W}_n$. Set $G(y) = F(yQ)' - Q\,F(y)$ $(y \in \mathfrak{P})$. Then $G \in \mathfrak{W}_{n-1}$.*

Proof. Let (1) hold. It is obvious that there are $C_j \in \mathfrak{P}$ such that $G(y) = \sum_{j=0}^{n} y^{(j)} C_j$. Since $C_n = S_n Q - Q S_n$, we have $G \in \mathfrak{F}_{n-1}$. Now we observe that $G((Q^i)') \equiv (i/(i+1))\, F((Q^{i+1})')$ for each positive integer i.

20. Lemma. Let $F \in \mathfrak{V}_n$, $P \in \mathfrak{P}$. Let P/Q^{n+1} have a rational primitive. Then $F(P) \equiv 0$.

Proof. It is well-known that there is an $f \in \mathfrak{P}$ such that $P/Q^{n+1} = (f/Q^n)'$; thus $P = f'Q - nfQ'$. By 18 we have $F(P) \equiv 0$.

21. Lemma. Let $F \in \mathfrak{W}_n$. Let $P \in \mathfrak{P}$, $F(P) \equiv 0$. Then P/Q^{n+1} has a rational primitive.

Proof. It is easy to see that the assertion holds, if $n = 0$. Now let k be a natural number such that the assertion holds for $n = k - 1$. Let $F \in \mathfrak{W}_k$, $P \in \mathfrak{P}$, $F(P) \equiv 0$. There are $f, g \in \mathfrak{P}$ such that $P = kfQ' - gQ$. Set $P_1 = kfQ' - f'Q$, $P_2 = f' - g$. Then $P = P_1 + QP_2$. By 18 we have $F(P_1) \equiv 0$ so that $F(QP_2) \equiv 0$. Let G be as in 19. Then $G \in \mathfrak{W}_{k-1}$ and $G(P_2) \equiv 0$ so that, by induction assumption, P_2/Q^k has a rational primitive. Obviously $P_1/Q^{k+1} = (-f/Q^k)'$ and $P/Q^{k+1} = P_1/Q^{k+1} + P_2/Q^k$. Therefore the assertion holds also for $n = k$.

22. Theorem. Let $P \in \mathfrak{P}$, $F \in \mathfrak{W}_n$. Then P/Q^{n+1} has a rational primitive if and only if $F(P) \equiv 0$.

(This follows at once from 20 and 21.)

Remark 1. It is very easy to construct the matrix $M = (m_{ik})$ by means of which the mapping Φ has been defined. We have $m_{ik} = \beta_{ik} Q^{(k-i+1)}$, where $\beta_{0k} = n$ and $\beta_{ik} = n\binom{k}{i} - \binom{k}{i-1}$ for $i = 1, \ldots, k$ $(k = 0, \ldots, n)$; in particular, $\beta_{kk} = n - k$. The numbers β_{ik} with $0 < i < k$ can be obtained from the obvious relations $\beta_{r,s+1} = \beta_{rs} + \beta_{r-1,s}$ $(1 \leqq r \leqq s;\ s = 1, \ldots, n - 1)$. Moreover $\beta_{in} = n\binom{n}{i} - \binom{n}{i-1} = \binom{n}{i-1}[n(n - i + 1) - i]/i = \binom{n}{i-1}[(n - i)(n + 1)]/i = \binom{n+1}{i}(n - i)$ $(i = 1, \ldots, n)$. Thus, if $n + 1$ is a prime, the numbers $\beta_{1n}, \ldots, \beta_{nn}$ are its multiples. For example, if $n = 4$, $\Phi(y)$ is the determinant

$$\begin{vmatrix} 4Q' & 4Q'' & 4Q''' & 4Q^{(4)} & 4Q^{(5)} \\ 0 & 3Q' & 7Q'' & 11Q''' & 15Q^{(4)} \\ 0 & 0 & 2Q' & 9Q'' & 20Q''' \\ 0 & 0 & 0 & Q' & 10Q'' \\ y & y' & y'' & y''' & y^{(4)} \end{vmatrix}.$$

Now let n be an arbitrary natural number. It follows from the definition of a determinant that $\Phi(y)$ is the sum of terms of the form

$$cQ^{(k_0-0+1)}Q^{(k_1-1+1)} \ldots Q^{(k_{n-1}-(n-1)+1)}y^{(k_n)}, \tag{11}$$

where c is an integer, $\{k_0, k_1, \ldots, k_n\} = \{0, 1, \ldots, n\}$ and $k_i \geqq i$ for $i = 0, \ldots, n$. Let us write $k_n = j$. Since $\sum_{i=0}^{n} (k_i - i) = 0$, we have $\sum_{i=0}^{n-1} (k_i - i + 1) + j - n + 1 = \sum_{i=0}^{n} (k_i - i + 1) = n + 1$ so that $j + \sum_{i=0}^{n-1} (k_i - i + 1) = 2n$. Hence (11) can now be expressed also in the form $cy^{(j)}(Q')^{j_1}(Q'')^{j_2} \cdots (Q^{(n+1)})^{j_{n+1}}$, where j_r are nonnegative integers, $j_1 + \ldots + j_{n+1} = n$ and $j + j_1 + 2j_2 + \ldots + (n + 1)j_{n+1} = 2n$. We see that the expressions V_j defined by (6) have the form described in the introduction.