CONTINUITY OF LIFTINGS

W. M. MIKULSKI, Krakow

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Summary. Conditions are given under which $L(M)\,\sigma_m(v_m)$ tend to $L(M)\,\sigma(v)$, where L is a lifting, M a manifold, σ_m and σ are sections defined in a neighbourhood of $x \in M$ such that $j^\infty_x(\sigma_m)$ tend to $j^\infty_x(\sigma)$, and v_m is a sequence of points over x tending to v.

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Let F and G be two natural bundles over n-dimensional manifolds. Let H be a natural bundle over dim (GR^n)-dimensional manifolds. ([4]). If U is an open subset of an n-manifold M, then a mapping $\sigma: U \to FM$ (or $\varrho: (\pi^G_M)^{-1}(U) \to HGM$) of class C^∞ such that $(\pi^F_M) \circ \sigma = \mathrm{id}_U$ $(\pi^H_{GM}) \circ \varrho = \mathrm{id}_{(\pi^G_M)^{-1}(U)})$ is called a section of $\pi^F_M: FM \to M$ $(\pi^H_{GM}: HGM \to GM)$. If M is an n-manifold, we denote by $\mathscr{F}M(\mathscr{H}\mathscr{G}M)$ the set of section of $FM \to M$ $(HGM \to GM)$. If φ is an embedding of an n-manifold M into an n-manifold N, we define $\varphi_*: \mathscr{F}M \to \mathscr{F}N$ and $(G\varphi)_*: \mathscr{H}\mathscr{G}N \to \mathscr{H}\mathscr{G}N$ by $\varphi_*\sigma = F\varphi \circ \sigma \circ \varphi^{-1}$ and $(G\varphi)_*\,\varrho = (HG\varphi) \circ \varrho \circ (G\varphi)^{-1}$. With each n-manifold M we associate a mapping $L(M): \mathscr{F}M \to \mathscr{H}\mathscr{G}M$, which is natural for embeddings. That is to say, for each embedding φ of an n-manifold M into an n-manifold N, we have $L(N) \circ \varphi_* = (G\varphi)_* \circ L(M)$.

A family $L = \{L(M)\}$ is called an (n, F, G, H)-lifting.

Examples. (1) Let F and H be two natural bundles over n-manifolds. Let G be the identity functor over n-manifolds. Let $D = \{D(M)\}$ be a natural differential operator ([6]) such that for each n-manifold M, $D(M): \mathscr{F}M \to \mathscr{H}M$. Then D is an (n, F, G, H)-lifting. In particular, if F is the functor of positive-defined symmetric $(0, 2)$-tensors and H is the functor of (p, q)-tensors, then D is called a natural tensor ([1]). Hence natural tensors are liftings.

(2) Let F be the functor of tangent bundles (or $(0,0)$-tensors) over n-manifolds. Let G be a natural bundle over n-manifolds. Let H be the functor of tangent bundles (or $(0, 0)$-tensors) over dim (GR^n)-manifolds. Let $L = \{L(M)\}$ be a lifting of vector fields to G (or a lifting of functions to G) (see [2], [3]). Then L is an (n, F, G, H)-lifting.

The main theorem of this paper reads as follows.

Theorem. *Let L be an (n, F, G, H)-lifting. Let M be an n-manifold and $\sigma \in \mathscr{F}M$ a section defined on a neighbourhood of $x \in M$ and satisfying the following condition:*

(*) There exists a vector field X defined on a neighbourhood of x such that $X(x) \neq 0$ and $j_x^\infty(L_X\sigma) = j_x^\infty(0)$. Moreover, let $X(x) \neq 0$ and $j_x^\infty(L_X\sigma) = j_x^\infty(0)$.

Let $\sigma_m \in \mathscr{F}M$ $(m = 1, 2, 3, \ldots)$ be a sequence of sections such that $j_x^\infty(\sigma_m)$ tend to $j_x^\infty(\sigma)$ if m tends to infinity. Let $v_m \in (\pi_M^G)^{-1}(x)$ $(m = 1, 2, 3, \ldots)$ be a sequence of points tending to v. Then $L(M)\,\sigma_m(v_m)$ tend to $L(M)\,\sigma(v)$.

Remark. $L_X\sigma$ is the Lie derivative of σ with respect to X. If $y \in \operatorname{dom}(X) \cap \cap \operatorname{dom}(\sigma)$, then $L_X\,\sigma(y)$ is the vector from $T_{\sigma(y)}FM$ given by the curve $t \to (\varphi_{-t})_* \,.\, \sigma(y)$, where $\{\varphi_t\}$ is a local 1-parameter group of X.

If φ is an embedding of an n-manifold M into an n-manifold N, then $\varphi_*(L_X\sigma) = L_{\varphi_* x}\varphi_*\sigma$ (see [6]). We denote by 0 the mapping given by $M \ni y \to 0 \in T_{\sigma(y)}FM$.

Remark. The counterexample of D. B. A. Epstein [1, p. 638–641] shows why we insist that σ should satisfy (*).

From now on, we denote by π the given map from GR^n to R^n. We write F_0 instead of $(\pi_{R^n}^F)^{-1}(0)$ and G_0 instead of $\pi^{-1}(0)$. Let $s = \dim(F_0)$. If $x \in R^n$, we denote by τ_x the translation by $x(\tau_x\colon R^n \to R^n, \tau_x(y) = x + y)$. We have the C^∞-diffeomorphism $T\colon R^n \times F_0 \to FR^n$ given by $(x, f) \to F\,\tau_x(f)$. We write L instead of $L(R^n)$. We denote by P the projection $R^n \times F_0 \to F_0$, and by $p\colon R^n \to R$ the projection $(x_1, \ldots, x_n) \to x_1$.

We prove two lemmas.

Lemma 1. *Let $\sigma_1, \sigma_2 \in \mathscr{F}R^n$ be two sections such that $0 \in \operatorname{dom}(\sigma_t)$ $(t = 1, 2)$ and $j_0^\infty(\sigma_1) = j_0^\infty(\sigma_2)$. Then $L\sigma_1$ is equal to $L\sigma_2$ on G_0.*

Proof. Choose a chart (U, ψ) on F_0 such that $P \circ T^{-1} \circ \sigma_0(0) \in U$. Putting $f_t = \psi \circ P \circ T^{-1} \circ \sigma_t$ $(t = 1, 2)$ we find that $j_0^\infty(f_1) = j_0^\infty(f_2)$. By Whitney's extension theorem [5] there exist a C^∞-mapping $f\colon R^n \to R^s$ and an open neighbourhood W of 0 such that $f = f_t$ on $V_t = \{(x_1, \ldots, x_n) \in \overline{W}\colon (-1)^t x_1 \geqq n|x_i| \text{ for } 2 \leqq i \leqq n\}$ for $t = 1, 2$. Let $\bar\sigma \in \mathscr{F}R^n$ be given by $\bar\sigma(x) = T(x, \psi^{-1} \circ f(x))$. Then $\bar\sigma = \sigma_t$ on V_t for $t = 1, 2$. Hence $L\bar\sigma = L\sigma_t$ on $\pi^{-1}(\operatorname{int} V_t)$ for $t = 1, 2$. Since $G_0 \subset \operatorname{cl}(\pi^{-1}(\operatorname{int} V_t))$ we obtain that $L\sigma_1 = L\sigma_2$ on G_0.

Lemma 1 is proved.

Lemma 2. *Let $\sigma \in \mathscr{F}R^n$ be a section such that $0 \in \operatorname{dom}(\sigma)$ and $j_0^\infty(L_{\partial/\partial x_1}\sigma) = j_0^\infty(0)$. Then there exist a section $\tilde\sigma \in \mathscr{F}R^n$ and a chart (U, φ) on F_0 such that $\tilde\sigma(0) \in U$, $j_0^\infty(\tilde\sigma) = j_0^\infty(\sigma)$ and $\partial/\partial x_1 \tilde f \equiv 0$, where $\tilde f = \varphi \circ P \circ T^{-1} \circ \tilde\sigma$.*

Proof. Choose a chart (U, φ) on F_0 such that $\sigma(0) \in U$. Let $\psi = (T \circ (\mathrm{id}_{R^n} \times \varphi^{-1}))^{-1}$. Putting $f = \varphi \circ P \circ T^{-1} \circ \sigma$, we find $\varepsilon > 0$ such that