Časopis pro pěstování matematiky, roč. 106 (1981), Praha

ON TWO GRAPH-THEORETICAL PROBLEMS FROM THE CONFERENCE AT NOVÁ VES U BRANŽEŽE

BOHDAN ZELINKA, Liberec

(Received October 10, 1979)

At the Czechoslovak Conference on Graph Theory at Nová Ves u Branžeže in May 1979 some problems were suggested by the participants. In this paper we shall deal with two of them.

I.

M. FIEDLER presented the following problem:

A bipartite graph (bigraph) $B = (N_1, N_2, H)$, both of whose vertex classes N_1, N_2 have the same finite cardinality $|N_1| = |N_2|$, will be called completely connected, if the following condition holds: whenever M is a non-empty proper subset of N_1, then the set $M' = \{k \in N_2 \mid \exists i \in M, (i, k) \in H\}$ fulfils $|M'| > |M|$.

Characterize critical completely connected bigraphs, i.e. such completely connected bigraphs which cease to be completely connected after deleting an arbitrary edge.

If $X \subseteq N_1$, then (following [1]) by $\Gamma_B(X)$ we shall denote the set of all vertices of B which are adjacent to at least one vertex of X. Throughout the next, we shall tacitly assume that each bigraph considered has at least four vertices.

We prove some lemmas.

Lemma 1. *Every circuit of an even length is a critical completely connected bigraph.*

Proof is straightforward.

Lemma 2. *Let $B^* = (N_1^*, N_2^*, H^*)$ be a completely connected bigraph, let $v_1 \in N_1^*$, $v_2 \in N_2^*$. Connect v_1 with v_2 by a path C of an odd length at least 3 whose inner vertices do not belong to B^*. If v_1 and v_2 are joined by an edge in B^*, delete this edge. Then the graph B thus constructed is a completely connected bigraph.*

Proof. The graph B is evidently a bigraph. Let $B = (N_1, N_2, H)$, $N_1^* \subset N_1$, $N_2^* \subset N_2$. Now let X be a non-empty proper subset of N_1. If $X \in N_1^* - \{v_1\}$, then

$\Gamma_B(X) = \Gamma_{B^*}(X)$ and, as B^* is completely connected, $|\Gamma_B(X)| > |X|$. If X is a proper subset of N_1^* and $v_1 \in X$, then $\Gamma_B(X) \subseteq (\Gamma_{B^*}(X) - \{v_2\}) \cup \{u\}$, where u is the vertex of C adjacent to v_1. We have $|\Gamma_B(X)| = |\Gamma_{B^*}(X)| > |X|$. If $X = N_1^*$, then $\Gamma_B(X) = N_2^* \cup \{u\}$ and $|\Gamma_B(X)| = |N_2| + 1 = |N_1| + 1 > |X|$. If $X \in N_1 - N_1^*$, then consider a circuit which is the union of C with a path connecting v_1 and v_2 in B^*. The set X is a proper subset of the intersection of the vertex set of this circuit with N_1, hence Lemma 1 implies that $|\Gamma_B(X)| > |X|$. Thus suppose $X \cap N_1^* = X^* \neq \emptyset$, $X - N_1^* = X^{**} \neq \emptyset$. If $X^* \subseteq N_1^* - \{v_1\}$, then $\Gamma_B(X) = \Gamma_B(X^*) \cup \Gamma_B(X^{**})$, $|\Gamma_B(X^*)| > |X^*|$, $|\Gamma_B(X^{**})| > |X^{**}| + 1$ and $|\Gamma_B(X^*) \cap \Gamma_B(X^{**})| \leqq 1$, because this intersection cannot contain any vertex other than v_2. This implies $|\Gamma_B(X)| \geqq |X^*| + |X^{**}| + 1 > |X|$. If $v_1 \in X^*$, $X^* \neq N_1$, then $|\Gamma_B(X^*)| \geqq |X^*| + 1$, $|\Gamma_B(X^{**})| \geqq |X^{**}| + 1$. If in the graph B^* the vertex v_2 is adjacent to no vertex of $X^* - \{v_1\}$, then $v_2 \notin \Gamma_B(X^*)$ and the set $\Gamma_B(X^*) \cap \Gamma_B(X^{**})$ can contain at most one vertex, namely u, and we have again $|\Gamma_B(X)| > |X|$. If in B^* the vertex v_2 is joined with another vertex of X^* than v_1, then also $v_2 \in \Gamma_B(X^*)$ and $\Gamma_B(X^*) = \Gamma_{B^*}(X^*) \cup \{v_2\}$; hence $|\Gamma_B(X^*)| \geqq |X^*| + 2$ and evidently again $|\Gamma_B(X^{**})| \geqq |X^{**}| + 1$. The set $\Gamma_B(X^*) \cap \Gamma_B(X^{**}) = \{u, v_2\}$ and hence $|\Gamma_B(X)| > |X|$. Finally, if $X = N_1^*$, then $X^{**} \neq N_1 - N_1^*$ (because X is a proper subset of N_1). Let w be a vertex of $N_1 - N_1^*$ which does not belong to X^{**}. To each vertex $x \in X^{**}$ we assign a vertex $\varphi(x)$ of $\Gamma_B(X^{**})$ so that $\varphi(x)$ is the vertex of C adjacent to x and lying between x and w. Evidently φ is an injection of X^{**} into $\Gamma_B(X^{**}) - (N_2 \cup \{u\})$ and thus $|\Gamma_B(X^{**}) - (N_2 \cup \{u\})| \geqq X^{**}$. We have $\Gamma_B(X^*) = N_2 \cup \{u\}$, hence $|\Gamma_B(X^*)| \geqq |X^*| + 1$, which yields $|\Gamma_B(X)| > |X|$. Therefore B is completely connected.

Lemma 3. *Let B be the graph described in Lemma 2. Let B^* be critical completely connected. If B^* contains the edge v_1v_2 or if in the graph $\hat{B}^*$ obtained from B^* by adding the edge v_1v_2 no edge except v_1v_2 can be deleted without loss of the complete connectedness, then B is critical completely connected, and vice versa.*

Proof. Let B^* contain v_1v_2. Let e be an arbitrary edge of B; by $B - e$ we denote the graph obtained from B by deleting e. If e belongs to B^*, then by $B^* - e$ we denote the graph obtained from B^* by deleting e. As B^* is critical, the graph $B^* - e$ is not completely connected. There exists a non-empty proper subset M of N_1^* such that $|\Gamma_{B^*-e}(M)| \leqq |M|$. If $v_1 \notin M$, then $\Gamma_{B-e}(M) = \Gamma_{B^*-e}(M)$ and $|\Gamma_{B-e}(M)| \leqq |M|$. If $v_1 \in M$, put $\tilde{M} = M \cup (N_1 - N_1^*)$. Then $\Gamma_{B-e}(\tilde{M}) \subseteq \Gamma_{B^*-e}(M) \cup (N_2 - N_2^*)$ and hence again $|\Gamma_{B-e}(\tilde{M})| \leqq |\tilde{M}|$. If e does not belong to B^*, then it is an edge of C and either is equal to v_1u, or is incident with a vertex of N_1 of the degree 2. If $e = v_1u$, then $\Gamma_{B-e}(N_1^*) = N_2^*$ nad $|\Gamma_{B-e}(N_1^*)| = |N_1^*|$. If e is incident with a vertex a of N_1 of the degree 2, then $|\Gamma_{B-e}(\{a\})| = 1 = |\{a\}|$. The proof for the case when the edge v_1v_2 exists is finished. Now let v_1, v_2 be non-adjacent in B^* and consider $\hat{B}^*$. If there exists an edge $e \neq v_1v_2$ of $\hat{B}^*$ such that $\hat{B}^* - e$ si completely connected, then also $B - e$ is completely connected and B is not critical. If there exists no such edge, then the proof is analogous to that in the preceding case.

Lemma 4. *Let B be a completely connected bigraph. Then B contains either a Hamiltonian circuit, or a factor consisting of an induced completely connected proper subgraph B^* and of a path C of an odd length at least* 3 *connecting two vertices of B^* and with inner vertices not belonging to B^*.*

Proof. Let $B = (N_1, N_2, H)$ be a completely connected bigraph. If B contains a circuit which is not Hamiltonian, then this circuit is a completely connected bigraph and so is the subgraph of B induced by its set of vertices. Hence if no proper induced subgraph of B is completely connected, the graph B contains a Hamiltonian circuit (because it must contain at least one circuit). Now let B contain at least one proper induced subgraph which is completely connected. From all such subgraphs we choose a subgraph B^* which is not a proper subgraph of another one. Let N_1^* (or N_2^*) be the intersection of the vertex set of B^* with N_1 (or N_2, respectively). As B^* is a completely connected graph, it is connected and $\Gamma_{B^*}(N_1^*) = N_2^*$. As B is completely connected, $|\Gamma_B(N_1^*)| > |N_1^*| = |N_2^*|$ and hence there exists at least one vertex of $N_2 - N_2^*$ adjacent to a vertex of N_1^* in B. Analogously $|\Gamma_B(N_1 - N_1^*)| > |N_1 - N_1^*| = |N_2 - N_2^*|$ and hence there exists at least one vertex of N_2 adjacent to a vertex of $N_1 - N_1^*$. Let U_1 be the set of all vertices of $N_1 - N_1^*$ which are adjacent to vertices of N_2^* and let U_2 be the set of all vertices of $N_2 - N_2^*$ which are adjacent to vertices of N_1^*. Suppose that each path in B connecting a vertex of U_1 with a vertex of U_2 contains a vertex of B^*. Then the subgraph of B induced by the set $(N_1 - N_1^*) \cup (N_2 - N_2^*)$ is disconnected and none of its connected components contains simultaneously a vertex of U_1 and a vertex of U_2. Let D be a connected component of this graph which does not contain a vertex of U_1, let P_1 (or P_2) be the intersection of its vertex set with N_1 (or N_2, respectively). Then $\Gamma_B(P_1) = P_2$. As B is completely connected, $|P_2| > |P_1|$. If $Q_1 = N_1 - (N_1^* \cup P_1)$, $Q_2 = N_2 - (N_2^* \cup P_2)$, then $|Q_1| > |Q_2|$. We have $\Gamma_B(N_1^* \cup Q_1) \subseteq N_2^* \cup Q_2$ and $|N_1^* \cup Q_1| > |N_2^* \cup Q_2|$, which is a contradiction. This implies that there exists a path C_0 connecting a vertex $u_1 \in U_1$ with a vertex $u_2 \in U_2$ which contains no vertex of B^*. Let B^{**} be the graph obtained from B^* by adding all vertices and edges of C_0, one edge joining u_1 with a vertex v_2 of N_2^* and one edge joining u_2 with a vertex v_1 of N_1^*. The graph B^{**} is completely connected according to Lemma 2; as B^* is its proper induced subgraph, the graph B^{**} is a factor of B with the described property.

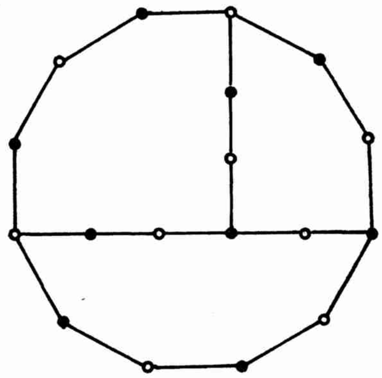
Now we prove a theorem.

Theorem 1. *Let $B = (N_1, N_2, H)$ be a critical completely connected bigraph. Then either B is a circuit, or there exists a critical completely connected bigraph $B^* = (N_1^*, N_2^*, H^*)$ and its vertices $v_1 \in N_1^*$, $v_2 \in N_2^*$ which satisfy one of the following conditions:*

(i) *The vertices v_1, v_2 are adjacent in B^* and B is obtained from B^* by deleting the edge v_1v_2 and connecting the vertices v_1, v_2 by a path of an odd length at least* 3 *whose inner vertices do not belong to B^*.*

(ii) *The vertices v_1, v_2 are not adjacent in B^*, the graph $\hat{B}^*$ obtained from B^* by adding the edge v_1v_2 ceases to be completely connected after deleting an arbitrary edge distinct from v_1v_2 and B is obtained from B^* by connecting the vertices v_1, v_2 by a path of an odd length at least 3 whose inner vertices do not belong to B^*.*

Proof. Let B_0 be a completely connected bigraph; we shall prove that it contains a factor B with the described properties. If B_0 contains a Hamiltonian circuit, then B is this circuit. If not, then according to Lemma 4 it contains a factor consisting of an induced completely connected proper subgraph B^* and of a path C of an odd length at least 3 connecting two vertices v_1, v_2 of B^* and with inner vertices not belonging to B^*. If v_1, v_2 are adjacent in B^*, find a critical completely connected factor B^*; if it contains v_1v_2, delete it. The union of this factor and C is the required factor B of B_0. If v_1, v_2 are not adjacent in B^*, add the edge v_1v_2 to B^* and denote the graph thus obtained from B^* by $\hat{B}^*$. Find a factor of $\hat{B}^*$ which is completely connected, contains v_1v_2 and ceases to be completely connected after deleting an arbitrary edge distinct from v_1v_2. (This can be done by successively deleting edges.) Then delete v_1v_2. The graph thus obtained from B_0 is the graph B. By Lemmas 3 and 4 such a graph B is critical completely connected. As an arbitrary completely connected bigraph B_0 contains such a factor, all critical completely connected bigraphs must have the described properties.



Thus a recursive characterization of critical completely connected bigraph is given. An example of such a bigraph is in Fig. 1; the vertices of N_1 are denoted by black dots, the vertices of N_2 by circles.

II.

A. PULTR presented the following problem:

We say that a (di)graph G is F-rigid (or A-rigid), if there exists no homomorphism (or isomorphism, respectively) $G \to G$ except the identical mapping. We say that