ON CERTAIN CLASSES OF p-VALENT FUNCTIONS

SHIGEYOSHI OWA, Osaka

(Received September 10, 1984)

Summary. We introduce the classes $T^*(n + p - 1, \alpha)$ of analytic p-valent functions with negative coefficients by using the symbol $D^{n+p-1} f(z)$ defined by $\{z^p/(1 - z)^{n+p}\} * f(z)$. The object of the present paper is to show distortion theorems and some closure theorems for functions $f(z)$ belonging to the class $T^*(n + p - 1, \alpha)$. Further, we consider the modified Hadamard product of functions $f(z)$ in $T^*(n + p - 1, \alpha)$.

Keywords: analytic function, p-valent function, distortion theorem, closure theorem, Hadamard product.

AMS (MOS) subject classification (1980): 30C45.

1. INTRODUCTION

Let $A(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in N = \{1, 2, 3, \ldots\}) \tag{1.1}$$

which are analytic in the unit disk $U = \{z: |z| < 1\}$. Further, let $f(z) \in A(p)$ and $g(z) \in A(p)$. Then we denote by $f * g(z)$ the Hadamard product of $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k} \quad (p \in N), \tag{1.2}$$

then

$$f * g(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k}. \tag{1.3}$$

With the aid of the Hadamard product, we introduce the symbol

$$D^{n+p-1} f(z) = \left\{\frac{z^p}{(1 - z)^{n+p}}\right\} * f(z) \tag{1.4}$$

for n, where n is any integer greater than $-p$. Then we can observe that

$$D^{n+p-1} f(z) = \frac{z^p (z^{n-1} f(z))^{(n+p-1)}}{(n + p - 1)!}. \tag{1.5}$$

In particular, the symbol $D^n f(z)$ was introduced by Ruscheweyh [7] and was named the nth order Ruscheweyh derivative of $f(z)$ by Al-Amiri [1].

Now, we introduce the following classes by using the symbol $D^{n+p-1} f(z)$.

Definition. We say that $f(z)$ *is in the class* $T^*(n + p - 1, \alpha)$, *if* $f(z)$ *defined by*

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (a_{p+k} \geqq 0) \tag{1.6}$$

satisfies the condition

$$Re \left\{ \frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} \right\} > \alpha \quad (z \in U) \tag{1.7}$$

for $p \in N$, $n > -p$ and $0 \leqq \alpha \leqq \frac{1}{2}$.

Particularly, $T^*(n + p - 1, \alpha)$ is a subclass of $T(n, \alpha)$ which was studied by Goel and Sohi [3]. Further, by using the symbol $D^{n+p-1} f(z)$, other classes were studied by Goel and Sohi [4], [5], Sohi [8], Fukui and Sakaguchi [2] and Owa [6].

2. DISTORTION THEOREMS

Theorem 1. *Let the function* $f(z)$ *be defined by* (1.6). *Then* $f(z)$ *is in the class* $T^*(n + p - 1, \alpha)$ *if and only if*

$$\sum_{k=1}^{\infty} \frac{(n + p + k - 1)! \{(n + p)(1 - \alpha) + k\}}{k!} a_{p+k} \leqq (n + p)! (1 - \alpha) . \tag{2.1}$$

Equality holds for the function $f(z)$ *given by*

$$f(z) = z^p - \frac{1 - \alpha}{(n + p)(1 - \alpha) + 1} z^{p+1} . \tag{2.2}$$

Proof. Suppose that the inequality (2.1) holds and let $|z| = 1$. Then we obtain

$$\left| \frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} - 1 \right| = \tag{2.3}$$

$$= \left| \frac{\sum_{k=1}^{\infty} (n + p + k - 1)(n + p + k - 2) \ldots k a_{p+k} z^k}{(n + p)! - (n + p) \sum_{k=1}^{\infty} (n + p + k - 1)(n + p + k - 2) \ldots (k + 1) a_{p+k} z^k} \right| \leqq$$

$$\leqq \frac{\sum_{k=1}^{\infty} (n + p + k - 1)(n + p + k - 2) \ldots k a_{p+k} |z|^k}{(n + p)! - (n + p) \sum_{k=1}^{\infty} (n + p + k - 1)(n + p + k - 2) \ldots (k + 1) a_{p+k} |z|^k} \leqq$$

$$\leqq \frac{\sum_{k=1}^{\infty}(n+p+k-1)(n+p+k-2)\dots k a_{p+k}}{(n+p)! - (n+p)\sum_{k=1}^{\infty}(n+p+k-1)(n+p+k-2)\dots(k+1)\,a_{p+k}} \leqq$$

$$\leqq 1-\alpha\,.$$

Hence we can see that the values of $D^{n+p}f(z)/D^{n+p-1}f(z)$ lie in a circle centered at $w = 1$ whose radius is $(1-\alpha)$. Consequently, we can observe that the function $f(z)$ satisfies the condition (1.7), hence $f(z)$ belongs to the class $T^*(n+p-1,\alpha)$.

For the converse, suppose that the function $f(z)$ belongs to the class $T^*(n+p-1,\alpha)$. Then we get

$$\text{(2.4)}\qquad Re\left\{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}\right\} =$$

$$= Re\left\{\frac{(n+p)! - \sum_{k=1}^{\infty}(n+p+k)(n+p+k-1)\dots(k+1)\,a_{p+k}z^k}{(n+p)! - (n+p)\sum_{k=1}^{\infty}(n+p+k-1)(n+p+k-2)\dots(k+1)\,a_{p+k}z^k}\right\}$$

$$> \alpha$$

for $z \in U$. Choose values of z on the real axis so that $D^{n+p}f(z)/D^{n+p-1}f(z)$ is real. Clearing the denominator in (2.4) and letting $z \to 1^-$ through real values, we have

$$\text{(2.5)}\qquad (n+p)! - \sum_{k=1}^{\infty}(n+p+k)(n+p+k-1)\dots(k+1)\,a_{p+k} \geqq$$

$$\geqq \alpha\{(n+p)! - (n+p)\sum_{k=1}^{\infty}(n+p+k-1)(n+p+k-2)\dots(k+1)\,a_{p+k}\}$$

which implies (2.1).

Finally, we can see that the function $f(z)$ given by (2.2) is an extreme one for the theorem. This completes the proof of the theorem.

Corollary 1. *Let the function $f(z)$ defined by (1.6) be in the class $T^*(n+p-1,\alpha)$. Then*

$$\text{(2.6)}\qquad a_{p+k} \leqq \frac{k!\,(n+p)!\,(1-\alpha)}{(n+p+k-1)!\,\{(n+p)(1-\alpha)+k\}}$$

for $k \geqq 1$. The equality holds for the function $f(z)$ of the form

$$\text{(2.7)}\qquad f(z) = z^p - \frac{k!\,(n+p)!\,(1-\alpha)}{(n+p+k-1)!\,\{(n+p)(1-\alpha)+k\}}\,z^{p+k}\,.$$

Theorem 2. *Let the function $f(z)$ defined by (1.6) be in the class $T^*(n+p-1,\alpha)$.*

Then

$$(2.8)\qquad |f(z)| \geqq |z|^p - \frac{1-\alpha}{(n+p)(1-\alpha)+1}|z|^{p+1}$$

and

$$|f(z)| \leqq |z|^p + \frac{1-\alpha}{(n+p)(1-\alpha)+1}|z|^{p+1}$$

for $z \in U$. The results are sharp.

Proof. Since $f(z)$ belongs to the class $T^*(n+p-1, \alpha)$, in view of Theorem 1 we obtain

$$(2.10)\qquad (n+p)!\{(n+p)(1-\alpha)+1\}\sum_{k=1}^{\infty} a_{p+k} \leqq$$

$$\leqq \sum_{k=1}^{\infty} \frac{(n+p+k-1)!\{(n+p)(1-\alpha)+k\}}{k!} a_{p+k} \leqq$$

$$\leqq (n+p)!(1-\alpha)$$

which gives

$$(2.11)\qquad \sum_{k=1}^{\infty} a_{p+k} \leqq \frac{1-\alpha}{(n+p)(1-\alpha)+1}.$$

Consequently, we see that

$$(2.12)\qquad |f(z)| \geqq |z|^p - |z|^{p+1}\sum_{k=1}^{\infty} a_{p+k} \geqq$$

$$\geqq |z|^p - \frac{1-\alpha}{(n+p)(1-\alpha)+1}|z|^{p+1}$$

and

$$(2.13)\qquad |f(z)| \leqq |z|^p + |z|^{p+1}\sum_{k=1}^{\infty} a_{p+k} \leqq$$

$$\leqq |z|^p + \frac{1-\alpha}{(n+p)(1-\alpha)+1}|z|^{p+1}$$

for $z \in U$.

Further, by taking the function $f(z)$ given by

$$(2.14)\qquad f(z) = z^p - \frac{1-\alpha}{(n+p)(1-\alpha)+1}z^{p+1}$$

we can observe that the results of the theorem are sharp.

Corollary 2. *Let the function $f(z)$ defined by (1.6) be in the class $T^*(n+p-1, \alpha)$.*

Then $f(z)$ is included in a disk with its center at the origin and radius r given by

$$r = \frac{(n + p + 1)(1 - \alpha) + 1}{(n + p)(1 - \alpha) + 1}. \tag{2.15}$$

Theorem 3. *Let the function $f(z)$ defined by (1.6) be in the class $T^*(n + p - 1, \alpha)$. Then*

$$|f'(z)| \geqq p|z|^{p-1} - \frac{(p + 1)(1 - \alpha)}{(n + p)(1 - \alpha) + 1} |z|^p \tag{2.16}$$

and

$$|f'(z)| \leqq p|z|^{p-1} + \frac{(p + 1)(1 - \alpha)}{(n + p)(1 - \alpha) + 1} |z|^p \tag{2.17}$$

for $z \in U$. The results are sharp.

Proof. Since $f(z)$ is in the class $T^*(n + p - 1, \alpha)$, by virtue of Theorem 1 we have

$$\begin{aligned} \frac{(n + p)!\,\{(n + p)(1 - \alpha) + 1\}}{p + 1} \sum_{k=1}^{\infty} (p + k)\, a_{p+k} \leqq \\ \leqq \sum_{k=1}^{\infty} \frac{(n + p + k - 1)!\,\{(n + p)(1 - \alpha) + k\}}{k!}\, a_{p+k} \leqq \\ \leqq (n + p)!\,(1 - \alpha) \end{aligned} \tag{2.18}$$

which implies

$$\sum_{k=1}^{\infty} (p + k)\, a_{p+k} \leqq \frac{(p + 1)(1 - \alpha)}{(n + p)(1 - \alpha) + 1}. \tag{2.19}$$

Hence, by using (2.19), we obtain

$$\begin{aligned} |f'(z)| &\geqq p|z|^{p-1} - |z|^p \sum_{k=1}^{\infty} (p + k)\, a_{p+k} \geqq \\ &\geqq p|z|^{p-1} - \frac{(p + 1)(1 - \alpha)}{(n + p)(1 - \alpha) + 1} |z|^p \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} |f'(z)| &\leqq p|z|^{p-1} + |z|^p \sum_{k=1}^{\infty} (p + k)\, a_{p+k} \leqq \\ &\leqq p|z|^{p-1} + \frac{(p + 1)(1 - \alpha)}{(n + p)(1 - \alpha) + 1} |z|^p \end{aligned} \tag{2.21}$$

for $z \in U$.

Further, we can see that the results of the theorem are sharp for the function $f(z)$ given by (2.14).

Corollary 3. *Let the function $f(z)$ defined by (1.6) be in the class $T^*(n + p - 1, \alpha)$. Then $f'(z)$ is included in a disk with its center at the origin and radius R given by*

$$R = \frac{(np + p^2 + p + 1)(1 - \alpha) + p}{(n + p)(1 - \alpha) + 1}. \tag{2.22}$$

3. CLOSURE THEOREMS

Theorem 4. *Let the functions*

$$f_i(z) = z^p - \sum_{k=1}^{\infty} a_{p+k,i} z^{p+k} \qquad (a_{p+k,i} \geqq 0) \tag{3.1}$$

be in the class $T^*(n + p - 1, \alpha)$ for every $i = 1, 2, 3, \ldots, m$. Then the function $h(z)$ defined by

$$h(z) = \sum_{i=1}^{m} c_i f_i(z) \qquad (c_i \geqq 0) \tag{3.2}$$

is also in the same class $T^*(n + p - 1, \alpha)$, where

$$\sum_{i=1}^{m} c_i = 1\,. \tag{3.3}$$

Proof. By means of the definition of $h(z)$, we can write

$$h(z) = z^p - \sum_{k=1}^{\infty} \Big(\sum_{i=1}^{m} c_i a_{p+k,i} \Big) z^{p+k}\,. \tag{3.4}$$

Now, since $f_i(z) \in T^*(n + p - 1, \alpha)$ for every $i = 1, 2, 3, \ldots, m$, we obtain

$$\sum_{k=1}^{\infty} \frac{(n + p + k - 1)!\,(n + p)(1 - \alpha) + k\}}{k!}\, a_{p+k,i} \leqq (n + p)!\,(1 - \alpha) \tag{3.5}$$

for every $i = 1, 2, 3, \ldots, m$, by virtue of Theorem 1. Consequently, with the aid of (3.5) we can see that

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(n + p + k - 1)!\,\{(n + p)(1 - \alpha) + k\}}{k!} \Big(\sum_{i=1}^{m} c_i a_{p+k,i} \Big) = \\ = \sum_{i=1}^{m} c_i \left\{ \sum_{k=1}^{\infty} \frac{(n + p + k - 1)!\,\{(n + p)(1 - \alpha) + k\}}{k!}\, a_{p+k,i} \right\} \leqq \\ \leqq \Big(\sum_{i=1}^{m} c_i \Big)(n + p)!\,(1 - \alpha) = (n + p)!\,(1 - \alpha)\,. \end{aligned} \tag{3.6}$$

This proves that the function $h(z)$ belongs to the class $T^*(n + p - 1, \alpha)$.

Theorem 5. *Let*

$$f_p(z) = z^p \tag{3.7}$$

and

$$f_{p+k}(z) = z^p - \frac{k!\,(n+p)!\,(1-\alpha)}{(n+p+k-1)!\,\{(n+p)(1-\alpha)+k\}}\,z^{p+k} \tag{3.8}$$

for $p \in N$, $n > -p$, $k \in N$ *and* $0 \leqq \alpha \leqq 1/2$. *Then* $f(z)$ *is in the class* $T^*(n + p - 1, \alpha)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=0}^{\infty} \lambda_{p+k} f_{p+k}(z)\,, \tag{3.9}$$

where $\lambda_{p+k} \geqq 0$ *and*

$$\sum_{k=0}^{\infty} \lambda_{p+k} = 1\,. \tag{3.10}$$

Proof. Assume that

$$f(z) = \sum_{k=0}^{\infty} \lambda_{p+k} f_{p+k}(z) = \tag{3.11}$$

$$= z^p - \sum_{k=1}^{\infty} \frac{k!\,(n+p)!\,(1-\alpha)}{(n+p+k-1)!\,\{(n+p)(1-\alpha)+k\}}\,\lambda_{p+k} z^{p+k}\,.$$

Then we get

$$\sum_{k=1}^{\infty} \left\{ \frac{(n+p+k-1)!\,\{(n+p)(1-\alpha)+k\}}{k!} \right. \tag{3.12}$$

$$\left. X \frac{k!\,(n+p)!\,(1-\alpha)}{(n+p+k-1)!\,\{(n+p)(1-\alpha)+k\}}\,\lambda_{p+k} \right\} \leqq (n+p)!\,(1-\alpha)\,.$$

By virtue of Theorem 1 this shows that $f(z)$ is in the class $T^*(n + p - 1, \alpha)$.

Conversely, assume that $f(z)$ belongs to the class $T^*(n + p - 1, \alpha)$. Again, by virtue of Theorem 1, we have

$$a_{p+k} \leqq \frac{k!\,(n+p)!\,(1-\alpha)}{(n+p+k-1)!\,\{(n+p)(1-\alpha)+k\}} \tag{3.13}$$

for $p \in N$, $n > -p$, $k \in N$ and $0 \leqq \alpha \leqq 1/2$. Next, setting

$$\lambda_{p+k} = \frac{(n+p+k-1)!\,\{(n+p)(1-\alpha)+k\}}{k!\,(n+p)!\,(1-\alpha)}\,a_{p+k} \tag{3.14}$$

for $p \in N$, $n > -p$, $k \in N$ and $0 \leqq \alpha \leqq 1/2$, and

$$\lambda_p = 1 - \sum_{k=1}^{\infty} \lambda_{p+k}\,, \tag{3.15}$$

we have the representation (3.9). This completes the proof of the theorem.