Časopis pro pěstování matematiky, roč. 106 (1981), Praha

ON TRANSFORMATIONS OF SETS IN R^n

HARRY I. MILLER[1]), Sarajevo

(Received November 26, 1979)

0. INTRODUCTION

In the first part of this work certain transformations of sets in R^n will be studied. We extend the results of various authors. An article by Neubrunn and Šalát [10] was the initial work in this area. Latter contributions were made in [12], [14] and [15].

The second part of this paper deals with analogues of theorems that appeared in [1], [4] and [5]. In particular, sets of the second category having the Baire property are studied here. Such sets and their duality with sets of positive measure have been studied extensively ([3], [6], [7], [8], [10], [11], [13] and [16]).

1. FAMILIES OF TRANSFORMATIONS IN R^n

Let $\mathscr{L}^n$ denote the collection of Lebesgue measurable subsets of R^n (n-dimensional Euclidean space). If $A \in \mathscr{L}^n$ then $|A|$ stands for the Lebesgue measure of the set A. Suppose that with each ω belonging to a metric space Ω a certain transformation of the family $\mathscr{L}^1$ into $\mathscr{L}^1$ is associated, this transformation being denoted by T_ω. Neubrunn and Šalát [10] considered families of transformations $\{T_\omega\}_{\omega \in \Omega}$ satisfying the following assumptions.

(i) There exists $\omega_0 \in \Omega$ such that for every closed interval $\langle a, b\rangle$ and every sequence $\{\omega_n\}_{n=1}^{\infty}$ of elements belonging to Ω and converging to ω_0,

$$\lim_{n\to\infty} (\inf T_{\omega_n}(\langle a, b\rangle)) = a\,, \quad \lim_{n\to\infty} (\sup T_{\omega_n}(\langle a, b\rangle)) = b$$

holds;

(ii) if $E, F \in \mathscr{L}^1$ and $E \subset F$ then for every $\omega \in \Omega$, $T_\omega(E) \subset T_\omega(F)$;

(iii) if $E \in \mathscr{L}^1$ and $\omega_n \to \omega_0$ (in Ω), then

[1]) This research is supported by the Foundation for Scientific Work of the Republic of Bosnia and Herzegovina.

$$\lim_{n\to\infty} |T_{\omega_n}(E)| = |T_{\omega_0}(E)| = |E| .$$

Consider the following examples.

Example 1.1. Set $\Omega = \mathbb{R}^1$. If $E \in \mathscr{L}^1$, then let $T_\omega(E) = E + \omega$ (i.e. the set of all numbers of the form $x + \omega$, $x \in E$). Taking 0 as ω_0 one can easily check that properties (i)–(iii) are satisfied.

Example 1.2. Set $\Omega = (0, 1\rangle$. If $E \in \mathscr{L}^1$, then let $T_\omega(E) = \omega E$ (i.e. the set of all numbers of the form ωx, $x \in E$). If we put $\omega_0 = 1$ then properties (i)–(iii) are satisfied. These examples appear in the work of Neubrunn and Šalát [10].

M. Pal [12] considered an extension of the families of transformations of Neubrunn and Šalát, namely, with each ω belonging to a metric space Ω he associated a transformation T_ω, mapping $\mathscr{L}^n$ into $\mathscr{L}^n$ in such a way that the family of transformations $\{T_\omega\}_{\omega\in\Omega}$ satisfied the following three conditions.

(I) There exists $\omega_0 \in \Omega$ such that for every closed sphere $K = S[a, r] \subset \mathbb{R}^n$ and every sequence $\{\omega_n\}$ $(\omega_n \in \Omega)$ converging to ω_0,

$$\lim_{n\to\infty} [\sup |a - T_{\omega_n}(K)|] = r \quad \text{holds} .$$

(II) If $E, F \in \mathscr{L}^n$ and $F \subset E$, then for every $\omega \in \Omega$, $T_\omega(F) \subset T_\omega(E)$.

(III) If $E \in \mathscr{L}^n$ and $\omega_n \to \omega_0$, then

$$\lim_{n\to\infty} |T_{\omega_n}(E)| = |T_{\omega_0}(E)| = |E| .$$

Here $|a - B|$ denotes the set $\{|a - b|; b \in B\}$ where $|a - b|$ is the ordinary Euclidean distance between a and b.

Clearly, Example 1.1 can be modified in the obvious way to Example 1.1′ by setting $\Omega = \mathbb{R}^n$ and taking $x + \omega$ to be the ordinary vector sum of x and ω. It is easy to see that Example 1.1′ satisfies properties (I)–(III).

Example 1.2 can be modified to Example 1.2′. In this case Ω remains unchanged, that is $\Omega = (0, 1\rangle$, and $\omega x = (\omega x_1, \ldots, \omega x_n)$ where $x = (x_1, \ldots, x_n)$. Then it is easy to check that Example 1.2′ satisfies properties (I)–(III). We now consider the following example.

Example 1.3′. Set $\Omega = (0, 1\rangle$. If $E \in \mathscr{L}^n$, then let $T_\omega(E) = \omega E$. If we put $\omega_0 = k$ for some k in the open interval (0, 1), then properties (I) and (III) fail to hold.

We will now list three properties, for a family $\{T_\omega\}_{\omega\in\Omega}$ of transformations on $\mathscr{L}^n$ into $\mathscr{L}^n$, satisfied by all three of the above examples (1.1′, 1.2′ and 1.3′) and show several consequences of the three properties.

We consider families of transformations $\{T_\omega\}_{\omega\in\Omega}$ on $\mathscr{L}^n$ into $\mathscr{L}^n$ (Ω a metric space) which satisfy the following conditions.

(a) If $E, F \in \mathscr{L}^n$ and $E \subset F$ then for every $\omega \in \Omega$, $T_\omega(E) \subset T_\omega(F)$.

(b) There exists $\omega_0 \in \Omega$, $a \in \mathbb{R}^n$ and k $(0 < k \leqq 1)$ such that

$$\lim_{n\to\infty} \frac{|T_{\omega_n}(S[a, r]) \cap S[a, kr]|}{|S[a, kr]|} = 1$$

for every $r > 0$ and for every sequence $\{\omega_n\}_{n=1}^{\infty}$ of elements belonging to Ω and converging to ω_0.

(c) There exists j $(0 < j \leqq 1)$ such that if $A, B \in \mathscr{L}^n$ and $A \subset B$, then $\limsup_{n\to\infty}$ . . $|T_{\omega_n}(B) \setminus T_{\omega_n}(A)| \leqq j \,.\, |B \setminus A|$ provided $\omega_n \to \omega_0$ (in Ω).

Theorem 1.1. *If $\{T_\omega\}_{\omega\in\Omega}$ is a family of transformations on $\mathscr{L}^n$ into $\mathscr{L}^n$ satisfying properties* (I), (II) *and* (III) *then $\{T_\omega\}_{\omega\in\Omega}$ satisfies properties* (a), (b) *and* (c).

Proof. Property (II) implies property (a). Properties (I) and (III) imply property (b) (with $k = 1$ and a any point in $\mathbb{R}^n$). Properties (II) and (III) imply property (c) with $j = 1$.

It is easy to see that the family of transformations given in Example 1.3′ satisfies properties (a), (b) and (c), where $a = 0$, $\omega_0 = k$ $(0 < k \leqq 1)$ and $j = k^n$ (to see the last equality consult [2], page 153). Therefore by Theorem 1.1 and our earlier remarks (i.e. that Example 1.3′ does not satisfy properties (I) and (III)) it follows that properties (a), (b) and (c) are strictly weaker than properties (I), (II) and (III).

M. Pal [12] proved the following theorem which extends Theorem 1.1 of Neubrunn and Šalát [10].

Theorem. *Let $\{T_\omega\}_{\omega\in\Omega}$ be a family of transformations satisfying conditions* (I), (II) *and* (III) *and let $\{\omega_n\}$ be a sequence converging to ω_0 (in Ω). Let A be a set of positive measure in $\mathbb{R}^n$. Then there exists a natural number N_0 such that for $n \geqq N_0$, $A \cap T_{\omega_n}(A)$ is a set of positive measure.*

We now show that this result remains true for families of transformations satisfying the (weaker) conditions (a), (b) and (c).

Theorem 1.2. *Suppose $A \subset \mathbb{R}^n$ has positive Lebesgue measure. If $\{T_\omega\}_{\omega\in\Omega}$ is a family of transformations on $\mathscr{L}^n$ into $\mathscr{L}^n$ satisfying conditions* (a), (b) *and* (c) *and a (in condition* (b)) *is a density point of A, then if $\{\omega_n\}$ is a sequence converging to ω_0, there exists a natural number N_0 such that for $n \geqq N_0$, $A \cap T_{\omega_n}(A)$ is a set of positive measure.*

Proof. Let $0 < \varepsilon < 1$ and let $\{\omega_m\}$ be a sequence converging to ω_0. Since a is a density point of A, there exists $r_\varepsilon > 0$ such that $0 < r \leqq r_\varepsilon$ implies

1) $\dfrac{|S[a, r] \cap A|}{|S[a, r]|} > 1 - \varepsilon$ or $|S[a, r]| - |A \cap S[a, r]| < \varepsilon \,.\, |S[a, r]|$.

By (c) there exists a natural number N_ε such that $m \geqq N_\varepsilon$ implies

2) $|T_{\omega_m}(S[a, r_\varepsilon]) \setminus T_{\omega_m}(S[a, r_\varepsilon] \cap A)| \leqq j \,.\, |S[a, r_\varepsilon] \setminus (S[a, r_\varepsilon] \cap A)| + \varepsilon \,.\, |S[a, r_\varepsilon]|$.

This in turn implies, in virtue of (a), that if $m \geqq N_\varepsilon$ then

3) $|T_{\omega_m}(S[a, r_\varepsilon])| - |T_{\omega_m}(S[a, r_\varepsilon] \cap A)| \leqq j \,.\, |S[a, r_\varepsilon] \setminus (S[a, r_\varepsilon] \cap A)| +$
$+ \varepsilon \,.\, |S[a, r_\varepsilon]|$.

Using 1) we see that for $m \geqq N_\varepsilon$ we have

4) $|T_{\omega_m}(S[a, r_\varepsilon])| - |T_{\omega_m}(S[a, r_\varepsilon] \cap A)| \leqq j \,.\, \varepsilon \,.\, |S[a, r_\varepsilon]| + \varepsilon \,.\, |S[a, r_\varepsilon]|$.

By (b) there exists a natural number $N'_\varepsilon > N_\varepsilon$ such that

5) $|T_{\omega_m}(S[a, r_\varepsilon]) \cap S[a, kr_\varepsilon]| > (1 - \varepsilon) \,.\, |S[a, kr_\varepsilon]|$ if $m \geqq N'_\varepsilon$.

From 4) and 5) it follows that if $m \geqq N'_\varepsilon$ then

6) $|T_{\omega_m}(S[a, r_\varepsilon] \cap A) \cap S[a, kr_\varepsilon]| > |S[a, kr_\varepsilon]| - \varepsilon \,.\, |S[a, kr_\varepsilon]| -$
$- (j + 1) \,.\, \varepsilon \,.\, |S[a, r_\varepsilon]|$,

or if $m \geqq N'_\varepsilon$ we have

7) $|T_{\omega_m}(S[a, r_\varepsilon] \cap A) \cap S[a, kr_\varepsilon]| > [k^n - (j + 2)\,\varepsilon] \,.\, |S[a, r_\varepsilon]|$.

Let ε be a fixed real number, $0 < \varepsilon < 1$, such that $[k^n(1 - \varepsilon) - (j + 2)\,\varepsilon] > 0$. Then this ε satisfies the inequality $[k^n - (j + 2)\,\varepsilon] \,.\, |S[a, r_\varepsilon]| > \varepsilon \,.\, k^n \,.\, |S[a, r_\varepsilon]|$. For the same ε, 1) yields

8) $|S[a, kr_\varepsilon]| - |A \cap S[a, kr_\varepsilon]| < \varepsilon \,.\, k^n \,.\, |S[a, r_\varepsilon]|$.

Therefore, because of 7) and 8), we have for our fixed ε

9) $|T_{\omega_m}(S[a, r_\varepsilon] \cap A) \cap A| > 0$ for each $m \geqq N'_\varepsilon > N_\varepsilon$,

completing the proof.

Saha and Ray [15] considered families $\{T_\omega\}_{\omega \in \Omega}$ of transformations of $\mathscr{L}^n$ into $\mathscr{L}^n$, which are more general than those satisfying properties (I), (II) and (III) of Pal [12]. In [9], the current author corrected several basic mistakes in the paper of Saha and Ray [15]. We now generalize the three conditions on families of transformations $\{T_\omega\}_{\omega \in \Omega}$ given in [9] and consider some of their consequences.

We will consider families of transformations $\{T_\omega\}_{\omega \in \Omega}$, where Ω is a metric space and $T_\omega : \mathscr{L}^n \to \mathscr{L}^n$ for each $\omega \in \Omega$ satisfy conditions (a), (b′) and (c), where (b′) denotes the following condition:

(b′) There exist $\omega_0 \in \Omega$, a, $b \in \mathbb{R}^n$, and k $(0 < k \leqq 1)$ such that

$$\lim_{n \to \infty} \frac{|T_{\omega_n}(S[b, r]) \cap S[a, kr]|}{|S[a, kr]|} = 1$$

for every $r > 0$ and for every sequence $\{\omega_n\}_{n=1}^{\infty}$ of elements belonging to Ω and converging to ω_0.

We now prove the following theorem, related to Theorem 2′ in [9] and Theorem 2 in [15].

Theorem 1.3. *Suppose A and B are two sets of positive measure in $\mathbb{R}^n$ and a is a point of density one in A, b is a point of density one in B and ω_0 is a point of Ω. Suppose $\{T_\omega\}_{\omega \in \Omega}$ is a family of transformations of $\mathscr{L}^n$ into $\mathscr{L}^n$ satisfying properties* (a), /b′) *and* (c) *with respect to the points* a, b ω_0 *mentioned above. Then, if* $\{\omega_n\}_{n=1}^{\infty}$ *is a sequence in* Ω *converging to* ω_0 *and* p *is a positive integer, there exists* p *strictly increasing integers* $n_1, n_2, \ldots, n_p$ *such that*

$$A \cap T_{\omega_{n_1}}(B) \cap T_{\omega_{n_2}}(B) \cap \ldots \cap T_{\omega_{n_p}}(B)$$

is a set of positive measure.

Proof. Let $0 < \varepsilon < 1$ and let $\{\omega_m\}$ be a sequence converging to ω_0. Since a is a density point of A and b is a density point of B there exists $r_\varepsilon > 0$ such that $0 < r < < r_\varepsilon$ implies

1) $|S[a, r]| - |A \cap S[a, r]| < \varepsilon \, . \, |S[a, r]|$ and

2) $|S[b, r]| - |B \cap S[b, r]| < \varepsilon \, . \, |S[b, r]|$.

Imitating the proof of Theorem 1.2 we can find a positive integer N'_ε such that

3) $|T_{\omega_m}(S[b, r_\varepsilon] \cap B) \cap S[a, kr_\varepsilon]| > (1 - \varepsilon)\,|S[a, kr_\varepsilon]| - (j + 1)\,\varepsilon \, . \, |S[a, r\varepsilon]|$ if $0 < \varepsilon < 1$ and $m \geqq N'$.

For each $i = 1, 2, \ldots, p$, there exists ε_i, $0 < \varepsilon_i < 1$, such that if $0 < \varepsilon < \varepsilon_i$ we have

4) $|T_{\omega_m}(S[b, r_\varepsilon] \cap B) \cap S[a, kr_\varepsilon]| > (1 - 1/(2 \, . \, 2^i))\,|S[a, kr_\varepsilon]|$ if $m \geqq N'_\varepsilon$ and

5) $|S[a, kr_\varepsilon]| - |A \cap S[a, kr_\varepsilon]| < 1/2 \, . \, 2^i)\,|S[a, kr_\varepsilon]$ if $0 < \varepsilon < \varepsilon_i$.

Equations 4) and 5) imply that

6) $|T_{\omega_m}(S[b, r_\varepsilon] \cap B) \cap (A \cap S[a, kr_\varepsilon])| > (1 - 1/2^i)\,|S[a, kr_\varepsilon]|$ if $0 < \varepsilon < \varepsilon_i$ and $m \geqq N'_\varepsilon$.

Let ε be a fixed real number satisfying

$$0 < \varepsilon < \min(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p)$$

and suppose that $n_1, n_2, \ldots, n_p$ are p distinct positive integers each greater than or equal to N'_ε.

Then it follows that

7) $|T_{\omega_{n_i}}(S[b, r_\varepsilon] \cap B) \cap (A \cap S[a, kr_\varepsilon])| > (1 - 1/2^i)\,|S[a, kr_\varepsilon]|$ if $i = 1, 2, \ldots, p$.

This yields

8) $|[\bigcap_{i=1}^{p} T_{\omega_{n_i}}(S[b, r_\varepsilon] \cap B)] \cap [A \cap S[a, kr_\varepsilon]]| > 0$,

which completes the proof.

We conclude this section by presenting the following theorem which is related to Theorem 3′ in [9] and Theorem 3 in [15].

Theorem 1.4. *Suppose* $A, B_1, B_2, \ldots, B_m$ *are sets of positive measures in* $\mathbb{R}^n$, a *is a point of density one in* A, b_i *is a point of density one in* B_i *for each* $i = 1, 2, \ldots, m$ *and* ω_0^i *is a point of* Ω *for each* $i = 1, 2, \ldots, m$. *Suppose* $\{T_\omega\}_{\omega\in\Omega}$ *is a family of transformations on* $\mathcal{L}^n$ *into* $\mathcal{L}^n$ *satisfying properties* (a), (b′) *and* (c) *with respect to the* *triple* (a, b_i, ω_0^i) *for each* $i = 1, 2, \ldots, m$. *If the sequence* $\{\omega_n^i\}_{n=1}^{\infty}$ *converges to* ω_0^i *for each* $i = 1, 2, \ldots, m$, *then there exists a positive integer* N *such that for* $n \geqq N$,

$$A \cap T_{\omega_n^1}(B_1) \cap T_{\omega_n^2}(B_2) \cap \ldots \cap T_{\omega_n^m}(B_m)$$

is a set of positive measure.

Proof. The proof of Theorem 1.4 is similar to that of Theorem 1.3 and will therefore be omitted.

2. BAIRE SETS IN $\mathbb{R}^n$

A set A in $\mathbb{R}^n$ is said to have the Baire property if it can be written in the from $A = (G \setminus P) \cup Q$, where G is an open set and P and Q are sets of the first category (i.e. countable unions of nowhere dense sets).

In this section we present two theorems. They are the Baire property analogues of results of Khan and Pal [4] and Mazumdar [5], respectively.

Theorem 2.1. *Let* A *and* B *be two Baire sets (i.e. sets possessing the Baire property) of the second category in* $\mathbb{R}^n$ *and let* $\alpha_1, \alpha_2, \ldots, \alpha_p$ $(\alpha_k \neq 0$ *for each* $k)$ *be real numbers. Then exist two spheres* K_1 *(with center at the origin) and* K_2, *such that for any system of* p *vectors* $z_1, z_2, \ldots, z_p$ *if* K_2 *and for any vector* $x \in K_1$, *there are vectors*

$$a(x; z_1, \ldots, z_p) \in A$$

and

$$b_k(x; z_1, \ldots, z_p) \in B$$
$$(k = 1, 2, \ldots, p)$$

such that

$$x = \frac{b_k(x; z_1, z_2, \ldots, z_p) - a(x; z_1, \ldots, z_p) - z_k}{\alpha_k}$$

for $k = 1, 2, \ldots, p$.

Proof. A and B can be written in the form $A = (G_1 \setminus P_1) \cup Q_1$ and $B = (G_2 \setminus P_2) \cup Q_2$, where G_1 and G_2 are open sets and P_1, P_2, Q_1 and Q_2 are sets of the first category in $\mathbb{R}^n$. Let $a \in G_1 \setminus P_1$ and $b \in G_2 \setminus P_2$ be two fixed points.

1) Let c denote $b - a$ and let $\alpha = \max(|\alpha_1|, \ldots, |\alpha_p|)$.

There exist positive real numbers r and s such that $r > s > 2(r - s)/3$ and such that

2) $K_A = S[a, r] \subset G_1$ and $K_B = S[b, s] \subset G_2$.

Define K_1 and K_2 as follows:

3) $K_1 = S[O, (r - s)/3\alpha]$ and $K_2 = S[c, (r - s)/3]$.

Suppose $x \in K_1$ and $z_1, z_2, \ldots, z_p \in K_2$.

4) Set $C = K_A \cap A$ and $C_k = (K_B \cap B) - \alpha_k x - z_k$ for $k = 1, \ldots, p$.

5) Let $X(x; z_1, \ldots, z_p)$ denote the set $C \cap C_1 \cap C_2 \cap \ldots \cap C_p$.

We proceed to show that $X(x; z_1, \ldots, z_p)$ is a set of the second Baire category.

The set $K_B - \alpha_k x - z_k$ has as its center a point whose distance from a is less than or equal to $2(r - s)/3$ since ($|w|$ denotes the length of w)

$$|a - b + \alpha_k x + z_k| = |a - b + \alpha_k x + (c + [(r - s)/3]\, \varepsilon)| =$$
$$= |\alpha_k x + [(r - s)/3]\, \varepsilon| \leqq (r - s)/3 + (r - s)/3 = 2((r - s)/3)$$

(where $|\varepsilon| \leqq 1$).

Furthermore, the radius of $K_B - \alpha_k x - z_k$ is s and $s > 2((r - s)/3)$. Therefore, for each $k = 1, \ldots, p$, the sphere $K_B - \alpha_k x - z_k$ contains a neighborhood of the point a. This in turn implies that each set C_k $(k = 1, \ldots, p)$ contains a neighborhood of a with the exception of a set of the first category, i.e.

6) $C_k \supset S[a, t_k] \setminus N_k$ for each $k = 1, \ldots, p$, where N_k is a set of the first Baire category and $t_k > 0$.

Therefore it follows that $X(x; z_1, \ldots, z_p)$ is a set of the second Baire category in $\mathbb{R}^n$ and hence it is not empty. So there exist vectors

$$a(x; z_1, \ldots, z_p) \in A \quad \text{and} \quad b_k(x; z_1, \ldots, z_p) \in B \quad (k = 1, \ldots, p)$$

such that

7) $a(x; z_1, \ldots, z_p) = b_1(x; z_1, \ldots, z_p) - z_1 - \alpha_1 x = \ldots = b_p(x; z_1, \ldots, z_p) - z_p - \alpha_p x$.

Therefore

8) $$x = \frac{b_k(x; z_1, \ldots, z_p) - a(x; z_1, \ldots, z_p) - z_k}{\alpha_k}$$

for each $k = 1, 2, \ldots, p$, which completes the proof.

We now prove the Baire property analogue of a result of Mazumdar [5] (which in turn is a generalization of a result of Das Gupta [1]).

Theorem 2.2. *Suppose that A and B, subsets of $\mathbb{R}^+$ (the set of all positive real numbers), are two Baire sets of the second category, i.e. $A = (G_1 \setminus P_1) \cup Q_1$ and $B = (G_2 \setminus P_2) \cup Q_2$ where G_1 and G_2 are non-empty open sets and P_1, Q_1 P_2, Q_2 are sets of the category. Suppose further that $p \in G_2 \setminus P_2$ and $q \in G_1 \setminus P_1$. If $(\alpha_n)_{n=1}^{\infty}$ is any sequence of positive real numbers converging to q/p (which will be denoted by α), then the set*

$X = \{x \in \mathbb{R}^+;\ x \in A$ and $x/\alpha_n \in B$ for infinitely many $n\}$ is a set of the second Baire category in $\mathbb{R}$.