Časopis pro pěstování matematiky, roč. 106 (1981), Praha

AN ANALOGUE OF A RESULT OF CARATHÉODORY*)

HARRY I. MILLER, Sarajevo

(Received August 8, 1978)

1. INTRODUCTION AND PRELIMINARIES

C. CARATHÉODORY has shown [3] that there exists a Lebesgue measurable function f, $f : \mathsf{R} \to \mathsf{R}$ (R the set of real numbers) such that $m(\{x \in I;\ f(x) \in J\}) > 0$ for each non-empty open interval I and each set J of positive Lebesgue measure. Here m denotes Lebesgue measure. S. BERMAN [2] has arrived at the same result using probabilistic methods.

The purpose of this note is to prove an analogue of the above mentioned result of Carathéodory by considering sets of the second Baire category rather than sets of positive Lebesgue measure. Specifically we will prove that there exists a Lebesgue measurable function g, $g : \mathsf{R} \to \mathsf{R}$ such that $\{x \in I;\ g(x) \in J\}$ is a set of the second Baire category in R for each non-empty open interval I and each set J which is of the second category (in the sense of Baire) in R.

The current author [4], using techniques of A. ABIAN [1], has proved the following

Theorem A. *Let P be a set, $\bar{P} = \mathfrak{c}$ ($\mathfrak{c}$ an infinite cardinal). Suppose $\mathcal{Q}$ is a collection of at most $\mathfrak{c}$ subsets of P, each having $\mathfrak{c}$ elements. Suppose further that $\mathcal{B}$ is a collection of subsets of P such that if $B \subset P$ with $Q \cap B \neq \emptyset$, $\forall Q \in \mathcal{Q}$, then $B \in \mathcal{B}$. Then P can be written as the union of $\mathfrak{k}$-many disjoint sets in $\mathcal{B}$ for each cardinal number $\mathfrak{k}$, $\mathfrak{k} \leqq \mathfrak{c}$.*

Here, and later in our work, we identify every cardinal $\mathfrak{k}$ with the set of all ordinals preceding $\mathfrak{k}$ (i.e. all ordinals having cardinal number less than $\mathfrak{k}$). Thus $\mathfrak{k}$ is a well-ordered set.

The following corollary, an application of Theorem A, will be useful later in our work.

Corollary. *Let P be a set of the second Baire category in R, the real line. Let $\mathfrak{c}$ be the cardinal of the continuum and let $\mathfrak{k}$ be any positive cardinal such that $\mathfrak{k} \leqq \mathfrak{c}$.*

---

*) This research was supported by the Republican Council for Scientific Work of Bosna and Hercegovina.

Then P can be expressed as a union of $\mathfrak{k}$-many pairwise disjoint subsets B_j of P, where B_j is of the second Baire category in R *for every $j < k$.*

For the proof of this corollary see [4].

2. RESULTS

The following lemma will be essential to the construction of the function g mentioned in the introduction.

Lemma. *Let $\{I_n\}_{n=1}^{\infty}$ denote the collection of non-empty open intervals with rational endpoints. Then there exists a sequence $\{D_n\}_{n=1}^{\infty}$ of subsets of* R *with the following properties:*

1) *D_n is of the second Baire category for each n.*
2) *$m(D_n) = 0$ for each n.*
3) *$D_n \subset I_n$ for each n.*
4) *$D_n \cap D_m = \emptyset$ if $n \neq m$.*
5) *$(R \setminus \bigcup_{i=1}^{n} D_i) \cap J$ is of the second Baire category $\forall J$, a non-empty open interval, and $\forall n$.*

Proof. The sequence $\{D_n\}_{n=1}^{\infty}$ will be constructed inductively. First we construct D_1. Suppose $I_1 = (a_1, b_1)$, $a_1 < b_1$, a_1 and b_1 rational numbers. There exists a set $D_1^1 \subset I_1$ such that

$m(D_1^1) = 0$,

D_1^1 is of the second Baire category in R,

$I_1 \setminus D_1^1$ is of the first Baire category in R.

(See [5], page 4.) Notice that D_1^1 fails to satisfy property 5). We will use Theorem A to partition D_1^1 into the union of two sets D_2^1 and D_3^1 each satisfying the five given properties. To see this let

$$\mathscr{B} = \{B;\ B \subset D_1^1 \text{ and } B \cap J \text{ is of the second category } \forall J, \text{ a non-empty open sub-interval of } I_1\}$$

and

$$\mathscr{Q} = \{(R \setminus \bigcup_{i=1}^{\infty} F_i) \cap J \cap D_1^1, \text{ where each } F_i \text{ is closed and nowhere dense and } J \text{ is an open non-empty sub-interval of } I_1\}.$$

Then $\overline{\overline{D_1^1}} = \mathfrak{c}$ (the cardinal of the continuum), $\overline{\overline{\mathscr{Q}}} \leqq \mathfrak{c}$ and $\overline{\overline{Q}} = \mathfrak{c}$ for every $Q \in \mathscr{Q}$ (as each Q is of the second category due to the properties of D_1^1). Further, if $B \subset D_1^1$ and $B \cap Q \neq \emptyset$ for every $Q \in \mathscr{Q}$ then $B \in \mathscr{B}$. To see this suppose that $B \cap J$ is of the first category for some J (a non-empty open sub-interval of I_1). Then $B \cap J = \bigcup_{i=1}^{\infty} X_i$ (with each X_i nowhere dense) or $B \cap J \subset \bigcup_{i=1}^{\infty} \mathrm{Cl}(X_i)$, which implies $(B \cap J) \cap \cap (\mathsf{R} \setminus \bigcup_{i=1}^{\infty} F_i) = \emptyset$, where $F_i = \mathrm{Cl}(X_i)$ is closed and nowhere dense. This contradicts the fact that $B \cap Q \neq \emptyset$ for every $Q \in \mathcal{Q}$.

Therefore, by Theorem A with $\mathfrak{k} = 2$ we obtain $D_1^1 = D_2^1 \cup D_3^1$ with $D_2^1 \cap D_3^1 = \emptyset$, and $D_2^1, D_3^1 \in \mathcal{B}$ (i.e. D_2^1 and D_3^1 each have second category intersection with each non-empty open sub-interval of I_1).

Take $D_1 = D_2^1$. Then clearly D_1 satisfies 1), 2), 3), and 5). Suppose now that $D_1, D_2, \ldots, D_n$ $(n \geqq 1)$ satisfy properties 1), 2), 3), 4), and 5). We proceed to construct D_{n+1}. If $I_{n+1} = (a_{n+1}, b_{n+1})$, where $a_{n+1} < b_{n+1}$ and a_{n+1}, b_{n+1} are rational numbers, then there exists a set $D_1^{n+1} \subset I_{n+1}$ such that

$m(D_1^{n+1}) = 0$,

D_1^{n+1} is of the second Baire category in R,

$I_{n+1} \setminus D_1^{n+1}$ is of the first Baire category in R.

The set $(D_1^{n+1} \cap J) \cap (\mathsf{R} \setminus \bigcup_{i=1}^{n} D_i)$ is of the second Baire category for each non-empty open sub-interval J of I_{n+1}, since $(\mathsf{R} \setminus \bigcup_{i=1}^{n} D_i) \cap J$ and $(J \setminus D_1^{n+1}) \cap (\mathsf{R} \setminus \bigcup_{i=1}^{n} D_i)$ are respectively of the second and first Baire category for each non-empty open sub-interval J of I_{n+1}.

Set $D_2^{n+1} = D_1^{n+1} \cap (\mathsf{R} \setminus \bigcup_{i=1}^{n} D_i)$, then we have

$m(D_2^{n+1}) = 0$,

$D_2^{n+1} \subset I_{n+1}$,

$D_2^{n+1} \cap D_i = \emptyset, \ i \in \{1, 2, \ldots, n\}$,

$D_2^{n+1} \cap J$ is of the second Baire category for each non-empty open sub-interval J of I_{n+1}.

By the argument (using Theorem A) in the $n = 1$ part of the proof, D_2^{n+1} can be partitioned into the disjoint union of two sets.

D_3^{n+1} and D_4^{n+1}, such that

$D_i^{n+1} \cap J$ is of the second Baire category for each non-empty open sub-interval J of I_{n+1}, where $i = 3, 4$.

Set $D_{n+1} = D_3^{n+1}$, then $m(D_{n+1}) = 0$, D_{n+1} is of the second Baire category, $D_{n+1} \subset I_{n+1}$, and $D_{n+1} \cap D_i = \emptyset$ for every $i \in \{1, 2, \ldots, n\}$.

Furthermore, if J is a non-empty open interval, then

$$(\mathsf{R} \setminus \bigcup_{i=1}^{n+1} D_i) \cap J = (\mathsf{R} \setminus \bigcup_{i=1}^{n} D_i) \cap (R \setminus D_{n+1}) \cap J \supset$$