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HEREDITARY RADICAL CLASSES OF LINEARLY ORDERED GROUPS

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The study of radical classes and semisimple classes of linearly ordered groups was begun by Chehata and Wiegandt [1]. The basic properties of the lattice $\mathscr{R}$ of all radical classes of linearly ordered groups were described in [3]; for analogous questions concerning semisimple classes cf. [4]. In the papers [5], [7] and [8] radical classes and semisimple classes of abelian linearly ordered groups were dealt with.

In [3] and [4] it was proved that the lattice $\mathscr{R}$ has no atoms, no antiatoms and fails to be modular.

A radical class $X \in \mathscr{R}$ is said to be hereditary if, whenever $G \in X$ and H is a convex subgroup of G, then $H \in X$. The collection of all hereditary radical classes will be denoted by $\mathscr{R}_h$.

In this note it will be shown that $\mathscr{R}_h$ (partially ordered by inclusion) is a complete distributive lattice. In fact, $\mathscr{R}_h$ fulfils the infinite distributive law

$$A \wedge (\bigvee B_i) = \bigvee (A \wedge B_i),$$

hence $\mathscr{R}_h$ is a Brouwer lattice. The corresponding dual infinite distributive law does not hold in $\mathscr{R}_h$. Further, it will be proved that $\mathscr{R}_h$ has infinitely many atoms and that the collection $\mathscr{P}$ of all prime intervals of the lattice $\mathscr{R}_h$ is a proper collection. Thus some properties of the lattice $\mathscr{R}_h$ are analogous to those of the lattice of all radical classes of l-groups [2] or the lattice of all torsion classes of l-groups (cf. Martinez [6]).

The collection of all principal elements of $\mathscr{R}_h$ will be denoted by $\mathscr{R}_{hp}$. It will be shown that if $X \in \mathscr{R}_h$, $Y \in \mathscr{R}_{hp}$ and $X \leqq Y$, then $X \in \mathscr{R}_{hp}$. If $I \neq \emptyset$ is a set and $\{X_i\}_{i \in I} \subset \mathscr{R}_{hp}$, then $\bigvee_{i \in I} X_i$ belongs to $\mathscr{R}_{hp}$ as well. (Let us remark that analogous results do not hold for principal elements of the lattice of all radical classes of abelian linearly ordered groups; cf. [5].)

1. BASIC NOTIONS

A collection X will be said to be propre if there exists a one-to-one mapping of the class of all cardinals into X.

The group operation in a linearly ordered group will be denoted by $+$; the commutativity of this operation is not assumed. We recall some definitions; cf. [1].

Let $\mathscr{G}$ be the class of all linearly ordered groups. When considering a subclass X of $\mathscr{G}$ we always suppose that X is closed with respect to isomorphisms and that the zero linearly ordered group $\{0\}$ belongs to X.

A subclass X of $\mathscr{G}$ is said to be closed with respect to transfinite extensions if, whenever $G \in \mathscr{G}$ and

$$\{0\} = G_1 \subseteq G_2 \subseteq \ldots \subseteq G_\alpha \subseteq \ldots \quad (\alpha < \delta)$$

is an ascending chain of convex normal subgroups of G such that

$$G_\beta / \bigcup_{\gamma < \beta} G_\gamma \in X \quad \text{for each} \quad \beta < \delta\,,$$

then $\bigcup_{\alpha<\delta} G_\alpha$ belongs to X.

We also say that the linearly ordered group $\bigcup_{\alpha<\delta} G_\alpha$ is a transfinite extension of linearly ordered groups $G'_\beta(\mathrm{b} < \delta)$, where G'_β is isomorphic to $G_\beta / \bigcup_{\gamma<\beta} G_\gamma$ for each $\beta < \delta$.

1.1. Definition. *A class X of linearly ordered groups is called a radical class, if*

(a) *X is closed under homomorphisms, and*

(b) *X is closed with respect to transfinite extensions.*

We denote by $\mathscr{R}$ the collection of all radical classes. Further, let $\mathscr{R}_h$ be the collection of all hereditary radical classes. Both $\mathscr{R}$ and $\mathscr{R}_h$ are partially ordered by inclusion. Then $\mathscr{G}$ is the greatest element in both $\mathscr{R}$ and $\mathscr{R}_h$; the trivial variety R_0 containing all one-element l-groups is the least element in both $\mathscr{R}$ and $\mathscr{R}_h$.

If $\{A_i\}_{i \in I}$ is a non-empty collection of hereditary radical classes, then $\bigcap_{i \in I} A_i$ also is a hereditary radical class. Thus $\mathscr{R}_h$ is a complete lattice. The lattice operations in $\mathscr{R}_h$ will be denoted by $\wedge$ and $\vee$. The operation $\wedge$ in $\mathscr{R}_h$ coincides with the intersection of classes.

Let $Y \subseteq \mathscr{G}$ and $G \in \mathscr{G}$. The intersection of all hereditary radical classes X with $Y \subseteq X$ will be denoted by $T_h(X)$. Similarly, the intersection of all hereditary radical classes Z with $G \in Z$ is denoted by $T_h(G)$; the hereditary radical class $T_h(G)$ is said to be principal. We denote by $\mathscr{R}_{hp}$ the collection of all principal hereditary radical classes.

2. THE OPERATION $\vee$ IN THE LATTICE $\mathscr{R}_h$

Let X be a subclass of $\mathscr{G}$. We denote by

Hom X — the class of all homomorphic images of linearly ordered groups belonging to X;

Sub X — the class of all convex subgroups of linearly ordered groups belonging to X;

Ext X — the class of all transfinite extensions of linearly ordered groups belonging to X.

Now we define for each ordinal $\varkappa$ the class $\text{Ext}_\varkappa X$ by induction as follows. We put $\text{Ext}_1 X = \text{Ext}\, X$; if $\varkappa > 1$, then we set

$$\text{Ext}_\varkappa X = \text{Ext} \bigcup_{\tau<\varkappa} \text{Ext}_\tau X .$$

Next we denote

$$\text{ext}\, X = \bigcup_\varkappa \text{Ext}_\varkappa X ,$$

where $\varkappa$ runs over the class of all ordinals.

2.1. Theorem. *Let X be a subclass of $\mathscr{G}$. Then $T_h(X) = \text{ext Hom Sub}\, X$.*

Proof. Denote ext Hom Sub $X = Z$. Clearly $Z \subseteq T_h(X)$ and $X \subseteq Z$. Hence it suffices to prove that Z is a hereditary radical class. Thus we have to verify that Z fulfils the following conditions: (i) Ext $Z \subseteq Z$, (ii) Sub $Z \subseteq Z$; (iii) Hom $Z \subseteq Z$.

For each subclass Z_1 of $\mathscr{G}$ we have Ext ext Z_1 = ext Z_1, hence (i) is valid. In [3] (Lemma 2.1) it was proved that for each subclass Z_2 of $\mathscr{G}$ the relation

$$\text{Hom ext Hom}\, Z_2 = \text{ext Hom}\, Z_2$$

holds; therefore (iii) holds as well.

Let $G \in Z$ and let H be a convex subgroup of G with $H \subset G$. Hence there is an ordinal $\varkappa$ such that $G \in \text{Ext}_\varkappa \text{Hom Sub}\, X$. Thus it suffices to verify that for each ordinal $\varkappa$ we have

$$\text{Sub Ext}_\varkappa \text{Hom Sub}\, X \subseteq \text{Ext}_\varkappa \text{Hom Sub}\, X . \tag{1}$$

a) Let $\varkappa = 1$. There is an ascending chain of convex normal subgroups

$$\{0\} = G_1 \subseteq G_2 \subseteq \ldots \subseteq G_\alpha \subseteq \ldots \quad (x < \delta) \tag{2}$$

of G such that

$$\bigcup_{\alpha<\delta} G_\alpha = G \tag{3}$$

and for each $\beta < \delta$, $G_\beta/\bigcup_{\gamma<\beta} G_\gamma \in \text{Hom Sub}\, X$. Let λ be the first ordinal with $\lambda < \delta$ and $G_\lambda \supseteq H$. Denote $H_\alpha = H \cap G_\alpha$ for each $\alpha < \delta$. Then $\{H_\alpha\}$ $(\alpha < \delta)$ is an ascending chain of convex normal subgroups of H and $\bigcup_{\alpha<\delta} H_\alpha = H$. If $\beta < \lambda$, then

$$G_\beta/\bigcup_{\gamma<\beta} G_\gamma = H_\beta/\bigcup_{\gamma<\beta} H_\gamma ;$$

if $\beta > \lambda$, then $H_\beta/\bigcup_{\gamma<\beta} H_\gamma = \{0\}$. In the case $\beta = \lambda$ we have

$$H_\beta/\bigcup_{\gamma<\beta} H_\gamma \in \text{Sub}\, \{G_\beta/\bigcup_{\gamma<\beta} G_\gamma\} \subseteq \text{Sub Hom Sub}\, X = \text{Hom Sub}\, X ,$$

thus for $\varkappa = 1$ the relation (1) holds. (We use the well-known relation Sub Hom $Y \subseteq$ $\subseteq$ Hom Sub Y which is valid for each $Y \subseteq \mathscr{G}$.)

b) Assume that $\varkappa > 1$ and that (1) holds for each ordinal less than $\varkappa$. Then there is an ascending chain of convex normal subgroups (2) of G such that (3) is valid and for each $\beta < \delta$ there is an ordinal $\tau(\beta) < \varkappa$ having the property

$$G_\beta / \bigcup_{\gamma<\beta} G_\gamma \in \operatorname{Ext}_{\tau(\beta)} \operatorname{Hom} \operatorname{Sub} X .$$

Let λ and H_α ($\alpha < \gamma$) be as in part a). The cases $b < \lambda$ and $b > \lambda$ are analogous as in a). Let $b = \lambda$. Then

$$H_\beta / \bigcup_{\gamma<\beta} H_\gamma \in \operatorname{Sub} \{G_\beta / \bigcup_{\gamma<\beta} G_\gamma\} \subseteq \operatorname{Sub} \operatorname{Ext}_{\tau(\beta)} \operatorname{Hom} \operatorname{Sub} X =$$
$$= \operatorname{Ext}_{\tau(\beta)} \operatorname{Hom} \operatorname{Sub} X ,$$

hence (1) is valid for each ordinal $\varkappa$, which completes the proof.

2.2. Theorem. *Let I be a nonempty class and for each $i \in I$ let X_i be a hereditary radical class. Then $\bigvee_{i\in I} X_i = \operatorname{ext} \bigcup_{i\in I} X_i$.*

Proof. From 2.1 it follows immediately that the relation

$$\bigvee_{i\in I} X_i = \operatorname{ext} \operatorname{Hom} \operatorname{Sub} \bigcup_{i\in I} X_i$$

is valid. Since X_i are hereditary radical classes, we have $\operatorname{Hom} \operatorname{Sub} X_i = X_i$, therefore $\bigvee_{i\in I} X_i = \operatorname{ext} \bigcup_{i\in I} X_i$.

From 2.2 and [3] (Thm. 2.3) we obtain:

2.2.1. Corollary. R_h *is a closed sublattice of the complete lattice* $\mathscr{R}$.

2.3. Theorem. *Let $A \in \mathscr{R}_h$, $\{B_i\}_{i\in I} \subseteq \mathscr{R}_h$. Then*

$$A \wedge \left(\bigvee_{i\in I} B_i\right) = \bigvee_{i\in I} (A \wedge B_i) .$$

Proof. It suffices to verify that $A \wedge (\bigvee_{i\in I} B_i) \leqq \bigvee_{i\in I} (A \wedge B_i)$. Let $G \in A \wedge (\bigvee_{i\in I} B_i)$. Hence $G \in A$ and $G \in \bigvee_{i\in I} B_i$. In view of 2.2, $G \in \operatorname{ext} \bigcup_{i\in I} B_i$. Thus G is constructed by the operation ext from certain linearly ordered groups G_{ij} ($i \in I$, $j \in K_i$) such that G_i belongs to B_i for each $i \in I$ and each $j \in K_i$.

According to the definition of ext, for each G_{ij} there exists a normal convex subgroup H_{ij} of G and a homomorphic image G'_{ij} of H_{ij} such that G'_{ij} is somorphic to G_{ij}. Because A is hereditary the linearly ordered group H_{ij} belongs to A and hence $G_{ij} \in A$. Thus $G_{ij} \in A \wedge B_i$ for each $i \in I$ and each $j \in K_i$. Therefore $G \in \operatorname{ext} \bigcup_{i\in I} . (A \wedge B_i) = \bigvee_{i\in I} (A \wedge B_i)$.

The following example shows that the relation

$$A \vee \left(\bigwedge_{i\in I} B_i\right) = \bigwedge_{i\in I} (A \vee B_i)$$

does not hold in general in the lattice $\mathscr{R}_n$. (The symbols $\Gamma_{j\in J} G_j$ and $G_1 \circ G_2$ denote lexicographic products of linearly ordered groups; cf., e.g., [5].)

2.4. Example. Let N be the set of all positive integers with the natural linear order. Let J be the linearly ordered set dual to N and for each $j \in J$ let G_j be an archimedean linearly ordered group, $G_j \neq \{0\}$, such that $G_{j(1)}$ and $G_{j(2)}$ fail to be isomorphic whenever $j(1)$ and $j(2)$ are distinct elements of J. For each $j \in J$ let $J_j = \{k \in J : k \leqq j\}$ (with the induced linear order). Put

$$\begin{aligned} G &= \Gamma_{j \in J} G_j\,, \\ G_{(j)} &= \Gamma_{k \in J_j} G_k \quad \text{for each} \quad j \in J\,, \\ A &= \bigvee_{j \in J} T_h(G_j)\,, \\ B_j &= T_h(G_{(j)}) \quad \text{for each} \quad j \in J\,. \end{aligned}$$

Then we have $G \notin A$, $\bigwedge_{j \in J} B_j = R_0$, hence

$$A \vee (\bigwedge_{j \in J} B_j) = A$$

and thus $G \notin A \vee (\bigwedge_{j \in J} B_j)$.

On the other hand, $G \in A \vee B_j$ for each $j \in J$, hence

$$G \in \bigwedge_{j \in J} (A \vee B_j)$$

and therefore $A \vee (\bigwedge_{j \in J} B_j) \neq \bigwedge_{j \in J} (A \vee B_j)$.

2.5. Lemma. *Let $X \subseteq \mathscr{G}$, $H \in T_h(X)$, $H \neq \{0\}$. Then there exists a convex subgroup H_1 of H with $H_1 \neq \{0\}$ such that $H_1 \in$ Hom Sub X.*

Proof. In view of 2.1 we have $H \in$ ext Hom Sub X, hence there is an ordinal τ such that $H \in \text{Ext}_\tau$ Hom Sub X. Thus there is an ordinal $\varkappa < \tau$ having the property that there exists a convex subgroup H' of H with $H' \neq \{0\}$ such that $H' \in \text{Ext}_\varkappa$ Hom Sub X.

Now let χ be the first ordinal having the property that there is a convex subgroup H'' of H with $H'' \neq \{0\}$ such that $H'' \in \text{Ext}_\chi$ Hom Sub X. Assume that $\chi > 1$. Then there is $\chi' < \chi$ such that there exists a convex subgroup $H^* \neq \{0\}$ of H'' with $H^* \in \text{Ext}_{\chi'}$ Hom Sub X. Since H^* is a convex subgroup of H, we have arrived at a contradiction. Hence $\chi = 1$. Therefore there is a convex subgroup $H_1 \neq \{0\}$ of H'' such that $H_1 \in$ Hom Sub X, which completes the proof.

3. ATOMS IN $\mathscr{R}_h$

3.1. Proposition. *Let $G \in \mathscr{G}$, $G \neq \{0\}$. Assume that G is archimedean. Then $T_h(G)$ is an atom in the lattice $\mathscr{R}_h$.*

Proof. We have $R_0 < T_h(G)$. Let $A \in \mathscr{R}_h$, $R_0 < A \leqq T_h(G)$. There exists $H \in A$ with $H \neq \{0\}$. In view of 2.1 we have $T_h(G) =$ ext Hom Sub $\{G\}$. Since G is archimedean, Hom Sub $\{G\}$ is the class of all linearly ordered groups G' such that either $G' = \{0\}$ or G' is isomorphic to G. Hence H can be constructed by the operation ext from a system of linearly ordered groups G_i $(i \in I)$ such that each G_i is isomorphic to G. Let $i \in I$ be fixed. There exists a normal convex subgroup H_i of G and a homomorphic image G'_i of H_i such that G'_i is isomorphic to G_i. Since A is hereditary, we have $H_i \in A$ and thus $G'_i \in A$. Therefore $G \in A$ and hence $A = T_h(G)$.

Because there is an infinite set of mutually nonisomorphic archimedean linearly ordered groups, 3.1 implies:

3.2. Corollary. *The class of all atoms of the lattice $\mathscr{R}_h$ is infinite.*

3.3. Proposition. *Let $X \in \mathscr{R}_h$, $X \neq R_0$. Then there exists an archimedean linearly ordered group $H \neq \{0\}$ such that $T_h(H) \leqq X$.*

Proof. There exists $G \in X$ such that $G \neq \{0\}$. Choose $g \in G$, $g > 0$ and let $\mathscr{H} = \{H_i\}_{i \in I}$ be the set of all convex subgroups of G not containing the element g. Let H_1 be the convex subgroup of G generated by g. Because the set $\mathscr{H}$ is linearly ordered, $\mathscr{H}$ has a unique maximal element H_2. Then H_2 is the largest proper convex subgroup of H_1. Hence H_2 is a normal subgroup in H_1. Therefore $H = H_1/H_2$ is o-simple and thus it is archimedean. Clearly $H \neq \{0\}$. Now we have $T_h(H) = T_h(H_1/H_2) \leqq T_h(G) \leqq T_h(X)$.

From 3.1 and 3.3 we infer:

3.4. Theorem. *Let $X \in \mathscr{R}_h$. Then the following conditions are equivalent:*

(i) *X covers R_0 in the lattice $\mathscr{R}_h$.*

(ii) *There is an archimedean linearly ordered group $H \neq \{0\}$ such that $X = T_h(G)$.*

Let A_0 be a set of non-zero archimedean linearly ordered groups such that (a) if G_1 and G_2 are distinct elements of A_0, then G_1 is not isomorphic to G_2, and (b) for each non-zero archimedean linearly ordered group G there is G' in A_0 such that G is isomorphic to G'. Put

$$X_0 = \bigvee_{G \in A_0} T_h(G) .$$

A collection X will be said to be small if there exists a set Y and a mapping of Y onto X.

3.5. Proposition. *Let $\mathscr{G}_1 = [R_0, X_0]$ (the interval taken in $\mathscr{R}_h$). Then*

(i) *$\mathscr{G}_1$ is a small collection;*

(ii) *$\mathscr{G}_1$ is a complete atomic Boolean algebra; the collection of atoms of $\mathscr{G}_1$ is $\{T_h(G)\}_{G \in A_0}$.*

Proof. $\mathscr{G}_1$ is obviously a complete lattice and in view of 2.3, $\mathscr{G}_1$ is distributive. From 3.4 it follows that $A'_0 = \{T_h(G)\}_{G \in A_0}$ is the collection of all atoms of $\mathscr{G}_1$. Let $R_0 \neq X \in \mathscr{G}_1$ and let $X' = \{T_h(G) : G \in A_0 \cap X\}$. Then

$$X = X \wedge X_0 = X \wedge (\bigvee_{G \in A_0} T_h(G)) = \bigvee_{G \in A_0} (X \wedge T_h(G)) = \\ = \bigvee_{G \in A_0 \cap X} (X \wedge T_h(G)) = \sup X' .$$

Moreover, if $X'' \subseteq A'_0$ and $\sup X'' = X$, then 2.3 implies that $X' = X''$. Hence $\mathscr{G}_1$ is isomorphic to the Boolean algebra of all subsets of the set A'_0.

3.6. Lemma. *Let $X \in \mathscr{G}_1$, $X \neq R_0$. Let I be a linearly ordered set isomorphic to the set of all negative integers (with the natural linear order). Let $G = \Gamma_{i \in I} G_i$, where each G_i belongs to $A_0 \cap X$. Assume that for each $G' \in A_0 \cap X$ and each $j \in I$ there is $i \in I$ with $i < j$ such that G' is isomorphic to G_i. Then*

(i) *$T_h(G)$ covers X,*

(ii) *$T_h(G)$ does not belong to $\mathscr{G}_1$,*

(iii) *$T_h(G) \wedge T_h(G') = R_0$ whenever $G' \in A_0$ and $G' \notin X$.*

Proof. We apply the same notations as in the proof of 3.5. For each $G' \in A_0 \cap X$ we have $T_h(G') \leqq T_h(G)$, hence $X = \bigvee_{G' \in A_0 \cap X} T_h(G') \leqq T_h(G)$. In view of 2.5, $T_h(G)$ does not belong to $\mathscr{G}_1$ and thus $X < T_h(G)$. Let $Y \in \mathscr{R}_h$, $X < Y \leqq T_h(G)$. There exists $H \in Y \setminus X$. Hence $H \in T_h(G)$. According to Thm. 2.1, H can be constructed from a subset S of the class Hom Sub $\{G\}$ by the operation ext. Because H does not belong to X, the set S must contain a linearly ordered group isomorphic to $\Gamma_{i \in I, i < j} G_i$ for some $j \in I$. Then we have $G \in Y$, whence $Y = T_h(G)$ and so (i) is valid. (iii) is a consequence of 2.1 and 2.3.

For each $X \in R_h$ we denote by $a(X)$ the collection of all $Y \in \mathscr{R}_h$ such that Y covers X in the lattice $\mathscr{R}_h$.

From 3.6 we immediately obtain:

3.7. Corollary. *Let $X \in \mathscr{G}_1$, $X \neq R_0$. Then there exists $Y \in a(X) \cap \mathscr{R}_{hp}$ such that $Y \notin \mathscr{G}_1$.*

The proof of the following proposition will be omitted (it can be established by using similar arguments as in the proof of 3.6).

3.8. Proposition. *Let $X \in \mathscr{G}_1$, $X \neq R_0$. Let I be as in 3.6 and let $G = \Gamma_{i \in I} G_i$, where each G_i belongs to $A_0 \cap X$. Then the following conditions are equivalent:*

(i) *$T_h(G)$ covers X;*

(ii) *for each $G' \in A_0 \cap X$ and each $j \in I$ there is $i \in I$ such that $i < j$ and G' is isomorphic to G_i.*

4. PRINCIPAL ELEMENTS OF $\mathscr{R}_h$

4.1. Proposition. *Let $X, Y \in \mathscr{R}_h$, $X \leqq Y$. Assume that Y is a principal element of $\mathscr{R}_h$. Then X is principal as well.*

Proof. Let $Y = T_h(G)$. In view of 2.1, $Y =$ ext Hom Sub $\{G\}$. There exists a set $S = \{H_i\}_{i \in I}$ of linearly ordered groups such that $S \subset$ Hom Sub $\{G\}$ and for each

$G_1 \in \operatorname{Hom} \operatorname{Sub} \{G\}$ there is $i \in I$ such that G_1 is isomorphic to H_i. Hence $Y = \operatorname{ext} \{H_i\}_{i \in I}$ and $X \subseteq \operatorname{ext} \{H_i\}_{i \in I}$. Thus there is $\emptyset \neq J \subseteq I$ such that $X = \operatorname{ext} \{H_i\}_{i \in J}$. We can assume that J is well-ordered (by using the Axiom of Choice). Put $H = \Gamma_{i \in J} H_i$. Then $H_i \in T_h(H)$ holds for each $i \in J$, hence $X = \operatorname{ext} \{H_i\}_{i \in J} = \bigvee_{i \in J} T_h(H_i) \leqq T_h(H)$. On the other hand, $H \in \operatorname{Ext} \{H_i\}_{i \in J}$ and so $T_h(H) \leqq T_h(\{H_i\}_{i \in J}) = X$. Thus $X = T_h(H) \in \mathscr{R}_{hp}$.

4.2. Proposition. *Let I be a nonempty set and for each $i \in I$ let X_i be a principal element of $\mathscr{R}_h$. Then $X = \bigvee_{i \in I} X_i$ is a principal element of $\mathscr{R}_h$ as well.*

Proof. There are $G_i \in \mathscr{G}$ such that $X_i = T_h(G_i)$. We clearly have $X = T_h(\{G_i\}_{i \in I}) = \operatorname{ext} \operatorname{Hom} \operatorname{Sub} \{G_i\}_{i \in I}$. There is a set $S = \{H_j\}_{j \in J} \subset \mathscr{G}$ such that (i) $S \subset \operatorname{Hom} \operatorname{Sub} \{G_i\}_{i \in I}$, and (ii) for each $G_1 \in \operatorname{Hom} \operatorname{Sub} \{G_i\}_{i \in I}$ there is $j \in J$ having the property that G_1 is isomorphic to H_j. Again, we can assume that J is well-ordered. Put $H = \Gamma_{j \in J} H_j$. It is easy to verify that $X = T_h(H)$, hence X is principal.

Let α be a cardinal. We denote by $I(\alpha)$ the first ordinal having the property that the set of all ordinals less than $I(\alpha)$ has the cardinality α. Let $J(\alpha)$ be the linearly ordered set dual to $I(\alpha)$.

Let $G \in \mathscr{G}$, $G \neq \{0\}$. We put

$$G_{(\alpha)} = \Gamma_{j \in J(\alpha)} G_j ,$$

where each G_j is isomorphic to G.

4.3. Lemma. *Let $G \in \mathscr{G}$, $G \neq \{0\}$, $\alpha >$ card G. Then $T_h(G) < T_h(G_{(\alpha)})$.*

Proof. We have $G \in \operatorname{Hom} \{G_{(\alpha)}\}$, hence $T_h(G) \leqq T_h(G_{(\alpha)})$. In view of 2.5, $G_{(\alpha)} \notin T_h(G)$. Hence $T_h(G) < T_h(G_{(\alpha)})$.

4.4. Corollary. *The class $\mathscr{R}_{hp}$ has no maximal element. In particular, $\mathscr{G}$ does not belong to $\mathscr{R}_{hp}$.*

Let $G \in \mathscr{G}$, $G \neq \{0\}$. In view of 4.3 there is a least cardinal $\beta = \beta(G)$ such that $T_h(G) < T_h(G_{(\beta(G))})$.

The following proposition shows that there are many prime intervals in the lattice $\mathscr{R}_h$.

4.5. Proposition. *Let $G \in \mathscr{G}$, $G \neq \{0\}$. Then $T_h(G)$ is covered by $T_h(G_{\beta(G)})$ in the lattice $\mathscr{R}_h$.*

Proof. We have $T_h(G) < T_h(G_{(\beta(G))})$. Let $X \in \mathscr{R}_h$, $T_h(G) < X \leqq T_h(G_{(\beta(G))})$. There exists $G_1 \in X \setminus T_h(G)$. Then $G_1 \in \operatorname{ext} \operatorname{Hom} \operatorname{Sub} \{G_{(\beta(G))}\}$. Hence there exists a set $S \subset \operatorname{Hom} \operatorname{Sub} \{G_{(\beta(G))}\}$ such that G_1 can be constructed by means of ext from the set S. In view of $G_1 \notin T_h(G)$ there is $H \in S$ such that H does not belong to Hom Sub . $\{G\}$. Therefore, from the construction of $G_{(\beta(G))}$ it follows that there is a convex