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DIFFERENTIAL EQUATIONS WITH INTERFACE CONDITIONS

ŠTEFAN SCHWABIK, Praha

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In this note, the theory of generalized linear differential equations will be applied to some linear systems of ordinary differential equations with interface conditions.

1. INTERFACE PROBLEMS AND GENERALIZED LINEAR DIFFERENTIAL EQUATIONS

Let us consider the ordinary linear differential system

$$\dot{\mathbf{x}} = \mathbf{F}(t)\mathbf{x} + \mathbf{g}(t), \quad t \in [0, 1] \tag{1.1}$$

where $\mathbf{F} : [0, 1] \to L(R_n)$ is an $n \times n$-matrix valued function and $\mathbf{g} : [0, 1] \to R_n$. Both $\mathbf{F}$ and $\mathbf{g}$ are assumed to be Lebesgue integrable on $[0, 1]$.

We consider the system (1.1) together with interface conditions

$$\mathbf{M}_j \mathbf{x}(t_j-) + \mathbf{N}_j \mathbf{x}(t_j+) = \mathbf{c}_j, \quad j = 1, 2, \ldots, k \tag{1.2}$$

where $0 = t_0 < t_1 < t_2 < \ldots < t_k < t_{k+1} = 1$, $\mathbf{M}_j, \mathbf{N}_j \in L(R_n, R_m)$ are real $m \times n$-matrices and $\mathbf{c}_j \in R_m$, $j = 1, \ldots, k$.

1.1. Definition. A function $\mathbf{x} : [a, b] \to R_n$, $[a, b] \subset [0, 1]$ is called a *solution of the interface problem* (1.1), (1.2) on $[a, b]$ if

a) $\mathbf{x}$ is absolutely continuous on every interval of the form $[a, b] \cap (t_{j-1}, t_j)$, $j = 1, \ldots, k + 1$,

b) $\mathbf{x}$ satisfies (1.1) almost everywhere in $[a, b]$,

c) for every $t_j \in (a, b)$, $j = 1, \ldots, k$ the interface condition (1.2) is satisfied.

Remark. Interface problems of this type are described and studied in various papers, see e.g. [1], [2], [3], [4], [7], [8]. The discontinuity of solutions of interface problems is caused by the nature of the conditions (1.2). The generalized linear differential equations have also the property that they admit discontinuous solutions, see e.g. [6]. This leads to the natural question about the connection between interface problems and generalized differential equations. The first question concerns the possibility of describing the interface problem (1.1), (1.2) by a generalized linear differential equation. If this is possible then we can ask for the counterpart of the results on generalized linear differential equations in the theory of interface problems.

We consider the interface problem (1.1), (1.2) for which the solution can be continued from the left to the right, i.e. we require that, given $\mathbf{x}(t-)$, it is possible to determine $\mathbf{x}(t+)$ (not uniquely in general). This requirement transfer to the interface conditions (1.2) in the sense that the solvability of the linear algebraic equation in $\mathbf{x}(t_j+)$

$$\mathbf{N}_j \mathbf{x}(t_j+) = \mathbf{c}_j - \mathbf{M}_j \mathbf{x}(t_j-)\,, \quad j = 1, \ldots, k$$

will be assumed for every given $\mathbf{x}(t_j-) \in R_n$. This means that we have to assume that

$$\mathbf{c}_j - \mathbf{M}_j \mathbf{x}^+ \in \mathrm{R}(\mathbf{N}_j)\,, \quad j = 1, \ldots, k$$

holds for any $\mathbf{x}^+ \in R_n$, where $\mathrm{R}(\mathbf{N}) \subset R_m$ stands for the range of an $m \times n$-matrix $\mathbf{N}$, i.e. $\mathrm{R}(\mathbf{N})$ is the linear span of all column vectors of the matrix $\mathbf{N}$. Hence we have the inclusion

$$\mathbf{c}_j + \mathrm{R}(\mathbf{M}_j) \subset \mathrm{R}(\mathbf{N}_j)\,, \quad j = 1, \ldots, k$$

which is clearly equivalent to

(1.3) $$\mathbf{c}_j \in \mathrm{R}(\mathbf{N}_j)\,, \quad j = 1, \ldots, k$$

(1.4) $$\mathrm{R}(\mathbf{M}_j) \subset \mathrm{R}(\mathbf{N}_j)\,, \quad j = 1, \ldots, k\,.$$

If the conditions (1.3) and (1.4) are not satisfied, then there exists a value $\mathbf{x}(t_j-)$ such that $\mathbf{x}(t_j+)$ cannot be determined in such a way that (1.2) holds.

1.1. Lemma. *The condition* (1.3) *holds if and only if there exist* $\mathbf{d}_j \in R_n$, $j = 1, \ldots, k$ *such that*

(1.5) $$\mathbf{N}_j \mathbf{d}_j = \mathbf{c}_j\,, \quad j = 1, \ldots, k\,.$$

The condition (1.4) *holds if and only if there exist* $\mathbf{D}_j \in L(R_n)$, $j = 1, \ldots, k$ *such that*

(1.6) $$\mathbf{M}_j + \mathbf{N}_j \mathbf{D}_j = \mathbf{0}\,, \quad j = 1, \ldots, k\,.$$

Proof. The first assertion is trivial. In order to prove the second statement, let us mention that the matrix equation $\mathbf{M}_j + \mathbf{N}_j \mathbf{X} = \mathbf{0}$ has a solution $\mathbf{X} \in L(R_n)$ if and only if rank $\mathbf{N}_j$ = rank $(\mathbf{N}_j, \mathbf{M}_j)$, i.e. every column of the matrix $\mathbf{M}_j$ depends linearly on the columns of the matrix $\mathbf{N}_j$. Hence the mentioned matrix equation has a solution $\mathbf{D}_j \in L(R_n)$ if and only if $\mathrm{R}(\mathbf{M}_j) \subset \mathrm{R}(\mathbf{N}_j)$.

In the sequel we assume that the interface conditions (1.2) satisfy (1.3) and (1.4) (or the equivalent conditions given by Lemma 1.1).

Remark. Let us mention that if the continuability conditions (1.3) and (1.4) are satisfied then for any $(\tilde{\mathbf{x}}, \tilde{t}) \in R_n \times [0, 1]$ and integrable $\mathbf{g} : [0, 1] \to R_n$ there exists a solution $\mathbf{x}(t)$ of the interface problem (1.1), (1.2) on the interval $[\tilde{t}, 1]$ such that $\mathbf{x}(\tilde{t}) = \tilde{\mathbf{x}}$. On the other hand if the conditions (1.3) and (1.4) are not satisfied then it is possible to show that there exists an initial point $(\tilde{\mathbf{x}}, \tilde{t}) \in R_n \times [0, 1]$ and an integrable function $\mathbf{g} : [0, 1] \to R_n$ such that the interface problem (1.1), (1.2) has no solution $\mathbf{x}$ defined on the whole interval $[\tilde{t}, 1]$ and such that $\mathbf{x}(\tilde{t}) = \tilde{\mathbf{x}}$.

Further, it is easy to see that if $m = n$ and $\det \mathbf{N}_j \neq 0$, $j = 1, \ldots, k$ then (1.3) and (1.4) are satisfied. According to Lemma 1.1 we have in this case $\mathbf{d}_j = \mathbf{N}_j^{-1}\mathbf{c}_j$ and $\mathbf{D}_j = -\mathbf{N}_j^{-1}\mathbf{M}_j$, $j = 1, \ldots, k$.

Let us now define, for $t \in [0, 1]$,

(1.7) $$\mathbf{A}(t) = \int_0^t \mathbf{F}(\tau)\, \mathrm{d}\tau + \sum_{j=1}^{k} (\mathbf{D}_j - \mathbf{I})\, \psi_{t_j}^+(t)$$

and

(1.8) $$\mathbf{f}(t) = \int_0^t \mathbf{g}(\tau)\, \mathrm{d}\tau + \sum_{j=1}^{k} \mathbf{d}_j\, \psi_{t_j}^+(t)$$

where $\psi_\alpha^+(t) = 0$ if $t \leqq \alpha$, $\psi_\alpha^+(t) = 1$ if $t > \alpha$, $\mathbf{d}_j$, $\mathbf{D}_j$, $j = 1, \ldots, k$ are determined by (1.5), (1.6), respectively, and $\mathbf{I}$ is the unit matrix in $L(R_n)$.

Evidently $\mathbf{A} : [0, 1] \to L(R_n)$, $\mathbf{f} : [0, 1] \to R_n$ are functions of bounded variation, left continuous on $[0, 1]$, i.e. $\Delta^- \mathbf{A}(t) = \mathbf{A}(t) - \mathbf{A}(t-) = \mathbf{0}$, $\Delta^- \mathbf{f}(t) = \mathbf{f}(t) - \mathbf{f}(t-) = \mathbf{0}$ for every $t \in (0, 1]$, $\Delta^+ \mathbf{A}(t) = \mathbf{A}(t+) - \mathbf{A}(t) = \mathbf{0}$, $\Delta^+ \mathbf{f}(t) = \mathbf{f}(t+) - \mathbf{f}(t) = \mathbf{0}$ for $t \in [0, 1)$, $t \neq t_j$ and

(1.9) $$\Delta^+ \mathbf{A}(t_j) = \mathbf{D}_j - \mathbf{I}\,, \quad \Delta^+ \mathbf{f}(t_j) = \mathbf{d}_j\,, \quad j = 1, \ldots, k\,.$$

We consider the generalized linear differential equation

(1.10) $$\mathrm{d}\mathbf{x} = \mathrm{d}[\mathbf{A}]\, \mathbf{x} + \mathrm{d}\mathbf{f}$$

(see [5], [6]). Let $\mathbf{x} : [a, b] \to R_n$, $[a, b] \subset [0, 1]$ be a solution of (1.10). By definition we have

(1.11) $$\mathbf{x}(\tau) = \mathbf{x}(\sigma) + \int_\sigma^\tau \mathrm{d}[\mathbf{A}(\varrho)]\, \mathbf{x}(\varrho) + \mathbf{f}(\tau) - \mathbf{f}(\sigma)$$

for every $\tau, \sigma \in [a, b]$ (the integral used in (1.11) is the Perron-Stieltjes integral). Using (1.7) and (1.8) we have by (1.11) for $\tau, \sigma \in [a, b] \cap [t_{j-1}, t_j]$, $j = 1, \ldots, k$

$$\mathbf{x}(\tau) = \mathbf{x}(\sigma) + \int_\sigma^\tau \mathbf{F}(\varrho)\, \mathbf{x}(\varrho)\, \mathrm{d}\varrho + \int_\sigma^\tau \mathbf{g}(\varrho)\, \mathrm{d}\varrho$$

and a straightforward argument shows that the solution $\mathbf{x} : [a, b] \to R_n$ of (1.10) satisfies a) and b) from Definition 1.1.

Using the results known for generalized linear differential equations (see [6], III.1) we have

$$\mathbf{x}(t-) = [\mathbf{I} - \Delta^- \mathbf{A}(t)]\,\mathbf{x}(t) - \Delta^- \mathbf{f}(t) = \mathbf{x}(t)\,, \quad t \in (a, b]$$

and

$$\mathbf{x}(t+) = [\mathbf{I} + \Delta^+ \mathbf{A}(t)]\,\mathbf{x}(t) + \Delta^+ \mathbf{f}(t)\,, \quad t \in [a, b)\,,$$

i.e. $\mathbf{x}(t+) = \mathbf{x}(t)$ if $t \in [a, b)$, $t \neq t_j$ and $\mathbf{x}(t_j+) = \mathbf{D}_j \mathbf{x}(t_j) + \mathbf{d}_j$ for $t_j \in [a, b)$. (The onesided limits of the solution $\mathbf{x}(t)$ exist, because every solution of (1.10) is of bounded variation.)

Hence by (1.5) and (1.6) we get

$$\mathbf{M}_j \mathbf{x}(t_j-) + \mathbf{N}_j \mathbf{x}(t_j) = \mathbf{M}_j \mathbf{x}(t_j) + \mathbf{N}_j(\mathbf{D}_j \mathbf{x}(t_j) + \mathbf{d}_j) =$$
$$= (\mathbf{M}_j + \mathbf{N}_j \mathbf{D}_j)\,\mathbf{x}(t_j) + \mathbf{N}_j \mathbf{d}_j = \mathbf{c}_j$$

for all $t_j \in (a, b)$, $j = 1, \ldots, k$ and the solution $\mathbf{x}$ of (1.10) satisfies also c) from Definition 1.1. In this way we can conclude that every solution of (1.10) is also a solution of the interface problem (1.1), (1.2).

Assume conversely that $\mathbf{x} : [a, b] \to R_n$ is a solution of the interface problem (1.1), (1.2) which is left continuous on $[a, b]$. Then we have

$$\mathbf{M}_j \mathbf{x}(t_j-) + \mathbf{N}_j \mathbf{x}(t_j+) = \mathbf{M}_j \mathbf{x}(t_j) + \mathbf{N}_j \mathbf{x}(t_j+) = \mathbf{c}_j = \mathbf{N}_j \mathbf{d}_j =$$
$$= (\mathbf{M}_j + \mathbf{N}_j \mathbf{D}_j)\,\mathbf{x}(t_j) + \mathbf{N}_j \mathbf{d}_j$$

for every $t_j \in (a, b)$, $j = 1, \ldots, k$, where $\mathbf{d}_j$, $\mathbf{D}_j$ are given in Lemma 1.1. Hence

$$\mathbf{N}_j\, \mathbf{x}(t_j\, +) = \mathbf{N}_j \mathbf{D}_j\, \mathbf{x}(t_j) + \mathbf{N}_j \mathbf{d}_j\,,$$

i.e.

$$\mathbf{x}(t_j+) - \mathbf{D}_j\, \mathbf{x}(t_j) - \mathbf{d}_j \in \mathrm{N}(\mathbf{N}_j)$$

where $\mathrm{N}(\mathbf{N}_j)$ denotes the null-space of the matrix $\mathbf{N}_j$. This yields for every $t_j \in (a, b)$ the equality

$$\mathbf{x}(t_j+) = \mathbf{D}_j\, \mathbf{x}(t_j) + \mathbf{d}_j + \mathbf{z}_j$$

with some $\mathbf{z}_j \in R_n$ such that $\mathbf{N}_j \mathbf{z}_j = \mathbf{0}$ ($\mathbf{z}_j \in \mathrm{N}(\mathbf{N}_j)$).

If we set $\tilde{\mathbf{d}}_j = \mathbf{d}_j + \mathbf{z}_j$, $j = 1, \ldots, k$ and define

$$\tilde{\mathbf{f}}(t) = \int_0^t \mathbf{g}(\tau)\, \mathrm{d}\tau + \sum_{j=1}^{k} \tilde{\mathbf{d}}_j\, \psi^+_{t_j}(t)\,, \quad t \in [a, b]$$

then it can be easily shown that for any $\sigma, \tau \in [a, b]$ we have

$$\mathbf{x}(\tau) - \mathbf{x}(\sigma) = \int_\sigma^\tau \mathbf{F}(\varrho)\, \mathbf{x}(\varrho)\, \mathrm{d}\varrho + \int_\sigma^\tau \mathbf{g}(\varrho)\, \mathrm{d}\varrho + \sum_{t_j \in [\sigma, \tau)} [(\mathbf{D}_j - \mathbf{I})\, \mathbf{x}(t_j) + \tilde{\mathbf{d}}_j] =$$
$$= \int_\sigma^\tau \mathrm{d}[\mathbf{A}(\varrho)]\, \mathbf{x}(\varrho) + \tilde{\mathbf{f}}(\tau) - \tilde{\mathbf{f}}(\sigma)$$

and consequently $\mathbf{x} : [a, b] \to R_n$ is a solution of the generalized linear differential equation

$$\mathrm{d}\mathbf{x} = \mathrm{d}[\mathbf{A}]\, \mathbf{x} + \mathrm{d}\tilde{\mathbf{f}}$$

which is of the form (1.10). The only difference is the form of $\tilde{\mathbf{f}}$ which differs from $\mathbf{f}$ given by (1.8) in the second term. In this way we have obtained the following

1.1. Theorem. *If the interface conditions* (1.2) *satisfy* (1.3) *and* (1.4) *then every solution of the generalized linear differential equation* (1.10) *with* $\mathbf{A}, \mathbf{f}$ *given by* (1.7), (1.8), *respectively, is a solution of the interface problem* (1.1), (1.2). *Conversely, every left continuous solution of the interface problem* (1.1), (1.2) *is a solution of* (1.10) *with* $\mathbf{A}, \mathbf{f}$ *given by* (1.7), (1.8) *where* $\mathbf{D}_j$, $\mathbf{d}_j$, $j = 1, \ldots, k$ *satisfy* (1.6), (1.5).

Remark. Let us mention that if the conditions (1.3) and (1.4) are satisfied then for every initial point $\tilde{\mathbf{x}} \in R_n$ and $\mathbf{g}$ integrable on $[0, 1]$ it is possible to construct a "train" composed of $k + 1$ pieces of Carathéodory solutions of the differential equation (1.1) on $[0, t_1]$, $[t_1, t_2]$, …, $[t_k, 1]$ in the sense of Conti [2] such that the solution $\mathbf{x}(t)$ on $[0, t_1]$ satisfies $\mathbf{x}(0) = \tilde{\mathbf{x}}$.

The left continuity of solutions of the interface problem (1.1), (1.2) is a requirement which can be easily satisfied for an arbitrary solution by changing its values at every point of discontinuity.

The class of generalized linear differential equations (1.10) corresponding to the problem (1.1), (1.2) depends on the null-space $\mathrm{N}(\mathbf{N}_j)$ of the matrices $\mathbf{N}_j$ (see Lemma 1.1 and the definition of $\mathbf{A}(t)$ and $\mathbf{f}(t)$ given in (1.7) and (1.8)). For example, the difference of any two functions $\mathbf{f}(t), \tilde{\mathbf{f}}(t)$ corresponding to the problem (1.1), (1.2) is of the form $\sum_{j=1}^{k} \mathbf{z}_j \psi^+_{t_j}(t)$ where $\mathbf{z}_j \in \mathrm{N}(\mathbf{N}_j)$; similarly for the matrices $\mathbf{A}(t)$ of the system (1.10).

1.2. Theorem. *Assume that the interface conditions satisfy* (1.3) *and* (1.4). *Let* $\mathbf{x}, \mathbf{y}$ *be two solutions of the interface problem* (1.1), (1.2) *defined on* $[a, b] \subset [0, 1]$, *left continuous on* $[a, b]$. *Then the difference* $\mathbf{z}(t) = \mathbf{x}(t) - \mathbf{y}(t)$, $t \in [a, b]$ *satisfies the generalized linear differential equation*

$$\mathrm{d}\mathbf{z} = \mathrm{d}[\mathbf{A}]\, \mathbf{z} + \mathrm{d}\mathbf{h} \tag{1.12}$$

where $\mathbf{A}$ *is given by* (1.7) *and* $\mathbf{h} : [0, 1] \to R_n$ *is a function of the form*

$$\mathbf{h}(t) = \sum_{j=1}^{k} \mathbf{z}_j \psi^+_{t_j}(t), \quad t \in [0, 1] \tag{1.13}$$

with $\mathbf{z}_j \in \mathrm{N}(\mathbf{N}_j)$, *i.e.* $\mathbf{N}_j \mathbf{z}_j = \mathbf{0}$, $j = 1, \ldots, k$.

Proof. Let $\mathbf{D}_j$, $j = 1, \ldots, k$ be given as in Lemma 1.1 and let $\mathbf{A}$ be defined by (1.7). By Theorem 1.1 the functions $\mathbf{x}, \mathbf{y}$ satisfy the equations $\mathrm{d}\mathbf{x} = \mathrm{d}[\mathbf{A}]\, \mathbf{x} + \mathrm{d}\mathbf{f}$, $\mathrm{d}\mathbf{y} = \mathrm{d}[\mathbf{A}]\, \mathbf{y} + \mathrm{d}\mathbf{f}^*$ where

$$\boldsymbol{f}(t) = \int_0^t \boldsymbol{g}(\tau)\,\mathrm{d}\tau + \sum_{j=1}^{k} \boldsymbol{d}_j\,\psi^+_{t_j}(t)\,, \quad \tilde{\boldsymbol{f}}(t) = \int_0^t \boldsymbol{g}(\tau)\,\mathrm{d}\tau + \sum_{j=1}^{k} \tilde{\boldsymbol{d}}_j\,\psi^+_{t_j}(t)\,,$$

$t \in [0, 1]$ with $\boldsymbol{N}_j \boldsymbol{d}_j = \boldsymbol{c}_j$, $\boldsymbol{N}_j \tilde{\boldsymbol{d}}_j = \boldsymbol{c}_j$, $j = 1, \ldots, k$. Hence $\boldsymbol{z}_j = \boldsymbol{d}_j - \tilde{\boldsymbol{d}}_j \in \mathrm{N}(\boldsymbol{N}_j)$, $j = 1, \ldots, k$ and

$$\boldsymbol{h}(t) = \boldsymbol{f}(t) - \tilde{\boldsymbol{f}}(t) = \sum_{j=1}^{k} \boldsymbol{z}_j\,\psi^t_{t_j}(t)\,, \quad t \in [0, 1]\,.$$

The difference $\boldsymbol{z}(t) = \boldsymbol{x}(t) - \boldsymbol{y}(t)$ evidently satisfies the equation

$$\mathrm{d}\boldsymbol{z} = \mathrm{d}[\boldsymbol{A}]\,\boldsymbol{z} + \mathrm{d}(\boldsymbol{f} - \tilde{\boldsymbol{f}})$$

and this yields the statement of the theorem.

1.1. Corollary. *If the conditions* (1.3), (1.4) *are satisfied and* $\boldsymbol{x} : [a, b] \to R_n$ *is a fixed left continuous solution of the interface problem* (1.1), (1.2) *on* $[a, b]$ *then an arbitrary left continuous solution of the interface problem* (1.1), (1.2) *is of the form* $\boldsymbol{x} + \boldsymbol{z}$, *where* $\boldsymbol{z} : [a, b] \to R_n$ *is a solution of the generalized linear differential equation* (1.12).

The proof of this statement easily follows from Theorems 1.1 and 1.2.

Let us now assume that instead of (1.4) the stronger condition

$$\mathrm{R}(\boldsymbol{M}_j) = \mathrm{R}(\boldsymbol{N}_j)\,, \quad j = 1, \ldots, k \tag{1.14}$$

is satisfied. This condition ensures the continuability of a solution of the interface problem from the left to the right as well as in the opposite direction.

1.2. Lemma. *Assume that* $\boldsymbol{M}$, $\boldsymbol{N}$ *are* $m \times n$*-matrices. Then the equality* $\mathrm{R}(\boldsymbol{M}) = \mathrm{R}(\boldsymbol{N})$ *holds for their ranges if and only if there exists a regular* $n \times n$*-matrix* $\boldsymbol{D}$ *such that*

$$\boldsymbol{M} + \boldsymbol{N}\boldsymbol{D} = \boldsymbol{0}\,. \tag{1.15}$$

Proof. Assume that $\mathrm{R}(\boldsymbol{M}) = \mathrm{R}(\boldsymbol{N})$. This is equivalent to the fact that the linear spans of the columns of $\boldsymbol{M}$ and $\boldsymbol{N}$ coincide. Let us write $\boldsymbol{M} = (\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n)$ and similarly $\boldsymbol{N} = (\boldsymbol{N}_1, \ldots, \boldsymbol{N}_n)$ where $\boldsymbol{M}_l$, $\boldsymbol{N}_l$ denote the l-th columns of $\boldsymbol{M}$, $\boldsymbol{N}$, respectively. It is known from linear algebra that there exists a regular $n \times n$-matrix $\boldsymbol{R}$ such that

$$\boldsymbol{M}\boldsymbol{R} = (\boldsymbol{M}_1, \ldots, \boldsymbol{M}_k, \boldsymbol{0}, \ldots, \boldsymbol{0})$$

where $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_k$ are linearly independent columns of $\boldsymbol{M}$, $k = \operatorname{rank} \boldsymbol{M}$ and similarly there is a regular $n \times n$-matrix $\boldsymbol{S}$ such that

$$\boldsymbol{N}\boldsymbol{S} = (\boldsymbol{N}_1, \ldots, \boldsymbol{N}_k, \boldsymbol{0}, \ldots, \boldsymbol{0})$$

where $\boldsymbol{N}_1, \boldsymbol{N}_2, \ldots, \boldsymbol{N}_k$ are linearly independent columns of $\boldsymbol{N}$, $k = \operatorname{rank} \boldsymbol{N} = \operatorname{rank} \boldsymbol{M}$. Since $\mathrm{R}(\boldsymbol{M}) = \mathrm{R}(\boldsymbol{N})$, the linear spans of the vectors $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_k$ and $\boldsymbol{N}_1, \ldots, \boldsymbol{N}_k$ are the same and consequently there is a regular $k \times k$-matrix $\boldsymbol{U}$ such that $(\boldsymbol{M}_1, \ldots, \boldsymbol{M}_k)\, \boldsymbol{U} = (\boldsymbol{N}_1, \ldots, \boldsymbol{N}_k)$. If we set

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{U}, & \boldsymbol{0} \\ \boldsymbol{0}, & \boldsymbol{I} \end{bmatrix}$$

then $\boldsymbol{MRT} = (\boldsymbol{M}_1, \ldots, \boldsymbol{M}_k, \boldsymbol{0}, \ldots, \boldsymbol{0})\, \boldsymbol{T} = (\boldsymbol{N}_1, \ldots, \boldsymbol{N}_k, \boldsymbol{0}, \ldots, \boldsymbol{0}) = \boldsymbol{NS}$ and $\boldsymbol{T}$ is a regular $n \times n$-matrix. If we set $\boldsymbol{D} = -\boldsymbol{ST}^{-1}\boldsymbol{R}^{-1}$ then $\boldsymbol{D}$ is a regular $n \times n$-matrix such such that $\boldsymbol{M} = -\boldsymbol{ND}$ and this proves the "only if" part of the lemma. The second implication is evident.

Lemma 1.2 yields immediately

1.3. Lemma. *The condition* (1.14) *holds if and only if there exist regular* $n \times n$-*matrices* $\boldsymbol{D}_j \in L(R_n)$ *such that*

$$\boldsymbol{M}_j + \boldsymbol{N}_j \boldsymbol{D}_j = \boldsymbol{0}\,, \quad j = 1, \ldots, k\,. \tag{1.16}$$

Let us now consider the generalized linear differential equations (1.10) with $\boldsymbol{A}(t)$ given by the relation (1.7) where the matrix $\boldsymbol{A}(t)$ is constructed by means of the regular $n \times n$-matrices $\boldsymbol{D}_j$, $j = 1, \ldots, k$ given by Lemma 1.3. In this case we have $\det(\boldsymbol{I} + \Delta^+\boldsymbol{A}(t)) \neq 0$ for every $t \in [0, 1)$ (see (1.9)) and also $\det(\boldsymbol{I} - \Delta^-\boldsymbol{A}(t)) = \det \boldsymbol{I} = 1$ for $t \in (0, 1]$. Hence by Theorem III.1.4 in [6], to every $(\tilde{\boldsymbol{x}}, \tilde{t}) \in R_n \times [0, 1]$ and $\boldsymbol{f} : [0, 1] \to R_n$ of bounded variation on $[0, 1]$ there exists a unique solution of (1.10) defined on $[0, 1]$ such that $\boldsymbol{x}(\tilde{t}) = \tilde{\boldsymbol{x}}$. This yields the following result.

1.4. Theorem. *If the interface conditions* (1.2) *satisfy* (1.3) *and* (1.14) *then the conclusions of theorem* 1.1 *hold. Moreover, to the interface problem* (1.1), (1.2) *there exists a generalized linear differential equation* (1.10) *which is uniquely solvable on the interval* $[0, 1]$ *for every initial point* $(\tilde{\boldsymbol{x}}, \tilde{t}) \in R_n \times [0, 1]$ *and every right hand side* $\boldsymbol{f}$ *of bounded variation on* $[0, 1]$.

Every solution of a generalized linear differential equation of the form (1.10), where the matrices $\boldsymbol{D}_j$, $j = 1, \ldots, k$ occuring in the definition (1.7) of the matrix $\boldsymbol{A}$ are regular can be given by the variation-of-constants formula (see III.2.14 in [6]), i.e.

$$\boldsymbol{x}(t) = \boldsymbol{X}(t)\, \boldsymbol{c} - \boldsymbol{X}(t) \int_0^t \mathrm{d}_s[\boldsymbol{X}^{-1}(s)]\, \boldsymbol{f}(s) + \boldsymbol{f}(t)\,, \quad t \in [0, 1] \tag{1.17}$$

where $\boldsymbol{X} : [0, 1] \to L(R_n)$ is the uniquely determined solution of the matrix equation

$$\boldsymbol{X}(t) = \boldsymbol{I} + \int_0^t \mathrm{d}[\boldsymbol{A}(r)]\, \boldsymbol{X}(r)\,, \quad t \in [0, 1] \tag{1.18}$$

called *the fundamental matrix* and $\boldsymbol{c} \in R_n$ is arbitrary.

For our purposes it is more convenient to have a formula for the solution using the conventional fundamental matrix of the differential equation (1.1).

Assume that $t \in (t_j, t_{j+1}]$, $j = 1, \ldots, k$. Then by (1.18) we have

$$\mathbf{X}(t) = \mathbf{X}(t_j) + \int_{t_j}^{t} \mathrm{d}[\mathbf{A}(r)]\, \mathbf{X}(r)$$

and taking into account the definition of $\mathbf{A}(t)$ from (1.7) we get

$$\mathbf{X}(t) = \mathbf{X}(t_j) + \int_{t_j}^{t} \mathrm{d}\left[\int_0^r \mathbf{F}(\varrho)\, \mathrm{d}\varrho + \sum_{j=1}^{k} (\mathbf{D}_j - \mathbf{I})\, \psi^+_{t_j}(r)\right] \mathbf{X}(r) =$$

$$= \mathbf{X}(t_j) + \int_{t_j}^{t} \mathbf{F}(r)\, \mathbf{X}(r)\, \mathrm{d}r + (\mathbf{D}_j - \mathbf{I})\, \mathbf{X}(t_j) =$$

$$= \mathbf{D}_j\, \mathbf{X}(t_j) + \int_{t_j}^{t} \mathbf{F}(r)\, \mathbf{X}(r)\, \mathrm{d}r\,.$$

Hence for $t \in (t_j, t_{j+1}]$, $j = 1, \ldots, k$ the fundamental matrix $\mathbf{X}(t)$ satisfies

$$\mathbf{X}(t) = \mathbf{U}(t, t_j)\, \mathbf{D}_j\, \mathbf{X}(t_j) \tag{1.19}$$

where $\mathbf{U}(t, \tau)$ is the fundamental matrix corresponding to the equation (1.1), i.e. for every $t, \tau \in [0, 1]$ we have

$$\mathbf{U}(t, \tau) = \mathbf{I} + \int_{\tau}^{t} \mathbf{F}(r)\, \mathbf{U}(r, \tau)\, \mathrm{d}r\,.$$

Using (1.19) we have

$$\begin{aligned} \mathbf{X}(t) &= \mathbf{U}(t, 0) \quad \text{for} \quad t \in [0, t_1]\,, \\ \mathbf{X}(t) &= \mathbf{U}(t, t_j)\, \mathbf{D}_j\, \mathbf{X}(t_j) = \\ &= \mathbf{U}(t, t_j)\, \mathbf{D}_j\, \mathbf{U}(t_j, t_{j-1})\, \mathbf{D}_{j-1} \ldots \mathbf{D}_1\, \mathbf{U}(t_1, 0) \quad \text{for} \quad t \in (t_j, t_{j+1}]\,. \end{aligned} \tag{1.20}$$

Integrating by parts in the integral in (1.17) (see I.4.33 in [6]) we get

$$\mathbf{x}(t) = \mathbf{X}(t)\, \mathbf{c} + \mathbf{f}(t) - \mathbf{X}(t)\left[-\int_0^t \mathbf{X}^{-1}(s)\, \mathrm{d}\mathbf{f}(s) + \mathbf{X}^{-1}(t)\, \mathbf{f}(t) - \mathbf{X}^{-1}(0)\, \mathbf{f}(0) - \right.$$

$$\left. - \sum_{0 \leqq \tau < t} \Delta^+ \mathbf{X}^{-1}(\tau)\, \Delta^+ \mathbf{f}(\tau)\right] =$$

$$= \mathbf{X}(t)\, \mathbf{c} + \mathbf{X}(t) \int_0^t \mathbf{X}^{-1}(s)\, \mathrm{d}\mathbf{f}(s) + \mathbf{X}(t) \sum_{0 \leqq \tau < t} \Delta^+ \mathbf{X}^{-1}(\tau)\, \Delta^+ \mathbf{f}(\tau)\,.$$

The definition (1.8) of $\mathbf{f}$ yields further

$$\mathbf{x}(t) = \mathbf{X}(t)\, \mathbf{c} + \mathbf{X}(t) \int_0^t \mathbf{X}^{-1}(s)\, \mathbf{g}(s)\, \mathrm{d}s + \mathbf{X}(t) \sum_{0 \leqq \tau < t} (\mathbf{X}^{-1}(\tau)\, \Delta^+ \mathbf{f}(\tau) + \tag{1.21}$$

$$+ \Delta^+ \mathbf{X}^{-1}(\tau)\, \Delta^+ \mathbf{f}(\tau)) =$$

$$= \mathbf{X}(t)\, \mathbf{c} + \mathbf{X}(t) \int_0^t \mathbf{X}^{-1}(s)\, \mathbf{g}(s)\, \mathrm{d}s + \mathbf{X}(t) \sum_{0 \leqq \tau < t} \mathbf{X}^{-1}(\tau +)\, \Delta^+ \mathbf{f}(\tau) =$$

$$= \mathbf{X}(t)\,\mathbf{c} + \mathbf{X}(t) \int_0^t \mathbf{X}^{-1}(s)\,\mathbf{g}(s)\,\mathrm{d}s + \mathbf{X}(t) \sum_{0<t_j<t} \mathbf{X}^{-1}(t_j)\,\mathbf{D}_j^{-1}\mathbf{d}_j$$

because $\Delta^+ \mathbf{f}(\tau) = \mathbf{0}$ for $\tau \neq t_j$, $j = 1, \ldots, k$, $\Delta^+ \mathbf{f}(t_j) = \mathbf{d}_j$ and $\mathbf{X}(t_j+) = \mathbf{D}_j\,\mathbf{X}(t_j)$, $j = 1, \ldots, k$ (see (1.19)).

It is a matter of routine to insert (1.20) into (1.21) and to derive the following result, which describes the solution of (1.10) in terms of the interface problem (1.1), (1.2).

1.4. Theorem. *If the $n \times n$-matrices $\mathbf{D}_j$, $j = 1, \ldots, k$ occuring in the definition (1.7) of $\mathbf{A}(t)$ are regular, then every solution $\mathbf{x} : [0, 1] \to R_n$ of the generalized linear differential equation (1.10) is given by the formula ($\mathbf{c} \in R_n$ is arbitrary)*

$$\text{(1.22)} \qquad \mathbf{x}(t) = \mathbf{U}(t, 0)\,\mathbf{c} + \mathbf{U}(t, 0) \int_0^t \mathbf{U}(0, s)\,\mathbf{g}(s)\,\mathrm{d}s \quad \textit{for} \quad t \in [0, t_1]\,,$$

$$\mathbf{x}(t) = \mathbf{U}(t, t_j)\,\mathbf{D}_j\,\mathbf{U}(t_j, t_{j-1})\,\mathbf{D}_{j-1} \ldots \mathbf{D}_1\,\mathbf{U}(t_1, 0)\left(\mathbf{c} + \int_0^{t_1} \mathbf{U}(0, s)\,\mathbf{g}(s)\,\mathrm{d}s\right) +$$

$$+ \sum_{l=1}^{j-1} \mathbf{U}(t, t_j)\,\mathbf{D}_j\,\mathbf{U}(t_j, t_{j-1}) \ldots \mathbf{D}_{l+1}\,\mathbf{U}(t_{l+1}, t_l)\left(\mathbf{d}_l + \int_{t_l}^{t_{l+1}} \mathbf{U}(t_l, s)\,\mathbf{g}(s)\,\mathrm{d}s\right) +$$

$$+ \mathbf{U}(t, t_j)\left(\mathbf{d}_j + \int_{t_j}^t \mathbf{U}(t_j, s)\,\mathbf{g}(s)\,\mathrm{d}s\right) \quad \textit{for} \quad t \in (t_j, t_{j+1}]\,, j = 1, \ldots, k \;\; (t_{k+1} = 1)\,.$$

1.5. Theorem. *If the conditions (1.3) and (1.14) are satisfied for the interface conditions (1.2) then the formula (1.22) yields a left continuous solution of the interface problem (1.1), (1.2) provided $\mathbf{D}_j$, $j = 1, \ldots, k$ are the regular $n \times n$-matrices from Lemma 1.3. Moreover, every left continuous solution of the interface problem (1.1), (1.2) can be written in the form*

$$\text{(1.23)} \qquad \mathbf{x}(t) = \mathbf{U}(t, 0)\,\mathbf{c} + \mathbf{U}(t, 0) \int_0^t \mathbf{U}(0, s)\,\mathbf{g}(s)\,\mathrm{d}s \quad \textit{for} \quad t \in [0, t_1]\,,$$

$$\mathbf{x}(t) = \mathbf{U}(t, t_j)\,\mathbf{D}_j\,\mathbf{U}(t_j, t_{j-1})\,\mathbf{D}_{j-1} \ldots \mathbf{D}_1\,\mathbf{U}(t_1, 0)\left(\mathbf{c} + \int_0^{t_1} \mathbf{U}(0, s)\,\mathbf{g}(s)\,\mathrm{d}s\right) +$$

$$+ \sum_{l=1}^{j-1} \mathbf{U}(t, t_j)\,\mathbf{D}_j\,\mathbf{U}(t_j, t_{j-1}) \ldots \mathbf{D}_{l+1}\,\mathbf{U}(t_{l+1}, t_l)\left(\mathbf{d}_l + \mathbf{z}_l + \int_{t_l}^{t_{l+1}} \mathbf{U}(t_l, s)\,\mathbf{g}(s)\,\mathrm{d}s\right) +$$

$$+ \mathbf{U}(t, t_j)\left(\mathbf{d}_j + \mathbf{z}_j + \int_{t_j}^t \mathbf{U}(t_j, s)\,\mathbf{g}(s)\,\mathrm{d}s\right) \quad \textit{for} \quad t \in (t_j, t_{j+1}]\, j = 1, \ldots, k\,,$$

where $\mathbf{c} \in R_n$ is arbitrary, $\mathbf{d}_j \in R_n$ are such that $\mathbf{N}_j\mathbf{d}_j = \mathbf{c}_j$ and $\mathbf{z}_j \in \mathrm{N}(\mathbf{N}_j)$, $j = 1, \ldots, k$.

Proof. By Theorem 1.3 the function given by the formula (1.22) in Theorem 1.4 is a solution of the interface problem (1.1), (1.2). By Corollary 1.1 every left continuous solution of the interface problem (1.1), (1.2) can be expressed in the form of the sum of the function from (1.22) and an arbitrary solution $\mathbf{z}$ of the equation

(1.12). Using the variation-of-constants formula for the equation (1.12) we can evaluate the function $\boldsymbol{z}$ (the procedure is the same as for the solution of the equation (1.10) in Theorem 1.4):

$$\boldsymbol{z}(t) = \boldsymbol{U}(t, 0)\, \boldsymbol{c} \quad \text{for} \quad t \in [0, t_1],$$

$$\boldsymbol{z}(t) = \boldsymbol{U}(t, t_j)\, \boldsymbol{D}_j\, \boldsymbol{U}(t_j, t_{j-1})\, \boldsymbol{D}_{j-1} \dots \boldsymbol{D}_1\, \boldsymbol{U}(t_1, 0)\, \boldsymbol{c} + \\ + \sum_{l=1}^{j-1} \boldsymbol{U}(t, t_j)\, \boldsymbol{D}_j\, \boldsymbol{U}(t_j, t_{j-1}) \dots \boldsymbol{D}_{l+1}\, \boldsymbol{U}(t_{l+1}, t_l)\, \boldsymbol{z}_l + \\ + \boldsymbol{U}(t, t_j)\, \boldsymbol{z}_j \quad \text{for} \quad t \in (t_j, t_{j+1}], \quad j = 1, \dots, k$$

where $\boldsymbol{z}_j \in \mathrm{N}(\boldsymbol{N}_j)$, $j = 1, \dots, k$ and $\boldsymbol{c} \in R_n$ is arbitrary. Adding the function $\boldsymbol{z}$ to the function from (1.22) we get (1.23).

Remark. Let us assume in addition that the null-spaces of the matrices $\boldsymbol{N}_j$ satisfy

$$\mathrm{N}(\boldsymbol{N}_j) = \{\boldsymbol{0}\}, \quad j = 1, \dots, k. \tag{1.24}$$

Then the matrices $\boldsymbol{D}_j$, $j = 1, \dots, k$ are uniquely determined by (1.16). Indeed, if we have $\boldsymbol{M}_j + \boldsymbol{N}_j\boldsymbol{D}_j = \boldsymbol{0}$ as well as $\boldsymbol{M}_j + \boldsymbol{N}_j\tilde{\boldsymbol{D}}_j = \boldsymbol{0}$ for some j, then $\boldsymbol{N}_j(\boldsymbol{D}_j - \tilde{\boldsymbol{D}}_j) = \boldsymbol{0}$ and consequently $\boldsymbol{D}_j = \tilde{\boldsymbol{D}}_j$. Hence also the matrix-valued function $\boldsymbol{A}(t)$ is uniquely determined by (1.7). Moreover, to every $\boldsymbol{c}_j \in \mathrm{R}(\boldsymbol{N}_j)$ there is a uniquely determined $\boldsymbol{d}_j$ such that $\boldsymbol{N}_j\boldsymbol{d}_j = \boldsymbol{c}_j$ and consequently also the function $\boldsymbol{f}(t)$ is given uniquely by (1.8).

If the interface conditions (1.2) are such that (1.3), (1.14), (1.24) hold then evidently there is a one-to-one correspondence between the interface problem (1.1), (1.2) and the generalized linear differential equation (1.10) with $\boldsymbol{A}, \boldsymbol{f}$ given by (1.7), (1.8), respectively.

The situation when (1.3), (1.14), (1.24) are satisfied occurs for instance, when $m = n$, $\det \boldsymbol{N}_j \neq 0$, $\det \boldsymbol{M}_j \neq 0$, $j = 1, \dots, k$. Of this type are the so called "shock conditions" $\boldsymbol{x}(t_j-) - \boldsymbol{x}(t_j+) = \boldsymbol{c}_j$, $j = 1, \dots, k$ or conditions of the form $\boldsymbol{x}(t_j-) + \boldsymbol{N}_j\, \boldsymbol{x}(t_j+) = \boldsymbol{c}_j$ with $\det \boldsymbol{N}_j \neq 0$, $j = 1, \dots, k$.

2. BOUNDARY VALUE PROBLEMS WITH INTERFACE CONDITIONS

In this section we turn our attention to the two-point boundary value problem for interface problems, i.e. we consider the system

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(t)\, \boldsymbol{x} + \boldsymbol{g}(t), \quad t \in [0, 1], \tag{2.1}$$

$$\boldsymbol{M}_j\, \boldsymbol{x}(t_j) + \boldsymbol{N}_j\, \boldsymbol{x}(t_j+) = \boldsymbol{c}_j, \quad j = 1, \dots, k, \tag{2.2}$$

$$\boldsymbol{M}\, \boldsymbol{x}(0) + \boldsymbol{N}\, \boldsymbol{x}(1) = \boldsymbol{r} \tag{2.3}$$

with $\boldsymbol{F}$, $\boldsymbol{g}$, $\boldsymbol{M}_j$, $\boldsymbol{N}_j$, $\boldsymbol{c}_j$, $j = 1, \dots, k$ given in Section 1 and $\boldsymbol{M}, \boldsymbol{N} \in L(R_n, R_m)$, $\boldsymbol{r} \in R_m$. Let us assume that the interface conditions (2.2) satisfy (1.3), (1.14) and (1.24), i.e. $\boldsymbol{c}_j \in \mathrm{R}(\boldsymbol{N}_j)$, $\mathrm{R}(\boldsymbol{M}_j) = \mathrm{R}(\boldsymbol{N}_j)$, $\mathrm{N}(\boldsymbol{N}_j) = \{\boldsymbol{0}\}$, $j = 1, \dots, k$.

Under these conditions it is possible to associate with (2.1), (2.2) a generalized linear differential equation

$$\mathrm{d}\mathbf{x} = \mathrm{d}[\mathbf{A}]\,\mathbf{x} + \mathrm{d}\mathbf{f} \tag{2.4}$$

which is uniquely solvable for every initial point $(\tilde{\mathbf{x}}, \tilde{t}) \in R_n \times [0, 1]$. (See Theorem 1.3.) We consider the boundary value problem (2.4), (2.3). For problems of this type some results are known (see [5], [6]). Our aim is to modify these results to the problem (2.1), (2.2), (2.3).

Let us define

$$\mathbf{B}(t) = \int_0^t \mathbf{F}(\tau)\,\mathrm{d}\tau + \sum_{j=1}^{k} (\mathbf{D}_j - \mathbf{I})\,\psi^-_{t_j}(t)\,, \quad t \in [0, 1] \tag{2.5}$$

where $\psi^-_\alpha(t) = 0$ if $t < \alpha$, $\psi^-_\alpha(t) = 1$ if $t \geqq \alpha$, $\mathbf{D}_j$, $j = 1, \dots, k$ are determined by (1.6) and are unique since (1.24) is assumed, (1.14) implies the regularity of $\mathbf{D}_j$ $j = 1, \dots, k$. We have evidently $\mathbf{B}(t+) - \mathbf{A}(t+) = \mathbf{B}(t-) - \mathbf{A}(t-) = \mathbf{0}$ for all $t \in (0, 1)$, $\mathbf{A}(0) = \mathbf{B}(0)$, $\mathbf{A}(1) = \mathbf{B}(1)$, $\Delta^+\mathbf{B}(t)\,\Delta^+\mathbf{A}(t) = \Delta^-\mathbf{B}(t)\,\Delta^-\mathbf{A}(t)$ for all $t \in (0, 1)$ and

$$\det(\mathbf{I} + \Delta^+\mathbf{A}(t))\det(\mathbf{I} + \Delta^-\mathbf{B}(t))\det(\mathbf{I} - \Delta^-\mathbf{A}(t)) \neq 0$$

since $\mathbf{I} + \Delta^+\mathbf{A}(t_j) = \mathbf{I} + \Delta^-\mathbf{B}(t_j) = \mathbf{D}_j$, $j = 1, \dots, k$ and $\mathbf{D}_j$ are regular $n \times n$-matrices, $\Delta^-\mathbf{A}(t) = \mathbf{0}$ for all $t \in (0, 1]$ and $\Delta^+\mathbf{A}(t) = \Delta^-\mathbf{B}(t) = \mathbf{0}$ for $t \in (0, 1)$, $t \neq t_j$, $j = 1, \dots, k$. The matrix valued function $\mathbf{A}$ is given by (1.7) in Section 1.

Using the results from [6] (Theorem III.5.5) we obtain the following result:

The boundary value problem (2.4), (2.3) possesses a solution if and only if

$$\mathbf{y}^*(1)\,\mathbf{f}(1) - \mathbf{y}^*(0)\,\mathbf{f}(0) - \int_0^1 \mathrm{d}[\mathbf{y}^*(t)]\,\mathbf{f}(t) = \boldsymbol{\lambda}^*\mathbf{r} \tag{2.6}$$

for any solution $(\mathbf{y}, \boldsymbol{\lambda})$ of the homogeneous system

$$\mathrm{d}\mathbf{y} = -\mathrm{d}[\mathbf{B}^*]\,\mathbf{y}\,, \tag{2.7}$$

$$\mathbf{y}(0) + \mathbf{M}^*\boldsymbol{\lambda} = \mathbf{0}\,, \quad \mathbf{y}(1) - \mathbf{N}^*\boldsymbol{\lambda} = \mathbf{0} \tag{2.8}$$

(a star denotes the transpose to a matrix).

The properties of $\mathbf{B} : [0, 1] \to L(R_n)$ ensure that for every $(\tilde{\mathbf{y}}, \tilde{t}) \in R_n \times [0, 1]$ the equation (2.7) has a uniquely determined solution $\mathbf{y} : [0, 1] \to R_n$ satisfying $\mathbf{y}(\tilde{t}) = \tilde{\mathbf{y}}$.

Let us consider a solution $\mathbf{y} : [0, 1] \to R_n$ of (2.7). Using the results of III.1 in [6] we have

$$\mathbf{y}(t+) = [\mathbf{I} + \Delta^+(-\mathbf{B}^*(t))]\,\mathbf{y}(t) = (\mathbf{I} - \Delta^+\mathbf{B}^*(t))\,\mathbf{y}(t) = \mathbf{y}(t)\,, \quad t \in [0, 1)$$

and

$$\mathbf{y}(t-) = [\mathbf{I} - \Delta^-(-\mathbf{B}^*(t))]\,\mathbf{y}(t) = (\mathbf{I} + \Delta^-\mathbf{B}^*(t))\,\mathbf{y}(t) = \mathbf{y}(t)$$

for $t \in (0, 1]$, $t \neq t_j$, $j = 1, \dots, k$,

$$\mathbf{y}(t_j-) = (\mathbf{I} + \Delta^-\mathbf{B}^*(t_j))\,\mathbf{y}(t_j) = \mathbf{D}_j^*\,\mathbf{y}(t_j)\,, \quad j = 1, \dots, k\,,$$

i.e.

$$\mathbf{y}(t_j-) - \mathbf{D}_j^* \mathbf{y}(t_j) = \mathbf{0}.$$

Further it is evident that $\mathbf{y}$ is absolutely continuous on every interval of the form (t_{j-1}, t_j), $j = 1, \ldots, k+1$ and $\mathbf{y}$ satisfies the ordinary differential equation

$$\dot{\mathbf{y}} = -\mathbf{F}^*(t)\,\mathbf{y}$$

almost everywhere in $[0, 1]$. This follows from the fact that we have by definition

$$\mathbf{x}(\tau) = \mathbf{x}(\sigma) + \int_\sigma^\tau \mathrm{d}[-\mathbf{B}^*(\varrho)]\,\mathbf{y}(\varrho) = \mathbf{x}(\sigma) - \int_\sigma^\tau \mathbf{F}^*(\varrho)\,\mathbf{y}(\varrho)\,\mathrm{d}\varrho$$

for any $\tau, \sigma \in (t_{j-1}, t_j)$, $j = 1, \ldots, k$. Hence every solution $\mathbf{y} : [0, 1] \to R_n$ of (2.7) is a solution of the interface problem

(2.9) $$\dot{\mathbf{y}} = -\mathbf{F}^*(t)\,\mathbf{y},$$

(2.10) $$\mathbf{y}(t_j-) - \mathbf{D}_j^* \mathbf{y}(t_j) = \mathbf{0}.$$

It is easy to check that conversely every solution of the interface problem (2.9), (2.10) is a solution of (2.7).

Taking this fact into account we reformulate the solvability condition (2.6) as follows.

2.1. Theorem. *Assume that the interface conditions* (2.2) *satisfy* (1.3), (1.14) *and* (1.24). *Then the boundary value problem* (2.1), (2.2), (2.3) *has a solution if and only if*

(2.11) $$\int_0^1 \mathbf{y}^*(t)\,\mathbf{g}(t)\,\mathrm{d}t + \sum_{j=1}^k \mathbf{y}^*(t_j)\,\mathbf{d}_j = \boldsymbol{\lambda}^*\mathbf{r}$$

for any solution $(\mathbf{y}, \boldsymbol{\lambda})$ *of the homogeneous problem* (2.9), (2.10) *with the parametric boundary conditions*

(2.12) $$\mathbf{y}(0) = -\mathbf{M}^*\boldsymbol{\lambda}, \quad \mathbf{y}(1) = \mathbf{N}^*\boldsymbol{\lambda}.$$

Proof. Using the integration-by-parts formula (see Theorem I.4.33 in [6]) and taking into account the form of $\mathbf{f}$ given in (1.8) we have

$$\mathbf{y}^*(1)\,\mathbf{f}(1) - \mathbf{y}^*(0)\,\mathbf{f}(0) - \int_0^1 \mathrm{d}[\mathbf{y}^*(t)]\,\mathbf{f}(t) = \int_0^1 \mathbf{y}^*(t)\,\mathrm{d}[\mathbf{f}(t)] =$$
$$= \int_0^1 \mathbf{y}^*(t)\,\mathrm{d}\left[\int_0^t \mathbf{g}(\tau)\,\mathrm{d}\tau + \sum_{j=1}^k \mathbf{d}_j\,\psi_{t_j}^+(t)\right] = \int_0^1 \mathbf{y}^*(t)\,\mathbf{g}(t)\,\mathrm{d}t + \sum_{j=1}^k \mathbf{y}^*(t_j)\,\mathbf{d}_j.$$

This together with (2.6) yields (2.11).

Remark. The parametric boundary value problem (2.9), (2.10), (2.12) plays the role of an adjoint problem to the problem (2.1), (2.2), (2.3).

If $0 \leqq m \leqq 2n$ and rank $\mathbf{N}$ = rank $(\mathbf{M}, \mathbf{N}) = m$ then there exist uniquely determined matrices $\mathbf{P}, \mathbf{Q} \in L(R_{2n-m}, R_n)$ and $\mathbf{P}^c, \mathbf{Q}^c \in L(R_m, R_n)$ such that

$$\det \begin{bmatrix} \mathbf{P}^c, & \mathbf{P} \\ \mathbf{Q}^c, & \mathbf{Q} \end{bmatrix} = 0$$

and $-\mathbf{MP}^c + \mathbf{NQ}^c = \mathbf{I}$, $-\mathbf{MP} + \mathbf{NQ} = \mathbf{0}$; $\mathbf{P}, \mathbf{Q}$ are the so called adjoint matrices associated with $[\mathbf{M}, \mathbf{N}]$ and $\mathbf{P}^c, \mathbf{Q}^c$ the complementary adjoint matrices associated with $[\mathbf{M}, \mathbf{N}]$. Using this concepts and the results from III.5.18 in [6] we obtain the following theorem.

2.2. Theorem. *Let the assumptions of Theorem* 2.1 *be fulfilled. Then the boundary value problem* (2.1), (2.2), (2.3) *has a solution if and only if*

$$\int_0^1 \mathbf{y}^*(t)\, \mathbf{g}(t)\, \mathrm{d}t + \sum_{j=1}^{k} \mathbf{y}^*(t_j)\, \mathbf{d}_j = [\mathbf{y}^*(0)\, \mathbf{P}^c + \mathbf{y}^*(1)\, \mathbf{Q}^c]\, \mathbf{r} \tag{2.13}$$

for any solution of the system (2.9), (2.10) *with the homogeneous boundary condition*

$$\mathbf{P}^*\, \mathbf{y}(0) + \mathbf{Q}^*\, \mathbf{y}(1) = \mathbf{0} \tag{2.14}$$

where $\mathbf{P}, \mathbf{Q}, \mathbf{P}^c, \mathbf{Q}^c$ *are the adjoint and the complementary adjoint matrices associated with* $[\mathbf{M}, \mathbf{N}]$.

The interface problem (2.9), (2.10) together with the boundary condition (2.14) represents the nonparametric form of the adjoint problem to (2.1), (2.2), (2.3).

Let us now assume that for the interface conditions (2.2) the assumption (1.24) is not satisfied, i.e. $\mathrm{N}(\mathbf{N}_j) \neq \{\mathbf{0}\}$ for some $j = 1, \ldots, k$. In this case, to the interface problem (2.1), (2.2) there is a variety of generalized linear differential equations of the form

$$\mathrm{d}\mathbf{x} = \mathrm{d}[\mathbf{A}]\, \mathbf{x} + \mathrm{d}(\mathbf{f} + \mathbf{h}) \tag{2.15}$$

where $\mathbf{h}(t) = \sum_{j=1}^{k} \mathbf{z}_j\, \psi_{t_j}^{+}(t)$, $t \in [0, 1]$ and $\mathbf{z}_j \in \mathrm{N}(\mathbf{N}_j)$, $j = 1, \ldots, k$ are arbitrary. In this case every solution of the interface problem (2.1), (2.2) is also a solution of an equation of the form (2.15) (see Theorem 1.2).

This leads to

2.3. Theorem. *Assume that the interface conditions* (2.2) *satisfy* (1.3) *and* (1.14). *Then the boundary value problem* (2.1), (2.2), (2.3) *has a solution if and only if there exist* $\mathbf{z}_j \in \mathrm{N}(\mathbf{N}_j)$, $j = 1, \ldots, k$ *such that*

$$\int_0^1 \mathbf{y}^*(t)\, \mathbf{g}(t)\, \mathrm{d}t + \sum_{j=1}^{k} \mathbf{y}^*(t_j)\, (\mathbf{d}_j + \mathbf{z}_j) = \boldsymbol{\lambda}^* \mathbf{r} \tag{2.16}$$

for any solution $(\mathbf{y}, \boldsymbol{\lambda})$ *of the homogeneous problem* (2.9), (2.10) *with the boundary conditions* (2.12).

Remark. A similar result can be derived if the nonparametric adjoint problem (2.9), (2.10), (2.14) is used (see Theorem 2.2). The authors in [5] use a different approach to adjoint equations for the generalized linear differential equation (2.4). The adjoint problem to (2.1), (2.2), (2.3) in the sense of [5] has the form (see Remark 3.9 in [5])

$$\mathrm{d}\mathbf{z} + \mathrm{d}[\mathbf{A}^*\mathbf{z}] - \mathbf{A}\,\mathrm{d}[\mathbf{z}] = \mathbf{0} \tag{2.17}$$

$$\mathbf{z}(0) = -\mathbf{M}^*\boldsymbol{\lambda}\,, \quad \mathbf{z}(1) = \mathbf{N}^*\boldsymbol{\lambda} \tag{2.18}$$

where (2.17) stands for the equation

$$\mathbf{z}(t) - \mathbf{z}(s) + \mathbf{A}^*(t)\,\mathbf{z}(t) - \mathbf{A}^*(s)\,\mathbf{z}(s) - \int_s^t \mathbf{A}^*(r)\,\mathrm{d}\mathbf{z}(r) = \mathbf{0} \tag{2.19}$$

which has to be satisfied for every $s, t \in [0, 1]$ provided $\mathbf{z} : [0, 1] \to R_n$ is a solution of (2.17).

It can be shown by integration by parts that

$$\int_s^t \mathbf{A}^*(r)\,\mathrm{d}[\mathbf{z}(r)] = -\int_s^t \mathrm{d}[\mathbf{A}^*(r)]\,\mathbf{z}(r) + \mathbf{A}^*(t)\,\mathbf{z}(t) - \mathbf{A}^*(s)\,\mathbf{z}(s) - \sum_{s \leqq \tau < t} \Delta^+\mathbf{A}(\tau)\,\Delta^+\mathbf{z}(\tau)$$

for every solution $\mathbf{z} : [0, 1] \to R_n$ and any $s, t \in [0, 1]$. Hence (2.19) implies

$$\mathbf{z}(t) - \mathbf{z}(s) + \int_s^t \mathrm{d}[\mathbf{A}^*(r)]\,\mathbf{z}(r) + \sum_{s \leqq \tau < t} \Delta^*\mathbf{A}^*(\tau)\,\Delta^+\mathbf{z}(\tau) = \mathbf{0}$$

for all $s, t \in [0, 1]$. Hence we have in our case

$$\mathbf{z}(t) = \mathbf{z}(s) - \int_s^t \mathrm{d}[\mathbf{A}^*(r)]\,\mathbf{z}(r) = \mathbf{z}(s) - \int_s^t \mathbf{F}^*(r)\,\mathbf{z}(r)\,\mathrm{d}r$$

for any $s, t \in (t_{j-1}, t_j)$ and the solution of (2.17) satisfies the differential equation

$$\dot{\mathbf{z}} = -\mathbf{F}^*(t)\,\mathbf{z}$$

a.e. in $[0, 1]$. Moreover, the solution $\mathbf{z}$ is left continuous in $[0,1]$ and

$$\mathbf{z}(s + \delta) = \mathbf{z}(s) - \int_s^{s+\delta} \mathrm{d}[\mathbf{A}^*(r)]\,\mathbf{z}(r) - \sum_{s \leqq \tau < s+\delta} \Delta^+\mathbf{A}(\tau)\,\Delta^+\mathbf{z}(\tau)$$

for every $s \in [0, 1)$ and $\delta > 0$ such that $s + \delta \leqq 1$. Passing to the limit $\delta \to 0+$ we get

$$\mathbf{z}(s+) = \mathbf{z}(s) - \Delta^+\mathbf{A}^*(s)\,\mathbf{z}(s) - \Delta^+\mathbf{A}^*(s)\,\Delta^+\mathbf{z}(s) = \mathbf{z}(s) - \Delta^+\mathbf{A}^*(s)\,\mathbf{z}(s+)\,.$$

Hence $\mathbf{z}(s+) = \mathbf{z}(s)$ for $s \in [0, 1)$, $s \neq t_j$, $j = 1, \ldots, k$ and

$$\mathbf{z}(t_j) = (\mathbf{I} + \Delta^+\mathbf{A}^*(t_j))\,\mathbf{z}(t_j+) = \mathbf{D}_j^*\,\mathbf{z}(t_j+)$$

for all $j = 1, \ldots, k$.

This implies that the equation (2.17) is equivalent to the interface problem

$$\dot{\mathbf{z}} = -\mathbf{F}^*(t)\,\mathbf{z}\,,$$

$$\mathbf{z}(t_j) - \mathbf{D}_j^*\,\mathbf{z}(t_j+) = \mathbf{0}\,, \quad j = 1, \ldots, k\,.$$

If we compare this problem with the problem (2.9), (2.10) then we can observe that the only difference is the fact that in the former case the solution is assumed to be left continuous while in the latter one it is right continuous. The solvability conditions given by Theorem 2.3 remain the same (see Remark 3.9 in [5]).

In [5] the authors derive also Green's function for boundary value problems of the form (2.4), (2.3). Using the results from [5] we obtain the following result.

Let $\mathbf{X}(t) : [0, 1] \to L(R_n)$ be the fundamental matrix satisfying the matrix equation

$$\mathbf{X}(t) = \mathbf{I} + \int_0^t \mathrm{d}[\mathbf{A}(r)]\,\mathbf{X}(r)\,, \quad t \in [0, 1]\,.$$

Assume that $m = n$ and $\det \mathbf{D} \neq 0$ where

$$\mathbf{D} = \mathbf{M}\,\mathbf{X}(0) + \mathbf{N}\,\mathbf{X}(1)\,.$$

Then for any $\mathbf{g}$, $\mathbf{M}_j$, $\mathbf{N}_j$, $\mathbf{c}_j$, $j = 1, \ldots, k$ and $\mathbf{r} \in R_n$ the boundary value problem (2.1), (2.2), (2.3) possesses a unique solution $\mathbf{x} : [0, 1] \to R_n$ and this solution is given by

$$\mathbf{x}(t) = \mathbf{X}(t)\,\mathbf{D}^{-1}\mathbf{r} + \int_0^1 \mathrm{d}_s[\mathbf{G}(t, s)]\,\mathbf{f}(s) \quad \text{on} \quad [0, 1] \tag{2.20}$$

where

$$\mathbf{G}(t, s) = \begin{cases} -\mathbf{X}(t)\,\mathbf{D}^{-1}\mathbf{M}\,\mathbf{X}(0)\,\mathbf{X}^{-1}(s)\,, & 0 \leqq s < t \leqq 1\,, \\ \mathbf{X}(t)\,\mathbf{D}^{-1}\mathbf{N}\,\mathbf{X}(1)\,\mathbf{X}^{-1}(s)\,, & 0 \leqq t \leqq s \leqq 1 \end{cases} \tag{2.21}$$

is the corresponding Green's function.

Example. Let $y(t)$ denote the bending of a beam fixed at the endpoints, let $h(t)$ stand for the (piecewise continuous) load and let q be a point load in the middle of the beam. The problem of determining $y(t)$ can be described as follows.

Find solutions of the equation

$$y^{(4)}(t) = h(t)\,, \quad t \in [0, 1]\,, \quad h \in L(0, 1) \tag{2.22}$$

which possess continuous derivatives up to the order 3 on $[0, 1] - \{\frac{1}{2}\}$ and satisfy the conditions

$$y(\tfrac{1}{2}+) = y(\tfrac{1}{2}-)\,, \quad \dot{y}(\tfrac{1}{2}+) = \dot{y}(\tfrac{1}{2}-)\,, \quad \ddot{y}(\tfrac{1}{2}+) = \ddot{y}(\tfrac{1}{2}-)\,, \tag{2.23}$$
$$\dddot{y}(\tfrac{1}{2}+) = \dddot{y}(\tfrac{1}{2}-) + q$$

and

$$y(0) = \dot{y}(0) = y(1) = \dot{y}(1) = 0\,, \tag{2.24}$$

By means of the transformation $x_1 = y$, $x_2 = \dot{y}$, $x_3 = \ddot{y}$, $x_4 = \dddot{y}$ the problem (2.22), (2.23), (2.24) can be written in the form of the system

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{g}\,, \tag{2.25}$$

$$\mathbf{x}((\tfrac{1}{2})-) - \mathbf{x}((\tfrac{1}{2})+) = \mathbf{c}_1\,, \tag{2.26}$$

$$\mathbf{M}\,\mathbf{x}(0) + \mathbf{N}\,\mathbf{x}(1) = \mathbf{0} \tag{2.27}$$

where

$$\mathbf{F} = \begin{bmatrix} 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \\ 0, & 0, & 0, & 0 \end{bmatrix}, \quad \mathbf{g}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ h(t) \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ q \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix},$$

$$\mathbf{N} = \begin{bmatrix} 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \end{bmatrix}.$$

This is a boundary value problem for an interface problem of the form considered in this paper. The interface condition (2.26) can be written in the form

$$\mathbf{M}_1\,\mathbf{x}(\tfrac{1}{2})-) + \mathbf{N}_1\,\mathbf{x}((\tfrac{1}{2})+) = \mathbf{c}_1$$

where $\mathbf{M}_1 = -\mathbf{N}_1 = \mathbf{I} \in L(R_4)$. Hence if we set $\mathbf{D}_1 = \mathbf{I}$, we have $\mathbf{M}_1 + \mathbf{N}_1\mathbf{D}_1 = \mathbf{0}$. Moreover, $\mathbf{N}_1\mathbf{d}_1 = -\mathbf{d}_1 = \mathbf{c}_1$, i.e. $\mathbf{d}_1 = -\mathbf{c}_1$. The fundamental matrix for the generalized linear differential equation corresponding to the interface problem given above is of the form

$$\mathbf{X}(t) = e^{\mathbf{F}t} = \begin{bmatrix} 1, & t, & \tfrac{1}{2}t^2, & \tfrac{1}{6}t^3 \\ 0, & 1, & t, & \tfrac{1}{2}t^2 \\ 0, & 0, & 1, & t \\ 0, & 0, & 0, & 1 \end{bmatrix}, \qquad t \in [0, 1]\,.$$

Further, it is easy to evaluate

$$\mathbf{D} = \mathbf{M}\,\mathbf{X}(0) + \mathbf{N}\,\mathbf{X}(1) = \begin{bmatrix} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 1, & 1, & \tfrac{1}{2}, & \tfrac{1}{6} \\ 0, & 1, & 1, & \tfrac{1}{2} \end{bmatrix}$$

and $\det \mathbf{D} = \frac{1}{12} \neq 0$. Hence we can construct Green's function as was described in the previous part of this section. It is a matter of routine to evaluate

$$\mathbf{D}^{-1} = \begin{bmatrix} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ -6, & -4, & 6, & -2 \\ 12, & 6, & -12, & 6 \end{bmatrix}.$$

A straightforward multiplication of matrices makes it possible to determine Green's function $\mathbf{G}(t, s)$ for the boundary value problem (see (2.21)) and to write

$$\mathbf{x}(t) = \int_0^1 \mathrm{d}_s[\mathbf{G}(t, s)]\, \mathbf{f}(s) \tag{2.28}$$

for the solution. Since the solution y of the original problem (2.22), (2.23), (2.24) satisfies $y(t) = x_1(t)$, we need to know only the first component of the vector $\mathbf{x}(t)$ on the right hand side of (2.28). Since only the fourth component of $\mathbf{f}$ is different from zero we get

$$y(t) = x_1(t) = \int_0^1 \mathrm{d}_s[G_{14}(t, s)]\, f_4(s)$$

where G_{14} is the element in the first row and the fourth column of $G(t, s)$. The determination of Green's function by (2.21) gives in our situation

$$G_{14}(t, s) = \quad s^3/6 - s^3t^2/2 + s^3t^3/3 - s^2t/2 + s^2t^2 - s^2t^3/2 \quad \text{for} \quad 0 \leqq s < t \leqq 1\,,$$

$$G_{14}(t, s) = -t^2s^3/2 + t^2s^2 - t^2s/2 + t^3s^3/3 - t^3s^2/2 + t^3/6 \quad \text{for} \quad 0 \leqq t \leqq s \leqq 1\,.$$

Further we have

$$\begin{aligned} y(t) = {} & \int_0^1 \mathrm{d}_s[G_{14}(t, s)]\, f_4(s) = -\int_0^1 G_{14}(t, s)\, \mathrm{d}f_4(s) + G_{14}(t, 1)\, f_4(1) - \\ - {} & G_{14}(t, 0)\, f_4(0) = \\ = {} & -\int_0^1 G_{14}(t, s)\, \mathrm{d}\left(\int_0^s h(\tau)\, \mathrm{d}\tau - q\, \psi^+_{1/2}(s)\right) + G_{14}(t, 1)\left(\int_0^1 h(\tau)\, \mathrm{d}\tau - q\right) = \\ = {} & -\int_0^1 G_{14}(t, s)\, h(s)\, \mathrm{d}s + G_{14}(t, \tfrac{1}{2})\, q - G_{14}(t, 1)\, q + G_{14}(t, 1)\int_0^1 h(\tau)\, \mathrm{d}\tau = \\ = {} & -\int_0^1 G_{14}(t, s)\, h(s)\, \mathrm{d}s + G_{14}(t, \tfrac{1}{2})\, q \end{aligned}$$

where the integration-by-parts formula for Perron-Stieltjes integrals is used and the equality $G_{14}(t, 1) = 0$ is taken into account. Substituting into this formula, we get the following explicit expression for the solution of the problem (2.22), (2.23), (2.24), $t \in [0, 1]$:

$$\begin{aligned} y(t) = {} & (-t^3/3 + t^2/2 - 1/6)\int_0^t s^3\, h(s)\, \mathrm{d}s + (t^3/2 - t^2 + t/2)\int_0^t s^2\, h(s)\, \mathrm{d}s + \\ & + (-t^3/3 + t^2/2)\int_t^1 s^3\, h(s)\, \mathrm{d}s + (t^3/3 - t^2)\int_t^1 s^2\, h(s)\, \mathrm{d}s + (t^2/2)\int_t^1 s\, h(s)\, \mathrm{d}s - \\ & - (t^3/6)\int_t^1 h(s)\, \mathrm{d}s + K(t)\, q \end{aligned}$$

where

$$K(t) = \begin{cases} -\frac{1}{16}t^2 + \frac{1}{12}t^3 & \text{for} \quad t \leqq \frac{1}{2}\,, \\ \frac{1}{48} - \frac{1}{8}t + \frac{3}{16}t^2 - \frac{1}{12}t^3 & \text{for} \quad t > \frac{1}{2}\,. \end{cases}$$