CONNECTIONS ON HIGHER ORDER TANGENT BUNDLES

TOMÁŠ KLEIN, Zvolen

(Received November 26, 1979)

1. PROLONGATION COFUNCTORS

Denote by $\mathcal{M}$ the category of differentiable manifolds and mappings, by $\mathcal{FM}$ the category of fibred manifolds and by $\mathcal{VB}$ the category of differentiable vector bundles.

As usual, T denotes the functor of $\mathcal{M}$ into $\mathcal{VB} \subset \mathcal{FM}$ transforming any manifold M into its tangent bundle TM and any map $f : M \to M_1$ into the induced tangent map

$$f_* = Tf : TM \to TM_1 .$$

Definition 1. A functor $p : \mathcal{M} \to \mathcal{FM}$ will be called *a prolongation functor*, if pM is a fibred manifold over M for any manifold M and

$$pf : pM \to pM_1$$

is a morphism of fibred manifolds over $f : M \to M_1$ for any mapping f (cf. [4]).

Every $f : M \to M_1$ determines the cotangent map f^* transforming any form $\omega \in T^*_{f(x)}M_1$ into $f^*\omega \in T^*_x M$. Let π, π_1, π^* or π_1^* be the fibre projections of TM, TM_1, T^*M or T^*M_1, respectively. Denote by $f^{-1}T^*M_1$ the induced bundle over M, i.e.

$$f^{-1}T^*M = \{(x, \omega) \in M \times T^*M_1;\ \pi_1^*\omega = f(x)\} .$$

Then we can define

$$T^*f : f^{-1}T^*M_1 \to T^*M$$

by $T^*f(x, \omega) = f^*\omega \in T^*_x M$. Obviously, T^*f is a fibre morphism over the identity of M (the so-called base-preserving morphism).

Given two fibred manifolds $\pi : E \to M_1$, $\pi_1 : E_1 \to M_1$ over the same base, a base-preserving morphism $\varphi : E \to E_1$ and a map $f : M \to M_1$, the induced morphism

$$f^{-1}\varphi : f^{-1}E \to f^{-1}E_1 \tag{1}$$

is defined by

$$(x, e) \mapsto (x, \varphi(e)) \quad \text{with} \quad \pi(e) = f(x) .$$

Definition 2. *A prolongation confunctor* $p : \mathcal{M} \to \mathcal{FM}$ is a rule transforming any manifold M into a fibred manifold pM over M and any map $f : M \to M_1$ into a base-preserving morphism

$$pf : f^{-1}pM_1 \to pM$$

such that

$$p(id_M) = id_{pM} \quad \text{for all} \quad M , \tag{2}$$

$$p(g \circ f) = pf \circ f^{-1}pg \tag{3}$$

for all $f : M \to M_1$ and $g : M_1 \to M_2$.

If the values of a prolongation cofunctor p lie in the subcategory $\mathcal{VB} \subset \mathcal{FM}$, then p is said to be a prolongation cofunctor of $\mathcal{M}$ into $\mathcal{VB}$.

Lemma 1. *T^* is a prolongation cofunctor of $\mathcal{M}$ into $\mathcal{VB}$.*

Proof is obvious.

In differential geometry, several prolongation cofunctors can be obtained by using the following general construction of the jet theory.

Consider two manifolds M, Q and a point $q \in Q$. The set $J^r(M, Q)_q = J^r_q M$ of all r-jets of M into Q with the target q is a fibred manifold over M. Consider further a mapping

$$J^r_q f : f^{-1} J^r(M_1, Q)_q \to J^r(M, Q)_q \tag{4}$$

defined by the following rule. If $b \in f^{-1} J^r(M_1, Q)_q$, $b = (x, j^r_{f(x)}\varphi)$, then

$$(J^r_q f)(b) = j^r_x(\varphi \circ f) .$$

Theorem 1. *J^r_q is a prolongation cofunctor of $\mathcal{M}$ into $\mathcal{FM}$.*

Proof is straightforward.

Theorem 2. *If Q is a vector space and $q = 0$, then J^r_0 is a prolongation cofunctor of $\mathcal{M}$ into $\mathcal{VB}$.*

Proof. In this case, $J^r(M, Q)_0$ is a vector bundle by

$$j^r_x\varphi + j^r_x\psi = j^r_x(\varphi + \psi) , \quad k \, . \, j^r_x\varphi = j^r_x k\varphi , \quad k \in R .$$

For any $f : M_1 \to M$, $f(y) = x$, we have

$$(j^r_x\varphi + j^r_x\psi) \circ j^r_y f = j^r_y(\varphi \circ f + \psi \circ f) = j^r_y(\varphi \circ f) + j^r_y(\psi \circ f) ,$$
$$(j^r_x k\varphi) \circ j^r_y f = k \, . \, j^r_y(\varphi \circ f) , \quad \text{QED.}$$

Remark 1. Let V be a manifold. If we associate with any manifold M the fibred manifold $J^r(M, V) \to M$ and define the induced map $f^{-1}J^r(M_1, V) \to J^r(M, V)$ for any $f : M \to M_1$ similarly to (4), we also obtain a prolongation cofunctor $J^r_V : \mathcal{M} \to \mathcal{F}\mathcal{M}$.

Let $\pi : E \to M$, $\pi_1 : E_1 \to M_1$ be vector bundles and $\varphi : E \to E_1$ a linear morphism over $f : M \to M_1$. Let

$$\varphi^* : (f^{-1}E_1)^* \to E^*$$

be the mapping defined by

$$\varphi^*(x, \omega) = \varphi^*_x(\omega), \quad \pi_1\omega = f(x), \quad x \in M,$$

where φ^*_x is the dual map to $\varphi \mid E_x$. It is easy to verify that φ^* is also a differentiable map.

Considering a prolongation functor $p : \mathcal{M} \to \mathcal{V}\mathcal{B}$, we define $p^*M = (pM)^*$ (= the dual bundle of pM) for any manifold M and

(5) $$p^*f = (pf)^* : f^{-1}p^*M_1 \to p^*M$$

for any $f : M \to M_1$. One easily finds

Lemma 2. *For any maps $f : M \to M_1$, $g : M_1 \to M_2$ we have*

$$p^*(g \circ f) = p^*f \circ f^{-1}p^*g .$$

Thus, p^* is a prolongation cofunctor $\mathcal{M} \to \mathcal{V}\mathcal{B}$. We shall say that the prolongation cofunctor p^* is dual to the prolongation functor p.

Conversely, given two vector bundles $E \to M$, $F \to M_1$, a map $f : M \to M_1$ and a base-preserving linear morphism $\psi : f^{-1}E \to F$, we define $\psi^* : E^* \to F^*$ by requiring that

(6) $$\psi^*_x : E^*_x \to F^*_{f(x)}$$

be the dual map to $\psi_{f(x)} : (f^{-1}F)_x \to E_x$. Using local coordinates, we directly deduce

Lemma 3. *ψ^* is differentiable.*

Let $q : \mathcal{M} \to \mathcal{V}\mathcal{B}$ be a prolongation cofunctor. Define $q^*M = (qM)^*$ for any manifold M and $q^*f = (qf)^*$ for any $f : M \to M_1$. One verifies easily that $q^*(g \circ f) = (q^*g) \circ (q^*f)$, so that $q^* : \mathcal{M} \to \mathcal{V}\mathcal{B}$ is a prolongation functor.

Definition 3. *The prolongation functor q^** will be called *dual* to the prolongation cofunctor q.

Theorem 3. *For any prolongation functor $p : \mathcal{M} \to \mathcal{V}\mathcal{B}$ and any prolongation cofunctor $q : \mathcal{M} \to \mathcal{V}\mathcal{B}$ we have*

$$(p^*)^* = p, \quad (q^*)^* = q .$$

Proof is straightforward.

Let M be an n-dimensional manifold.

Definition 4. Any jet $A \in J^r_x(M, R)_0$ of M into reals with a source x and target 0 will be called an *r-covector* on M at x. The vector bundle $J^r(M, R)_0 \to M$ will be denoted by $T^{r*}M$ and called the *r-th order cotangent bundle* of M.

Let $A = j^r_{x_0}F \in T^{r*}M$. Without loss of generality we may assume that the coordinate form of F is

$$F = a_i x^i + \ldots + \frac{1}{r!} a_{i_1 \ldots i_r} x^{i_1} \ldots x^{i_r}.$$

In this way, any local chart (x^i) on M induces a local chart $(x^i_0, x_i, \ldots, x_{i_1 \ldots i_r})$ on $T^{r*}M$.

By *Theorem* 2, T^{r*} is a prolongation cofunctor of $\mathcal{M}$ into $\mathcal{VB}$ and we can construct the dual prolongation functor $T^r : \mathcal{M} \to \mathcal{VB}$.

Definition 5. The dual vector bundle

$$T^r M = (T^{r*}M)^*$$

is called the *r-th order tangent bundle of* M and the induced map $T^r f : T^r M \to T^r M_1$ is said to be the *r-th order tangent map* of $f : M \to M_1$.

By dualization, any local chart (x^i) on M induces a local chart $(x^i_0, x^i, \ldots, x^{i_1 \ldots i_r})$ on $T^r M$.

We remark that one also can construct the r-th order tensor bundles over M, see [6].

2. LINEAR MAPPINGS BETWEEN HIGHER ORDER TANGENT SPACES

Let β^r_k denote the canonical projection of r-jets into k-jets, $r > k$. The kernel of $\beta^r_{r-1} : T^{r*}M \to T^{r-1*}M$ is naturally identified with the *r-th symmetric tensor power* $\bigcirc^r T^*M$, so that we have an exact sequence

$$0 \to \bigcirc^r T^*M \to T^{r*}M \xrightarrow{\beta^r_{r-1}} T^{r-1*}M \to 0. \tag{7}$$

The dual sequence is

$$0 \to T^{r-1}M \to T^r M \to \bigcirc^r TM \to 0. \tag{8}$$

In particular, we have

$$TM \subset T^2 M \subset \ldots \subset T^{r-1}M \subset T^r M.$$

For any $f : M \to M_1$, the following diagram commutes:

$$(9) \qquad \begin{array}{ccccccccc} 0 \to & T^{r-1}M & \to & T^rM & \to & \bigcirc^r TM & \to 0 \\ & \downarrow T^{r-1}f & & \downarrow T^rf & & \downarrow \bigcirc^r Tf & \\ 0 \to & T^{r-1}M_1 & \to & T^rM_1 & \to & \bigcirc^r TM_1 & \to 0\,, \end{array}$$

see [3], [7].

Let g, h be real functions on M, $g(x) = h(x) = 0$. Since the r-th order partial derivatives of the product $g \,.\, h$ at x do not depend on the r-th order partial derivatives of g and h at x, we have a well – defined map

$$\mu : (T_x^{r-1*}M) \times (T_x^{r-1*}M) \to T_x^{r*}M\,,$$
$$(j_x^{r-1}g, j_x^{r-1}h) \mapsto j_x^r(g \,.\, h)\,.$$

Since μ is symmetric and bilinear, it can be viewed as a map $\mu : \bigcirc^2 T_x^{r-1*}M \to \to T_x^{r*}M$.

Lemma 4. *The sequence*

$$\bigcirc^2 T_x^{r-1*}M \xrightarrow{\mu} T_x^{r*}M \xrightarrow{\beta^r_1} T_x^{1*}M \to 0$$

is exact.

Proof. As g and h vanish at x, $g \,.\, h$ has all the first order partial derivatives at x equal to zero. Hence $Im\,\mu \subset Ker\,\beta^r_1$. Conversely, if $C \in Ker\,\beta^r_1$, it can be written in the form

$$C = j_0^r(x^i\, h_i(x))$$

with $h_i(0) = 0$. This implies $C \in Im\,\mu$, QED.

Let $K_x^r M = Ker\,\mu$ and $S_x^{r*}M = \bigcirc^2 T_x^{r-1*}M \mid K_x^r M$. Then we have an exact sequence

$$(10) \qquad 0 \to S_x^{r*}M \to T_x^{r*}M \to T_x^*M \to 0$$

and its dual

$$(11) \qquad 0 \to T_xM \to T_x^rM \to S_x^rM \to 0\,,$$

where

$$S_x^rM = (S_x^{r*}M)^* \subset \bigcirc^2 T_x^{r-1}M\,.$$

Lemma 5. *Let $f : M_1 \to M$, $f(y) = x$. Then*

$$\bigcirc^2 T^{r-1*}f(K_x^rM) \subset K_y^rM_1\,.$$

Proof. Let $u = (j_x^{r-1}g, j_x^{r-1}h) \in K_x^rM$, i.e. $0 = j_x^r(g \,.\, h) \in T_x^{r*}M$. If

$$\bigcirc^2 T^{r-1*} f(u) = (j_y^{r-1}\, g(f), j_y^{r-1}\, h(f))\,,$$

then

$$(j_y^{r-1}\, g(f), j_y^{r-1}\, h(f)) = j_y^r(g(f) \,.\, h(f)) = j_y^r(g \,.\, h)\,(f) = j_x^r(g \,.\, h) \circ j_y^r f\,,$$

where $\circ$ denotes the composition of jets. The coordinate formula yields

$$0 = j^r_x(g \,.\, h) \circ j^r_y f \in T^{r*}_y M_1\,, \quad \text{QED}\,.$$

Corollary 1. *The mapping $T^r f$ dual to $T^{r*} f$ has the property*

$$T^r f(S^r_y M_1) \subset S^r_x M\,.$$

Coordinate formulae for mapings $T^{r*}f$, $T^r f$.

Let (x^i) or (y^α) be local coordinates on M or M_1, respectively. Consider $F : M \to R$, $F(x) = 0$ and a mapping $f : M_1 \to M$ with the coordinate form $x^i = f^i(y^\alpha)$. Let

$$j^r_y f = (y^\alpha, x^i, f^i_\alpha, f^i_{\alpha_1\alpha_2}, \ldots, f^i_{\alpha_1\ldots\alpha_r})\,,$$

$$j^r_x F = (x^i, \bar{x}_i, \ldots, \bar{x}_{i_1\ldots i_r}) \in T^{r*}_x M\,,$$

$$T^{r*} f(x^i, \bar{x}_i, \ldots, \bar{x}_{i_1\ldots i_r}) = (y^\alpha, \bar{y}_\alpha, \ldots, \bar{y}_{\alpha_1\ldots\alpha_r})\,.$$

From the coordinate formula for the composition of jets, we obtain

$$\bar{y}_\alpha = \bar{x}_i f^i_\alpha\,,$$

$$\bar{y}_{\alpha_1\alpha_2} = \bar{x}_{i_1 i_2} f^{i_1}_{\alpha_1} f^{i_2}_{\alpha_2} + \bar{x}_{i_1} f^{i_1}_{\alpha_1\alpha_2}\,,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\bar{y}_{\alpha_1\ldots\alpha_r} = \bar{x}_{i_1\ldots i_r} f^{i_1}_{\alpha_1} \ldots f^{i_r}_{\alpha_r} + \bar{x}_{i_1\ldots i_{r-1}} (f^{i_1}_{\alpha_1\alpha_2} f^{i_2}_{\alpha_3} \ldots f^{i_{r-1}}_{\alpha_r} + \ldots + f^{i_1}_{\alpha_{r-1}\alpha_r} f^{i_2}_{\alpha_1} \ldots f^{i_{r-1}}_{\alpha_{r-2}}) + \ldots + \bar{x}_{i_1} f^{i_1}_{\alpha_1\ldots\alpha_r}\,.$$

Dualization yields the following coordinate formula for $T^r f$:

(12)
$$x^i = f^i_\alpha y^\alpha + f^i_{\alpha_1\alpha_2} y^{\alpha_1\alpha_2} + \ldots + f^i_{\alpha_1\ldots\alpha_r} y^{\alpha_1\ldots\alpha_r}\,,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x^{i_1\ldots i_{r-1}} = f^{i_1}_{\alpha_1} \ldots f^{i_{r-1}}_{\alpha_{r-1}} y^{\alpha_1\ldots\alpha_{r-1}} + (f^{i_1}_{\alpha_1\alpha_2} f^{i_2}_{\alpha_3} \ldots f^{i_{r-1}}_{\alpha_r} + \ldots + f^{i_1}_{\alpha_{r-1}\alpha_r} f^{i_2}_{\alpha_1} \ldots f^{i_{r-1}}_{\alpha_{r-2}})\, y^{\alpha_1\ldots\alpha_r}\,,$$
$$x^{i_1\ldots i_r} = f^{i_1}_{\alpha_1} \ldots f^{i_r}_{\alpha_r} y^{\alpha_1\ldots\alpha_r}\,.$$

Let

$$L^r : T^r_y M_1 \to T^r_x M$$

be an arbitrary linear mapping with the coordinate form

(13)
$$x^i = a^i_\alpha y^\alpha + a^i_{\alpha_1\alpha_2} y^{\alpha_1\alpha_2} + \ldots + a^i_{\alpha_1\ldots\alpha_r} y^{\alpha_1\ldots\alpha_r}\,,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x^{i_1\ldots i_r} = a^{i_1\ldots i_r}_{\alpha_1} y^{\alpha_1} + \ldots + a^{i_1\ldots i_r}_{\alpha_1\ldots\alpha_r} y^{\alpha_1\ldots\alpha_r}\,.$$

Definition 6. We shall say that L^r is an *r-mapping* with respect to $L^{r-1} : T^{r-1}_y M \to T^{r-1}_x M$, if L^r can be restricted to $T_y M_1 \to T_x M$ and the factor map

$$T^r_y M_1|_{T_y M_1} \to T^r_x M|_{T_x M} \quad \text{coincides with} \quad \mathrm{O}^2 L^{r-1}|_{S^r{}_y M_1}\,.$$

Theorem 4. *A linear mapping* $L^r = T^r_y M_1 \to T^r_x M$ *is of the form* $T^r_y f$ *iff*

a) L^r *can be restricted to* $L^{r-1} : T^{r-1}_y M_1 \to T^{r-1}_x M$,

b) L^{r-1} *is of the form* $T^{r-1}_y f$,

c) L^r *is an r-mapping with respect to* L^{r-1}.

Proof consists in direct evaluation in local coordinates, which we omit here.

3. REGULAR CONNECTION ON $T^r M$

Given a vector bundle $E \to M$, its first jet prolongation $J^1 E$ is also a vector bundle over M. A connection on E means any linear morphism $\Gamma : E \to J^1 E$ satisfying $\beta^1_0 \circ \Gamma = id_E$. If we have some local coordinates x^i, y^α on E, then the equations of Γ are

$$y^\alpha_i = \Gamma^\alpha_{\beta i}(x)\, y^\beta\,, \tag{14}$$

where y^α_i are the induced coordinates on $J^1 E$, [1], [5].

The set LE of all linear isomorphisms between the individual fibres of E is a Lie groupoid in the sense of Ehresmann. For every $\Phi \in LE$, $\Phi : E_x \to E_y$, we set $a\Phi = x$, $b\Phi = y$. Let $QLE \to M$ be the fibred manifold of all (first order) elements of connection on LE, i.e. every $A \in (QLE)_x$ is the 1 – jet at x of a local map φ of M into LE satisfying $a\,\varphi(t) = x$, $b\,\varphi(t) = t$ for all t and $\varphi(x) = id_{E_x}$. Every section $\gamma : M \to QLE$ determines a connection Γ on E as follows. If $\gamma(x) = j^1_x\, \Phi(t)$, then $\gamma(t)\,(y)$ is a local section of E for every $y \in E_x$ and we put $\Gamma(y) = j^1_x[\Phi(t)\,(y)]$. Given a subgroupoid $\Omega \subset LE$, a connection Γ on E is said to be an Ω – connection, if it is generated by a section $\gamma : M \to Q\Omega$, i.e. for every $x \in M$ there is a local map φ of M into Ω with $a\,\varphi(t) = x$, $b\,\varphi(t) = t$ and $\varphi(x) = id_{E_x}$ such that $\Gamma(y) = j^1_x[\varphi(t)\,(y)]$ for all $y \in E_x$.

In particular, let $\pi^r(M)$ denote the groupoid of all invertible r-jets of M into itself. Every element of $\pi^r(M)$ with a source x and target y determines a linear map of $T^r_x M$ into $T^r_y M$, so that $\pi^r(M)$ is a subgroupoid of $L(T^r M)$. A connection on $T^r M$ will be called regular, if it is a $\pi^r(M)$ – connection in the above sense. Using Theorem 4, we shall characterize the regular connections. However, we first explain some necessary general ideas.

Let $E_1 \to M$ be another vector bundle and Γ_1 a linear connection on E_1 with the equations

$$z^\lambda_i = \Gamma^\lambda_{\mu i}(x)\, z^\mu \tag{15}$$

in some local coordinates x^i, z^λ on E_1. According to [2], Γ and Γ_1 determine a connection $\Gamma \otimes \Gamma_1$ on the tensor product $E \otimes E_1$ with the following equations:

$$w^{\alpha\lambda}_i = \Gamma^\alpha_{\beta i} w^{\beta\lambda} + \Gamma^\lambda_{\mu i} w^{\alpha\mu}\,, \tag{16}$$

provided $w^{\alpha\lambda}$ are the induced coordinates on $E \otimes E_1$.