Časopis pro pěstování matematiky, roč. 105 (1980), Praha

EXISTENCE OF GENERALIZED SYMMETRIC RIEMANNIAN SPACES WITH SOLVABLE ISOMETRY GROUP*)

MILOŠ BOŽEK, Bratislava

(Received June 23, 1978)

The first existence theorem on generalized symmetric Riemannian spaces was proved by LEDGER and OBATA in [8].

Theorem A. *For every integer $k \geqq 3$ there is a compact generalized symmetric Riemannian space admitting a regular s-structure of order k and not admitting a regular s-structure of order* 2.

This result was strengthened by KOWALSKI, see [4].

Theorem B. *For every integer $k \geqq 2$ there is a compact generalized symmetric Riemannian space of order k such that the identity component of its full isometry group is semi-simple. In particular, if $k \geqq 3$, then such a space does not admit regular s-structures of orders $l = 2, \ldots, k - 1$.*

The main result of this paper is in a sense dual to that of Kowalski.

Main Theorem. *For every even integer $m \geqq 4$ there is an irreducible generalized symmetric Riemannian space of order m diffeomorphic to $\mathbf{R}^{m-1}$ and such that the identity component of its full isometry group is solvable.*

1. GENERALIZED SYMMETRIC RIEMANNIAN SPACES

We shall make use of the terminology of the paper [3]. All differentiable manifolds, mappings, tensor fields, etc. are of the class C^∞.

Let a connected Riemannian manifold (M, g) be given and let $x \in M$. A *symmetry at x* is any isometry s_x of (M, g) such that x is an isolated fixed point of s_x.

*) I am grateful to Dr. O. KOWALSKI for his valuable hints and comments.

A *regular s-structure* on (M, g) is a family $\{s_x \mid x \in M\}$ of symmetries of (M, g) briefly denoted by $\{s_x\}$, for which the following condition is fulfilled:

(R) For every $x, y \in M$ we have $s_x \circ s_y = s_z \circ s_x$, where $z = s_x(y)$.

Any regular s-structure $\{s_x\}$ on (M, g) determines a tensor field S of type (1,1) defined by $S_x = (s_x)_{*,x}$ for all $x \in M$. The tensor field S is called the *symmetry tensor field* of $\{s_x\}$; this tensor field is differentiable, see [3], Theorem 1.

A regular s-structure $\{s_x\}$ on (M, g) is said to be of *order* k, if k is the least integer for which $(s_x)^k = id$ for all $x \in M$.

A Riemannian manifold is called a *generalized symmetric Riemannian space*, shortly a *g.s. space*, if it admits at least one regular s-structure. Theorem 2 in [3] says that there is a regular s-structure of finite order on every g.s. space. The *order* of a g.s. space (M, g) is the least integer k such that (M, g) admits a regular s-structure of order k.

Finally, a *regular s-manifold* is a triple $(M, g, \{s_x\})$, where (M, g) is a g.s. space and $\{s_x\}$ is a fixed regular s-structure on (M, g).

2. THE ISOMETRY GROUPS OF A CERTAIN CLASS OF G.S. SPACES

Let (M, g) be a g.s. space and let o be a fixed point of M. We are going to use the following notation: V is the tangent vector space $T_o(M)$, $I(M)$ is the full isometry group of (M, g) and $I(M, o)$ its isotropy subgroup at o, $I(M)^0$ is the identity component of $I(M)$. It is known that the group $I(M)$ acts transitively on M, see [8], Theorem 1. Hence every g.s. space is a Riemannian homogeneous space. Particularly, the mapping $\pi : I(M)^0 \to M$ defined by $\pi(a) = a(o)$ is surjective and locally trivial. Finally, let $\hat{H}$ denote the image of the linear isotropy representation of $I(M, o)$ in V and $\hat{\mathfrak{h}}$ its Lie algebra. $\hat{H}$ is a group of linear transformations of V which is naturally isomorphic to $I(M, o)$.

In this section and in the next one, we shall investigate g.s. spaces with a finite group $I(M, o)$. We shall call such spaces *g.s. spaces with a finite isometry group at a point*. The following assertions are equivalent: The group $\hat{H}$ is finite; $\hat{\mathfrak{h}} = 0$; $\pi : I(M)^0 \to M$ is a covering; $\pi_{*,1} : T_1(I(M)^0) \to V$ is a vector space isomorphism.

Throughout this section, (M, g) always denotes a g.s. space with finite isometry group at a point.

There is exactly one Lie algebra structure on V such that $\pi_{*,1}$ is an algebra isomorphism. Recall that there is a scalar product g_o on the algebra V. Let us denote by L the group of all isometric automorphisms of the algebra V and put $L' = \{A \in L \mid Av \neq v$ for $v \neq 0\}$.

The main purpose of the present section is to prove the following two theorems.

Theorem 1. *The identity component $I(M)^0$ of the full isometry group $I(M)$ of*

(M, g) *is a solvable group. If M is simply connected, then* $I(M)^0$ *is diffeomorphic to M.*

Theorem 2. *For* (M, g) *simply connected, the map* $\{s_x\} \mapsto S_0$ *is a bijection between the set of all regular s-structures on* (M, g) *and the set* L.

First we shall need some preliminary results.

Lemma 2.1. *For a simply connected* (M, g), *the projection* $\pi : I(M)^0 \to M$ *is a diffeomorphism.*

Proof. The projection π is a covering. This covering is trivial, because M is simply connected. Our assertion follows now from the connectedness of both spaces $I(M)^0$ and M.

Proposition 2.1. $\hat{H} \subset L$. $\hat{H} = L$ *if M is simply connected.*

Proof. Clearly, every element of $\hat{H}$ is an isometry of the vector space V. To prove that it is also an automorphism of the algebra V, let us consider a transformation $\bar{f}$ of $I(M)^0$ for every $f \in I(M, o)$ defined by $\bar{f}(a) = f \circ a \circ f^{-1}$ for each $a \in I(M)^0$. $\bar{f}$ is an automorphism of the group $I(M)^0$ and $\pi \circ \bar{f} = f \circ \pi$, therefore $f_{*,o} = \pi_{*,1} \circ \bar{f}_{*,1} \circ \circ (\pi_{*,1})^{-1}$ is an automorphism of the algebra V, and $\hat{H} \subset L$.

Now, let M be simply connected and let F be an arbitrary element of L. The mapping $\bar{F} = (\pi_{*,1})^{-1} \circ F \circ \pi_{*,1}$ is an automorphism of the Lie algebra V. By Lemma 2.1, the group $I(M)^0$ is simply connected, therefore there is an automorphism $\bar{f}$ of $I(M)^0$ such that $\bar{f}_{*,1} = \bar{F}$. Since F preserves the scalar product g_o, $\bar{f}$ preserves the left-invariant metric π^*g. Hence $\bar{f}$ is an isometry of $(I(M)^0, \pi^*g)$ and the map $f = = \pi \circ \bar{f} \circ \pi^{-1}$ is an isometry of (M, g). Clearly $f_{*,o} = F$, i.e. $F \in \hat{H}$.

Proof of Theorem 1. Let $\{s_x\}$ be a regular s-structure on (M, g). By Proposition 2.1, the corresponding S_o is an automorphism of the algebra V, thus the Lie algebra V admits an automorphism of finite order with no non-zero fixed vector. According to [7] or [9], the algebra V is solvable. The algebra of the Lie group $I(M)^0$ is isomorphic to V, hence the group $I(M)^0$ is solvable. The second assertion of Theorem 1 is Lemma 2.1.

Lemma 2.2. *Let G be a connected Lie group with a left-invariant metric g and let* s_e *be an isometric automorphism of G such that the neutral element e of G is an isolated fixed point of* s_e. *Then the family* $\{s_x = L_x \circ s_e \circ L_x^{-1}\}$ *is a regular s-structure on* (G, g).

Proof. It is clear every transformation s_x, $x \in G$ is a symmetry of (G, g) at x. An easy calculation shows that $\{s_x\}$ satisfies the condition (R).

For every Riemannian s-manifold $(M, g, \{s_x\})$ let us denote by $\mathrm{Cl}(\{s_x\})$ the closure of the subgroup of $I(M)$ algebraically generated by the set $\{s_x\}$. It is shown, see [3]

Theorem A, that the group $\mathrm{Cl}(\{s_x\})$ is a transitive Lie group of isometries of (M, g). For the identity component of $\mathrm{Cl}(\{s_x\})$ we have

Proposition 2.2. $\mathrm{Cl}(\{s_x\})^0 = I(M)^0$ *for every regular s-structure* $\{s_x\}$ *on* (M, g).

Proof. The group $\mathrm{Cl}(\{s_x\})^0$ is a connected subgroup of $I(M)^0$ and both the groups cover M. Therefore $\mathrm{Cl}(\{s_x\})^0$ is an open subgroup of $I(M)^0$, thus $\mathrm{Cl}(\{s_x\})^0 = I(M)^0$.

If $\{s_x\}$ is a regular s-structure on (M, g), then according to [3] we have

$$s_x = f \circ s_o \circ f^{-1} \text{ for every } x \in M \text{ and every } f \in \mathrm{Cl}(\{s_x\})^0 \text{ such that } f(o) = x .$$

Hence Proposition 2.2 has the following

Corollary. *Let* (M, g) *and* $\{s_x\}$ *be as in Propositon* 2.2. *Then* $s_x = f \circ s_o \circ f^{-1}$ *for all* $x \in M$ *and for all* $f \in I(M)^0$ *such that* $f(o) = x$.

Proof of Theorem 2. The map $\{s_x\} \mapsto S_0$ is a composition of two maps, namely $\{s_x\} \mapsto s_o$ and $s_o \mapsto S_o = (s_o)_{*,o}$. The first map is injective by the foregoing Corollary and the second is injective because it is the linear isotropy representation of an isometry group. In virtue of Proposition 2.1, S_o is an element of the group L, moreover, it is an element of the set L', because it has no non-zero fixed vector. Thus the map $\{s_x\} \mapsto S_o$ is an injection from the set of all regular s-structures on (M, g) into the set L'.

To prove that this map is also surjective, let us consider an arbitrary element F of L'. As in the proof of Proposition 2.1, there is an isometric automorphism $\bar{s}$ of the group $I(M)^0$ such that $\bar{s}_{*,1} = (\pi_{*,1})^{-1} \circ F \circ \pi_{*,1}$. It is easy to see that $\bar{s}$ is a symmetry of $(I(M)^0, \pi^*g)$ at the identity 1. The family $\{s_x = \pi \circ L_a \circ \bar{s} \circ L_a^{-1} \circ \pi^{-1}\}$, where $a = \pi^{-1}(x)$, is a regular s-structure on (M, g) by Lemmas 2.1 and 2.2. We have $S_o = (s_o)_{*,o} = \pi_{*,1} \circ \bar{s}_{*,1} \circ (\pi_{*,1})^{-1} = F$, thus the map in question is surjective, which concludes the proof of Theorem 2.

3. CANONICAL CONNECTION ON G.S. SPACES WITH FINITE ISOMETRY GROUP AT A POINT

Let $(M, g, \{s_x\})$ be a Riemannian s-manifold, ∇ the Riemannian connection of (M, g) and S the symmetry tensor field of $\{s_x\}$. Following A. J. Ledger [1], we introduce a new connection $\tilde{\nabla}$ by the formulas

$$\tilde{\nabla}_X Y = \nabla_X Y - D(Y, X), \quad D(Y, X) = (\nabla S)(S^{-1} Y, (I - S)^{-1} X),$$

where X and Y are arbitrary vector fields on M. The connection $\tilde{\nabla}$ is called the *canonical connection of the Riemannian s-manifold* $(M, g, \{s_x\})$. O. Kowalski has shown in [6] that the connection $\tilde{\nabla}$ depends only on the regular s-structure $\{s_x\}$ and not on the Riemannian connection ∇:

Theorem C. *The canonical connection $\tilde{\nabla}$ of $(M, g, \{s_x\})$ is the only connection on M, for which the following conditions hold:*

(i) *All symmetries s_x, $x \in M$ are affine transformations of the affine manifold $(M, \tilde{\nabla})$.*

(ii) $\tilde{\nabla} S = 0$.

Thus, we can speak about the *canonical connection of the regular s-structure* $\{s_x\}$. To obtain some further properties of the canonical connection we shall start with

Proposition 3.1. *Let $M = G/H$ be a reductive homogeneous space with respect to a decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, and let $\bar{f}$ be an automorphism of G such that $\bar{f}(H) \subset H$ and $\bar{f}_{*,e}(\mathfrak{m}) \subset \mathfrak{m}$. Then the induced map $f : G/H \to G/H$ defined by $f(aH) = \bar{f}(a) H$ for all $a \in G$ is an affine transformation of M with respect to the canonical connection of the second kind $\tilde{\nabla}$.*

Proof. The subspace $\mathfrak{m}$ of $\mathfrak{g}$ is supposed to be identified in the natural way with the tangent vector space $T_o(M)$ of $M = G/H$ at the origin $o = H$.

The transformation f induces a transformation f_* of the set of all vector fields on M via

$$(f_* X)_p = f_{*,q}(X_q)\,, \quad \text{where} \quad q = f^{-1}(p)\,,$$

and a transformation f^* of the set of all affine connections on M via

$$(f^* \nabla)_X Y = f_*^{-1}(\nabla_{f_* X} f_* Y)\,.$$

Our assertion is equivalent to the equality $f^* \tilde{\nabla} = \tilde{\nabla}$. By [2], Corollary X.2.2 we have to prove that the connection $f^* \tilde{\nabla}$ is invariant and that

$$((f^* \tilde{\nabla})_X Y)_o = [\tilde{X}, Y]_o$$

for all vectors $X \in T_o(M)$ and all vector fields Y on M, where $\tilde{X}$ is an extension of the vector X to a vector field on M defined by

$$\tilde{X}_p = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\exp tX)(p) \quad \text{for all} \quad p \in M\,.$$

The first statement is equivalent to

$$a^*(f^* \tilde{\nabla}) = f^* \tilde{\nabla} \quad \text{for all} \quad a \in G$$

and follows from the relations $f \circ a = \bar{f}(a) \circ f$ and $a^*(f^* \tilde{\nabla}) = (f \circ a)^* \tilde{\nabla}$. The second statement is a consequence of the identities $f_* \tilde{X} = \widetilde{f_{*,o}(X)}$ and $f_*[\tilde{X}, Y] = [f_* \tilde{X}, f_* Y]$ which hold for every vector $X \in T_o(M)$ and every vector field Y on M.

Proposition 3.2. *Let $M = G/H$ be a reductive homogeneous space with the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, let g be a Riemannian metric on M. Let $\{s_x\}$ be a regular s-structure on (M, g) such that s_0 is an affine transformation of M with respect to* *the canonical connection of the second kind* $\bar{\nabla}$ *and such that* $s_x = f \circ s_e \circ f^{-1}$ *for all* $x \in M$ *and* $f \in G$ *with* $f(o) = x$. *Then the canonical connection* $\tilde{\nabla}$ *of the regular s-structure* $\{s_x\}$ *coincides with the connection* $\bar{\nabla}$.

Proof. The assumptions about $\{s_x\}$ imply that all symmetries s_x, $x \in M$ are affine transformations of $(M, \bar{\nabla})$ and that the symmetry tensor field S of $\{s_x\}$ is G-invariant. By [2], Proposition X.2.7, S is $\bar{\nabla}$-parallel. Our assertion follows from Theorem C.

Any g.s. space (M, g) with a finite isometry group at a point is a reductive homogeneous space $M = G/H$ where $G = I(M)^0$ and $\mathfrak{m} = \mathfrak{g}$. For every symmetry s of M at the origin o the induced map $\bar{s} : G \to G$, $a \mapsto s \circ a \circ s^{-1}$ is an automorphism of G preserving H and $\mathfrak{m}$ and such that $s(aH) = \bar{s}(a) H$ for all $a \in G$. This fact together with Propositions 3.1, 3.2 and with Corollary of Proposition 3.2 implies the first part of the next theorem. The second part of the theorem follows then from Theorem X.2.6 of [2].

Theorem 3. *All regular s-structures on a g.s. space* (M, g) *with a finite isometry group at a point have the same canonical connection* $\tilde{\nabla}$, *namely the projection of the Cartan* $(-)$*-connection of the group* $I(M)^0$ *via the covering map* $\pi : I(M)^0 \to M$. *For its torsion and curvature we have*

$$\tilde{T}_o(X, Y) = -[X, Y] \quad \textit{for every} \quad X, Y \in T_o(M) \, (= \mathfrak{g}) \,, \quad \textit{and} \quad \tilde{R} = 0 \,.$$

As an immediate consequence of Propositions 3.1, 3.2 and Proposition X.2.12 of [2] we obtain

Proposition 3.3. *Let* (G, g) *be a connected Lie group with a left-invariant Riemannian metric. Let* s_e *be an isometrical automorphism of* (G, g) *which has the neutral element* $e \in G$ *as an isolated fixed point. Then the canonical connection* $\tilde{\nabla}$ *of the regular s-structure* $\{s_x = L_x \circ s_e \circ L_x^{-1}\}$ *is the Cartan* $(-)$*-connection. For its torsion and curvature we have*

$$\tilde{T}_e(X, Y) = -[X, Y] \quad \textit{for all} \quad X, Y \in T_e(G) = \mathfrak{g} \,, \quad \textit{and} \quad \tilde{R} = 0 \,.$$

4. RIEMANNIAN S-MANIFOLD $(G_n, g, \{s_x\})$

Let $n \geqq 1$ be an integer. Let us consider a matrix group G_n consisting of all matrices of the form

$$\begin{Vmatrix} e^{u_0} & 0 & \dots & 0 & x_0 \\ 0 & e^{u_1} & \dots & 0 & x_1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e^{u_n} & x_n \\ 0 & 0 & \dots & 0 & 1 \end{Vmatrix},$$

where $(x_0, x_1, \dots, x_n, u_1, \dots, u_n) \in \mathbf{R}^{2n+1}$ is an arbitrary element and $u_0 = -u_1 - \dots$

$\ldots - u_n$. Thus, the group G_n is a Lie group diffeomorphic to the cartesian space $\mathbf{R}^{2n+1}(x_0, x_1, \ldots, x_n, u_1, \ldots, u_n)$. Suppose that the points of G_n are identified with the corresponding $(2n + 1)$-tuples. Particularly, for the neutral element e of G_n we have $e = (0, \ldots, 0)$.

Let us consider a Riemannian metric g on G_n defined by

$$g = \sum_{i=0}^{n} \mathrm{e}^{-2u_i}(\mathrm{d}x_i)^2 + a \sum_{\alpha,\beta=1}^{n} \mathrm{d}u_\alpha \, \mathrm{d}u_\beta \,, \quad a > 0 \,.$$

The metric g is left-invariant. For $n = 1$ or 2, the corresponding manifolds are known from the complete list of g.s. spaces in dimensions 3 or 5, see [5], §§ 4, 12.

Finally, let s_e be a transformation of G_n given by

$$\begin{aligned} s_e(x_0, x_1, \ldots, x_n, u_1, \ldots, u_n) &= \\ = (-x_n, x_0, x_1, \ldots, x_{n-1}, &-u_0, u_1, \ldots, u_{n-1}) \,, \end{aligned} \tag{1}$$

where $u_0 = -u_1 - \ldots - u_n$ again. The transformation s_e is an automorphism of the Lie group G_n and also a symmetry of (G_n, g) at e. By Lemma 2.2, the family

$$\{s_x = L_x \circ s_e \circ L_x^{-1}\} \tag{2}$$

is a regular s-structure on (G_n, g). Thus, we have defined a Riemannian s-manifold $(G_n, g, \{s_x\})$ for all $n \geqq 1$. All the symmetries s_x, $x \in G_n$ are of order $2n + 2$, hence the g.s. space (G_n, g) is of order at most $2n + 2$. In fact, it is exactly of order $2n + 2$ as will be shown in § 6.

In the rest of the present section some coordinate expressions will be given. Let us denote

$$\begin{aligned} X_i &= \mathrm{e}^{u_i} \frac{\partial}{\partial x_i} \,, \quad i = 0, 1, \ldots, n \,, \\ U_\alpha &= \frac{\partial}{\partial u_\alpha} \,, \quad \alpha = 1, \ldots, n \,. \end{aligned} \tag{3}$$

A direct calculation shows that the vector fields $X_0, X_1, \ldots, X_n, U_1, \ldots, U_n$ form a basis of the Lie algebra $\mathfrak{g}_n$ of the Lie group G_n and that the bracket operation in $\mathfrak{g}_n$ is given by the formulas

$$\begin{aligned} [X_i, X_j] &= [U_\alpha, U_\beta] = 0 \,, \\ [X_0, U_\alpha] &= X_0 \,, \quad [X_\alpha, U_\beta] = -\delta_{\alpha\beta} X_\alpha \end{aligned} \tag{4}$$

for all $i, j = 0, 1, \ldots, n$, $\alpha, \beta = 1, \ldots, n$. The coordinate expressions of the Riemannian metric g and the symmetry tensor field S of the regular s-structure $\{s_x\}$ with respect to the basis (3) are given by

$$\begin{aligned} g(X_i, X_j) &= \delta_{ij} \,, \\ g(X_i, U_\alpha) &= 0 \,, \\ g(U_\alpha, U_\beta) &= a(1 + \delta_{\alpha\beta}) \,, \end{aligned} \tag{5}$$

for all $i, j = 0, 1, \dots, n$, $\alpha, \beta = 1, \dots, n$,

(6)
$$S(X_i) = \begin{cases} X_{i+1} & \text{for} \quad i = 0, 1, \dots, n-1\,, \\ -X_0 & \text{for} \quad i = n\,, \end{cases}$$

$$S(U_\alpha) = \begin{cases} U_{\alpha+1} - U_1 & \text{for} \quad \alpha = 1, \dots, n-1\,, \\ -U_1 & \text{for} \quad \alpha = n\,. \end{cases}$$

5. THE GROUP $I(G_n, e)$ IS FINITE

We shall continue our study of the Riemannian s-manifolds $(G_n, g, \{s_x\})$. The tangent vector space $T_e(G_n)$ which is supposed to be identified with the Lie algebra $\mathfrak{g}_n$ will be denoted by V. Let $\tilde{\nabla}$ or ∇ be the canonical connection of $\{s_x\}$ or the Riemannian connection of (G_n, g), respectively. The Riemannian metric g and the Riemannian curvature R, the tensor field S, the torsion $\tilde{T}$ and the curvature $\tilde{R}$ of the canonical connection $\tilde{\nabla}$, and also the difference tensor field $D = \nabla - \tilde{\nabla}$, are invariant with respect to the left translations of G_n. Thus we can replace these tensor fields by their evaluations at the point $e \in G_n$. The corresponding tensors on $V = T_e(G_n)$ will be denoted by the same letters, for the sake of brevity. Tensors g and S are given by formulas (5) and (6), respectively, tensor $\tilde{T}$ and $\tilde{R}$ by Proposition 3.3. Tensors D and R can be calculated from the formulas

(7)
$$2\,g(D(X, Y), Z) = g(\tilde{T}(X, Y), Z) + g(\tilde{T}(X, Z), Y) + g(\tilde{T}(Y, T), X)\,,$$

(8)
$$R(X, Y)\,Z = \tilde{R}(X, Y)\,Z + D(D(Z, Y), X) - D(D(Z, X), Y) + \\ + D(Z, \tilde{T}(X, Y))\,,$$

which hold for all vectors $X, Y, Z \in V$, see [3] Lemma 4 and [1] Lemma 4.8. In order to calculate the tensors D and R it is convenient to consider the complexification $V^c = V \otimes_{\mathbf{R}} \mathbf{C}$ and to choose a basis of V^c consisting of eigenvectors of the transformation S.

The characteristic equation of S is

$$(x^{n+1} + 1)\,(x^n + x^{n+1} + \dots + x + 1) = 0\,.$$

Hence the eigenvalues of the transformation S are the complex numbers

$$z^{2j+1},\; z^{2\alpha}\,, \quad j = 0, 1, \dots, n\,, \quad \alpha = 1, \dots, n\,,$$

where

$$z = \cos\frac{\pi}{n+1} + \mathrm{i}\,.\,\sin\frac{\pi}{n+1}\,.$$

For the corresponding eigenvectors we can choose the complex vectors

(9)
$$Y_j = \sum_{i=0}^{n} z^{(2j+1)i} X_i\,, \quad j = 0, 1, \dots, n\,,$$

$$V_\alpha = \sum_{\beta=1}^{n} z^{2\alpha\beta} U_\beta\,, \qquad \alpha = 1, \ldots, n\,,$$

where $X_0, X_1, \ldots, X_n, U_1, \ldots, U_n$ are given by (3). The vectors (9) form a basis of the vector space V^c, because all eigenvalues of S are mutually different.

From now on, we shall identify the index set $\{0, 1, \ldots, n\}$ with the additive group $\mathbf{Z}_{n+1}$ in the natural way and denote

$$\bar{i} = -i - 1 \quad \text{for every} \quad i = 0, 1, \ldots, n\,,$$

$$\bar{\alpha} = -\alpha \qquad \text{for every} \quad \alpha = 1, \ldots, n\,.$$

Thus, for the complex conjugate vectors to those in (9) we have

$$\bar{Y}_i = Y_{\bar{i}}\,, \quad \bar{V}_\alpha = V_{\bar{\alpha}} \quad \text{for all } i \text{ and } \alpha\,. \tag{10}$$

Let us extend the tensors g, S, $\tilde{T}$, $\tilde{R}$, D and R to the complex vector space V^c without any change of the notation. From (5), (9), from Proposition 3.3 and from (4) and (9) we obtain

$$\begin{aligned} g(Y_i, Y_j) &= (n+1)\,\delta_{i\bar{j}}\,, \\ g(Y_i, V_\alpha) &= 0\,, \\ g(V_\alpha, V_\beta) &= (n+1)\,a\delta_{\alpha\bar{\beta}}\,, \end{aligned} \tag{11}$$

for all $i, j = 0, 1, \ldots, n$, $\alpha, \beta = 1, \ldots, n$,

$$\tilde{T}(Y_i, Y_j) = \tilde{T}(V_\alpha, V_\beta) = 0\,, \quad \tilde{T}(Y_i, V_\alpha) = Y_{i+\alpha} \tag{12}$$

for all $i, j = 0, 1, \ldots, n$, $\alpha, \beta = 1, \ldots, n$,

$$\tilde{R} = 0\,. \tag{13}$$

Relations (7), (11) and (12) imply

$$D(Y_i, Y_j) = \begin{cases} 0 & \text{if} \quad i = j\,, \\ \dfrac{1}{a} V_{i+j+1} & \text{if} \quad i \neq j\,, \end{cases} \tag{14}$$

$$D(Y_i, V_\alpha) = D(V_\alpha, V_\beta) = 0\,, \quad D(V_\alpha, Y_i) = -Y_{i+\alpha}\,.$$

Finally, from the relations (8), (12), (13) and (14) we obtain

$$R(Y_i, Y_j)\,Y_k = \begin{cases} -\dfrac{1}{a} Y_{i+j+k+1} & \text{if} \quad i \neq j\,, \quad k = \bar{i}\,, \\ \dfrac{1}{a} Y_{i+j+k+1} & \text{if} \quad i \neq j\,, \quad k = \bar{j}\,, \\ 0 \quad \text{if } i = j & \text{or} \quad \bar{i} \neq k \neq \bar{j}\,, \end{cases} \tag{15}$$

$$R(Y_i, V_\alpha)\,Y_j = \begin{cases} 0 & \text{if} \quad i + j + \alpha + 1 = 0\,, \\ \dfrac{1}{a} V_{i+j+\alpha+1} & \text{if} \quad i + j + \alpha + 1 \neq 0\,, \end{cases}$$

$$R(Y_i, V_\alpha) V_\beta = -Y_{i+\alpha+\beta},$$
$$R(Y_i, Y_j) V_\alpha = R(V_\alpha, V_\beta) Y_i = R(V_\alpha, V_\beta) V_\gamma = 0 .$$

The purpose of the present section is to prove the assertion announced in the title:

Proposition 5.1. *The isometry group $I(G_n, e)$ at the point e is finite for all $n \geqq 1$.*

This proposition can be reformulated. The group $I(G_n, e)$ is isomorphic to the linear isotropy group $\hat{H}$. By [5] Proposition 13.2, the Lie algebra $\hat{\mathfrak{h}}$ of the Lie group $\hat{H}$ is given by

$$\hat{\mathfrak{h}} = \{A \in \mathfrak{gl}(V) \mid A(g) = A(D^k(R)) = 0,\ k = 0, 1, 2, \ldots\} .$$

Therefore Proposition 5.1 is a consequence of the following

Proposition 5.2. $A(g) = A(R) = A(D(R)) = 0 \Rightarrow A = 0.$

Proof. Let $A \in \mathfrak{gl}(V)$ be an arbitrary endomorphism of the vector space V. The linear extension of A onto the complexification V^c of V will be denoted by the same symbol. With respect to the basis (9) we have

$$A(Y_i) = \sum_j A_i^j Y_j + \sum_\beta B_i^\beta V_\beta , \tag{16}$$
$$A(V_\alpha) = \sum_j C_\alpha^j Y_j + \sum_\beta D_\alpha^\beta V_\beta .$$

It remains to show that our assumptions on A yield $A_i^j = B_i^\beta = C_\alpha^j = D_\alpha^\beta = 0$ for all i, j and α, β.

Condition $A(g) = 0$ is equivalent to

$$g(A(X), Y) + g(X, A(Y)) = 0 \quad \text{for all} \quad X, Y \in V^c . \tag{17}$$

Let us substitute successively $X = \bar{Y}_i$, $Y = Y_j$ then $X = \bar{Y}_i$, $Y = V_\beta$ and finally $X = \bar{V}_\alpha$, $Y = V_\beta$ into (17). Using (10), (11) and (16) we get

$$A_{\bar{i}}^{\bar{j}} = -A_j^i , \tag{18a}$$

$$B_{\bar{i}}^{\bar{\beta}} = -\frac{1}{a} C_\beta^i , \tag{18b}$$

$$D_{\bar{\alpha}}^{\bar{\beta}} = -D_\beta^\alpha . \tag{18c}$$

Condition $A(R) = 0$ is equivalent to

$$A(R(X, Y) Z) - R(A(X), Y) Z - R(X, A(Y)) Z - R(X, Y) A(Z) = 0 \tag{19}$$

for all $X, Y, Z \in V^c$. As a rule, we shall apply the following procedure: After replacing X, Y, Z in (19) by suitable vectors of the basis (9) and making use of (15) and (16), the left hand side of (19) will be expressed as a linear combination of the basic vectors (9). So we get a system of linear homogeneous equations in A_i^j, B_i^β, C_α^j and D_α^β. Let us start with the substitution $X = Y_{i+\beta}$, $Y = \bar{V}_\beta$, $Z = \bar{Y}_i$ for arbitrary i and β. Comparing the coefficients at the vector $Y_{i+\beta}$, we get

$$- B_i^{\bar{\beta}} + \frac{1}{a} C_\beta^i = 0\,.$$

Using in addition (18b), we get

$$B_i^\beta = C_\beta^i = 0 \quad \text{for all i and } \beta\,. \tag{20}$$

Now, for every $i, j = 0, 1, \ldots, n$ we define a map $f_i^j : \mathbf{Z}_{n+1} \to \mathbf{C}$ putting $f_i^j(k) = A_{i+k}^{j+k} - A_i^j$, $k \in \mathbf{Z}_{n+1}$. Our next aim is to prove that all these mappings are homomorphisms of additive groups. It will be done in a series of lemmas.

Lemma 5.1. *For every $i, j, k = 0, 1, \ldots, n$ and $\alpha, \beta = 1, \ldots, n$ we have*

(i) $f_i^j(k) = D_\alpha^{\alpha+(j-i)} + D_{k-\alpha}^{k-\alpha+(j-i)}$ *if* $\alpha \neq j - i$, $\alpha \neq k + j - i$, $\alpha \neq k$.

(ii) $A_{i+k}^i = A_i^{i-k}$ *if* $n \geqq 2$.

(iii) $f_i^j(k) = D_k^{k+(j-i)}$ *if* $n \geqq 2$ *and* $k \neq 0$, $k \neq i - j$, $i \neq j$.

(iv) $f_i^j(k) = 0$ *if* $n \geqq 2$ *and* $k = 0$ *or* $k = i - j$.

(v) $f_i^i(k) = D_\alpha^\alpha + D_{k-\alpha}^{k-\alpha}$ *if* $k \neq \alpha$.

Proof. (i) Put $X = Y_i$, $Y = V_\alpha$, $Z = V_{k-\alpha}$ in (19), and calculate the coefficient at the vector Y_{j+k} on the left hand side.

(ii) For given i, k put $j = i - k$. Because $n \geqq 2$, there is an index $\alpha \in \{1, \ldots, n\}$ such that $\alpha \neq i - j$, consequently $\alpha \neq k$, $\alpha \neq k + j - i$. Hence, by (i) and (18c) we have

$$A_{i+k}^i - A_i^{i-k} = A_{i+k}^{j+k} - A_i^j = D_\alpha^{\alpha-k} + D_{-(\alpha-k)}^{-\alpha} = 0\,.$$

(iii) Put $X = Y_j$, $Y = V_k$, $Z = V_{i-j}$ in (19). In the same way as in the proof of (i) we get

$$A_{i+k}^{j+k} - A_j^{j-(i-j)} = D_k^{k+(j-i)}\,.$$

Now, in virtue of (ii),

$$A_j^{j-(i-j)} = A_{j+(i-j)}^j = A_i^j\,,$$

and (iii) holds.

(iv) The first case $k = 0$ is evident. The second is a simple consequence of (ii).

(v) This follows from (i) for $j = i$.

Lemma 5.2. *Let $n \geqq 2$ and let $i, j, r, s \in \{0, 1, \ldots, n\}$ be indices such that $j - i = s - r$. Then $f_i^j = f_r^s$.*

Proof. Because $n \geqq 2$, to every $k = 0, 1, \ldots, n$ there is an index α, more precisely $\alpha(k)$, such that $k \neq \alpha(k)$. Our assertion follows from Lemma 5.1 (iii), (iv), (v).

Lemma 5.3. *If $n \geqq 2$, then all the mappings $f_i^j : \mathbf{Z}_{n+1} \to \mathbf{C}$, $i, j = 0, 1, \ldots, n$ are homomorphisms of additive groups.*

Proof. We have to prove $f_i^j(u - v) = f_i^j(u) - f_i^j(v)$ for all $i, j, u, v \in \mathbf{Z}_{n+1}$. By Lemma 5.2 we get $f_i^j = f_{i-v}^{j-v}$, therefore

$$f_i^j(u - v) = A_{i+u-v}^{j+u-v} - A_i^j = A_{(i-v)+u}^{(j-v)+u} - A_{i-v}^{j-v} - A_{(i-v)+v}^{(j-v)+v} - A_{i-v}^{j-v} = f_{i-v}^{j-v}(u) - f_{i-v}^{j-v}(v) = f_i^j(u) - f_i^j(v) .$$

Corollary. $f_j^i(k) = 0$ *for every* $i, j, k \in \mathbf{Z}_{n+1}$, $n \geqq 2$.

Let us continue the proof of Proposition 5.2. By the foregoing Corollary we have

$$A_{i-k}^{j-k} = A_i^j \quad \text{for every} \quad i, j, k = 0, 1, \ldots, n , \quad n \geqq 2 .$$

Particularly, for $k = -i - j - 1$ it follows

$$A_i^j = A_{\bar{j}}^{\bar{i}} \quad \text{for every} \quad i, j = 0, 1, \ldots, n , \quad n \geqq 2 .$$

Combining this result with (18a), we get

(21) $$A_i^j = 0 \quad \text{for every} \quad i, j = 0, 1, \ldots, n , \quad n \geqq 2 .$$

Further, by Lemma 5.1 (iii) and (v) and by Corollary of Lemma 5.3, we have

(22) $$D_\alpha^\beta = 0 \quad \text{for all} \quad \alpha, \beta = 1, \ldots, n , \quad n \geqq 2 .$$

Therefore, Proposition 5.2 for $n \geqq 2$ is a consequence of (20), (21) and (22).

From now on let $n = 1$. Then $\bar{1} = 1$. From (18c) we have $D_1^1 = -D_{\bar{1}}^{\bar{1}}$, thus

(23) $$D_1^1 = 0 \quad \text{for} \quad n = 1 .$$

Further, put $X = Z = Y_0$ and $Y = Y_1$ in (19). Comparing the coefficient at the vector Y_1 we get $A_0^1 = 0$. Since $\bar{0} = 1$ and $\bar{1} = 0$ for $n = 1$, the formula (18a) yields

(24) $$A_0^1 = A_1^0 = 0 \quad \text{for} \quad n = 1 .$$

In the same way we obtain

(25) $$A_1^1 + A_0^0 = 0 \quad \text{for} \quad n = 1 .$$

By a direct calculation it can be seen that the conditions $A(g) = A(R) = 0$ do not imply $A_1^1 = A_0^0 = 0$ for $n = 1$.

To conclude the proof of Proposition 5.2 for $n = 1$, it is enough to prove

$$A(D_W(R)) = 0 \quad \text{for all} \quad W \in V^c \Rightarrow A_0^0 = 0 .$$

Here the endomorphism D_W of the vector space V^c defined by

$$D_W(X) = D(X, W) \quad \text{for all} \quad X \in V^c$$

acts on the tensor algebra $\mathscr{T}(V^c)$ as a derivation.

Let us substitute $X = Z = W = Y_0$, $Y = Y_1$ into $(A(D_W(R)))(X, Y, Z) = 0$ and let us develop the term on the left hand side making use of (14), (15) and (16). As a consequence of (23), (24) and (25) we get $4\, A_0^0 a^{-2} V_1 = 0$, therefore

$$A_0^0 = 0 \quad \text{for} \quad n = 1\,.$$

6. THE ORDER OF THE G.S. SPACE (G_n, g)

The manifold G_n introduced in § 4 is a simply connected Lie group with a finite isometry group $I(G_n, e)$ at the neutral element e. By Lemma 2.1 the natural projection $\pi : I(G_n)^0 \to G_n$ is a diffeomorphism. Moreover, it is a Lie group isomorphism, because the metric g is left-invariant. The corresponding Lie algebras will be identified via the induced isomorphism $\pi_{*,1}$. Now, by Theorem 2, the set of all regular s-structures on (G_n, g) is equivalent to the set L' consisting of all isometric automorphisms of the algebra $\mathfrak{g}_n$ without any non-zero fixed vector.

Our aim is to calculate the orders of all elements of L'. First we shall determine explicitly the group L of all isometric automorphisms of the algebra $\mathfrak{g}_n$. We shall start with some properties of the algebra $\mathfrak{g}_n$.

The following is immediate from (4):

Lemma 6.1. *The centre of the algebra $\mathfrak{g}_n$ is trivial.*

Let $\mathfrak{g}_n'$ and $\mathfrak{g}_n''$ denote the vector subspaces of the algebra $\mathfrak{g}_n$ generated by the sets $\{X_0, X_1, \ldots, X_n\}$ and $\{U_1, \ldots, U_n\}$, respectively. Clearly, the vector space $\mathfrak{g}_n$ is a direct sum of its mutually orthogonal subspaces $\mathfrak{g}_n'$ and $\mathfrak{g}_n''$. The formulas (4) yield

$$[\mathfrak{g}_n, \mathfrak{g}_n] \subset \mathfrak{g}_n'\,. \tag{26}$$

Lemma 6.2. *For every non-zero vector $X \in \mathfrak{g}_n'$ there is an index $\alpha \in \{1, \ldots, n\}$ such that $[X, U_\alpha] \neq 0$.*

Proof. Let $X = \sum_i c_i X_i$, $X \neq 0$ be an arbitrary element of $\mathfrak{g}_n$. If $c_0 \neq 0$, then $[X, U_\alpha] = c_0 X_0 - c_\alpha X_\alpha \neq 0$ for all $\alpha \in \{1, \ldots, n\}$. If $c_0 = 0$, then there is an index $\alpha \in \{1, \ldots, n\}$ with $c_\alpha \neq 0$, hence $[X, U_\alpha] = -c_\alpha X_\alpha \neq 0$.

It is obvious that all the 1-dimensional subspaces $(X_0), (X_1), \ldots, (X_n)$ generated by the vectors $X_0, X_1, \ldots, X_n$ are ideals of $\mathfrak{g}_n$.

Lemma 6.3. *The ideals $(X_0), (X_1), \ldots, (X_n)$ are the only 1-dimensional ideals of the algebra $\mathfrak{g}_n$.*

Proof. Suppose (X) to be a 1-dimensional ideal of $\mathfrak{g}_n$. Take the decomposition $X = X' + X''$, $X' \in \mathfrak{g}_n'$, $X'' \in \mathfrak{g}_n''$. By Lemma 6.1, there is a non-zero vector $Z \in \mathfrak{g}_n$ such that $[Z, X] \neq 0$. Further, there is a real number $k \neq 0$ such that $[Z, X] = kX = kX' + kX''$. Formula (26) yields $kX'' = 0$, so $X'' = 0$. Therefore $X \in \mathfrak{g}_n'$,

$X = \sum_i c_i X_i$. By Lemma 6.2, there is an index $\alpha \in \{1, \dots, n\}$ such that $[X, U_\alpha] \neq 0$. Because (X) is an ideal, there is a real number l such that $[X, U_\alpha] = lX$. This implies $lc_i = 0$ for all $i \neq 0, \alpha$, and $(l + 1) c_0 = (l - 1) c_\alpha = 0$, $l \neq 0$. It means that at most one of the numbers $c_0, c_1, \dots, c_n$ is non-zero, namely c_0 or c_α. Because $X \neq 0$, exactly one of the numbers $c_0, c_1, \dots, c_n$ does not vanish, thus $(X) = (X_i)$ for some $i \in \{0, 1, \dots, n\}$.

Now, let us consider the multiplicative group $(\mathbf{Z}_2)^{n+1} = \{\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n) \mid \varepsilon_i = \pm 1\}$ and the permutation group $\mathbf{S}_{n+1}$ of the set $\{0, 1, \dots, n\}$. Their semi-direct product will be defined via $(\varepsilon, \sigma) . (\delta, \tau) = (\varepsilon * \delta, \sigma \circ \tau)$, where $(\varepsilon * \delta)_i = \varepsilon_{\tau(i)} . \delta_i$ for all $(\varepsilon, \sigma), (\delta, \tau) \in (\mathbf{Z}_2)^{n+1} \times \mathbf{S}_{n+1}$ and $i = 0, 1, \dots, n$. It will be proved that the groups $(\mathbf{Z}_2)^{n+1} \times \mathbf{S}_{n+1}$ and L are isomorphic. In order to do it let us define a map $\Phi : (\mathbf{Z}_2)^{n+1} \times \mathbf{S}_{n+1} \to GL(\mathfrak{g}_n)$ by the formulas

$$\Phi(\varepsilon, \sigma) X_i = \varepsilon_i X_{\sigma(i)} ,$$

$$\Phi(\varepsilon, \sigma) U_\alpha = \begin{cases} U_{\sigma(\alpha)} & \text{if} \quad \sigma(0) = 0 , \\ U_{\sigma(\alpha)} - U_{\sigma(0)} & \text{if} \quad \sigma(0) \neq 0 , \quad \sigma(\alpha) \neq 0 , \\ -U_{\sigma(0)} & \text{if} \quad \sigma(0) \neq 0 , \quad \sigma(\alpha) = 0 \end{cases}$$

for all $i = 0, 1, \dots, n$ and $\alpha = 1, \dots, n$.

Proposition 6.1. *The map Φ is an injective group homomorphism and* im $\Phi = L$.

Proof. Clearly, Φ is injective. An easy calculation shows that Φ is a group homomorphism. The inclusion im $\Phi \subset L$ follows from the formulas (4) and (5). It remains only to prove the converse inclusion.

Let $B \in L$ be an arbitrary element. The automorphism B of the algebra $\mathfrak{g}_n$ maps every 1-dimensional ideal (X_i), $i = 0, 1, \dots, n$ into a 1-dimensional ideal of $\mathfrak{g}_n$. Thus, Lemma 6.3 yields that there is a permutation $\sigma \in \mathbf{S}_{n+1}$ and a function $\varepsilon : \{0, 1, \dots, n\} \to \mathbf{R}$, $\varepsilon(i) = \varepsilon_i$ such that

(27) $$BX_i = \varepsilon_i X_{\sigma(i)} \quad \text{for all} \quad i .$$

Since the transformation B preserves the scalar product g and since all the vectors $X_0, X_1, \dots, X_n$ are of the same length, we have $\varepsilon_i = \pm 1$ for all i, that is $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n) \in (\mathbf{Z}_2)^{n+1}$.

Another consequence of Lemma 6.3 says that the subspace $\mathfrak{g}'_n$ is invariant by L. Therefore, the orthogonal complement $\mathfrak{g}''_n$ to $\mathfrak{g}'_n$ is invariant by L, too. It follows that there is a real matrix (B^β_α), $\alpha, \beta = 1, \dots, n$ such that $BU_\alpha = \sum_\beta B^\beta_\alpha U_\beta$ for all $\alpha = 1, \dots, n$. Because B is an automorphism of the algebra $\mathfrak{g}_n$, it is $B[X_{\sigma^{-1}(\beta)}, U_\alpha] = [BX_{\sigma^{-1}(\beta)}, BU_\alpha]$ for all α, β. Using formulas (4) we get

$$B^\beta_\alpha = \begin{cases} 1 & \text{if} \quad \beta = \sigma(\alpha) , \\ 0 & \text{if} \quad \beta \neq \sigma(\alpha) , \\ -1 & \text{if} \quad \beta = \sigma(0) . \end{cases}$$

Comparing this result and the relation (27) on the one hand with the definition of the mapping Φ on the other we get $B = \Phi(\varepsilon, \sigma)$.

Corollary. *If two transformations $B, B' \in L$ agree at all the vectors $X_0, X_1, \ldots, X_n$, then they coincide.*

Recall that the transformation S of the vector space $T_e(G_n) = \mathfrak{g}_n$ has been introduced in § 4 (see formulas (6)).

Proposition 6.2. *An element $B \in L$ belongs to the set L' if and only if it is conjugated to the transformation S.*

Proof. The part „if" is evident. To prove the converse implication we shall need the following.

Lemma 6.4. *Let $B = \Phi(\varepsilon, \sigma) \in L$ be an element of L'. Then σ is a full cycle and $B^{n+1}(X_0) = -X_0$.*

Proof. Let k be the least positive integer for which $\sigma^k(0) = 0$ holds. Suppose $k < n + 1$. Denote $U'_\alpha = U_{\alpha'}$, where $\alpha' = \sigma^\alpha(0)$ for all $\alpha \in \{1, \ldots, k - 1\}$ (this set may be empty), and denote the elements of the set $\{U_1, \ldots, U_n\} - \{U'_1, \ldots, U'_{k-1}\}$ by $U'_k, \ldots, U'_n$. Vector

$$(n - k + 1) \sum_{\alpha=1}^{k-1} U'_\alpha - k \sum_{\beta=k}^{n} U'_\beta$$

is a fixed non-zero vector of the transformation B. This contradicts the assumption $B \in L'$. Thus $k = n + 1$. Therefore, σ is a full cycle and $B^{n+1}(X_0) = \pm X_0$. It remains to get rid of the sign plus. But if $B^{n+1}(X_0) = +X_0$, then the non-zero vector $X_0 + B(X_0) + \ldots + B^n(X_0)$ is preserved under the transformation $B \in L'$, which is a contradiction.

Let us return to the proof of Proposition 6.2. By the foregoing lemma, every element $B = \Phi(\varepsilon, \sigma) \in L'$ determines a couple $(\delta, \tau) \in (\mathbf{Z}_2)^{n+1} \times \mathbf{S}_{n+1}$ such that $B^i(X_0) = \delta_i X_{\tau(i)}$ for all i. Put $A = \Phi(\delta, \tau)$. Then we have

$$B \circ A(X_i) = B(\delta_i X_{\tau(i)}) = B^{i+1}(X_0) = \Phi(\delta, \tau)\, X_{i+1} = A \circ S(X_i)$$

for all $i = 0, 1, \ldots, n - 1$. In virtue of the second part of Lemma 6.4 we have

$$B \circ A(X_n) = B^{n+1}(X_0) = -X_0 = \Phi(\delta, \tau)\,(-X_0) = A \circ S(X_n)\,.$$

Therefore, by Corollary of Proposition 6.1, we obtain $B \circ A = A \circ S$.

Because the transfotmation S is of order $2n + 2$, Proposition 6.2 and Theorem 2 imply the following

Proposition 6.3. Every regular s-structure on (G_n, g) is of order $2n + 2$.

Let $G_n = G^{(0)} \times G^{(1)} \times \ldots \times G^{(r)}$ be the de Rham decomposition of the Riemannian manifold (G_n, g). By a result of HANO, see Theorem VI.3.5 in [2], the identity component $I(G_n)^0$ of the full isometry group of (G_n, g) satisfies

$$I(G_n)^0 = I(G^{(0)})^0 \times I(G^{(1)})^0 \times \ldots \times I(G^{(r)})^0 .$$

Recall that, according to the first part of § 5, we identify the Lie algebra of the Lie group $I(G_n)^0$ with the algebra $\mathfrak{g}_n$.

Proposition 7.1. *The Riemannian manifold (G_n, g) is irreducible for all $n \geqq 1$.*

Proof. If $n = 1$ then $\dim G_1 = 3$ and, by Proposition 6.3, the g.s. space (G_1, g) is of order 4. Therefore, by Proposition 14.1 in [5], the g.s. space (G_1, g) is irreducible.

Let $n \geqq 2$. Suppose that G_n is reducible, $G_n = G^1 \times G^2$. Then the Lie algebra $\mathfrak{g}_n$ is a direct sum of some non-trivial ideals $\mathfrak{g}^1$ and $\mathfrak{g}^2$. For every $X \in \mathfrak{g}_n$ denote by X^1 and X^2 the $\mathfrak{g}^1$- and $\mathfrak{g}^2$-components of X, respectively.

Lemma 7.1. *$X_i \in \mathfrak{g}^1 \cup \mathfrak{g}^2$ for all $i = 0, 1, \ldots, n$.*

Proof. According to formulas (4) we have

$$X_0^1 = [X_0^1, U_\alpha] , \tag{28}$$

$$X_\alpha^1 = -[X_\alpha^1, U_\alpha] \tag{29}$$

for all $\alpha = 1, \ldots, n$. By formulas (29) and (26), there are real numbers c_{ij}, $i, j = 0, 1, \ldots, n$ such that

$$X_i^1 = \sum_{j=0}^{n} c_{ij} X_j \quad \text{for all} \quad i . \tag{30}$$

From (28), (30) and (4) we obtain

$$X_0^1 = [\sum_{j=0}^{n} c_{0j} X_j, U_\alpha] = c_{00} X_0 - c_{0\alpha} X_\alpha$$

for all α. Because $n \geqq 2$, this yields $c_{0\alpha} = 0$ for all α and $X_0^1 = c_{00} X_0 = c_{00}(X_0^1 + X_0^2)$ hence $(c_{00} - 1) X_0^1 + c_{00} X_0^2 = 0$. It follows $c_{00} = 1$ or $c_{00} = 0$, which is equivalent to $X_0 \in \mathfrak{g}^1 \cup \mathfrak{g}^2$. In the same way, using formulas (29), (30) and (4) we get $X_\alpha \in \mathfrak{g}^1 \cup \mathfrak{g}^2$ for all $\alpha = 1, \ldots, n$.

Lemma 7.2. *$X_\alpha \in \mathfrak{g}^\varepsilon \Rightarrow U_\alpha \in \mathfrak{g}^\varepsilon$ for all $\alpha = 1, \ldots, n$ and $\varepsilon = 1, 2$.*

Proof. Let $\alpha \in \{1, \ldots, n\}$ be arbitrary but fixed. Let us suppose e.g. $X_\alpha \in \mathfrak{g}^1$.