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DISCRETE ANALOGUES OF WIRTINGER'S INEQUALITY FOR A TWO-DIMENSIONAL ARRAY

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In [4], G. PÓLYA and G. SZEGÖ studied the inequality

$$(*) \qquad \iint_D (f_x^2 + f_y^2)\,dx\,dy \geqq \Lambda^2 \iint_D f^2\,dx\,dy\,,$$

where $f = 0$ on the boundary C of the domain of integration D. In [2], H. D. BLOCK dealt with the corresponding discrete problem. The inequality is given for the two-dimensional array

$$\{x_{ij}\}_{\substack{i=1,\dots,m\\ j=1,\dots,n}}\,.$$

In [3] we have shown a new, simpler proof of the discrete analogues of Wirtinger's inequality in case of n numbers $x_1, \dots, x_n$. The proof was based on the use of trigonometric polynomials (see [1], pp. 13–20). The paper contains also some sharpenings of the inequalities obtained.

In the present paper, we establish the two-dimensional analogues of trigonometric polynomials. Using them we prove the discrete variations of $(*)$ in a similar way as in [3]. To simplify the proofs, the inequalities are studied for arrays of the form $\{x_{ij}\}_{i,j=1}^n$. The results for

$$\{x_{ij}\}_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$$

could be proved in the same way.

Using the results established in [3] we prove some inequalities for the "asymmetrical" case, i.e. inequalities involving the series

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}^2 \quad \text{and} \quad \sum_{i=1}^{n}\sum_{j=1}^{n} (x_{ij} - x_{i+1,j})^2\,.$$

1. LIST OF THEOREMS FROM [3] USED IN THE PAPER

Theorem 1.1. *Let $x_1, \dots, x_n$ be n real numbers such that*

$$(1.1) \qquad \sum_{i=1}^{n} x_i = 0\,.$$

Let us define $x_{n+1} = x_1$. *Then*

$$\sum_{i=1}^{n} (x_i - x_{i+1})^2 \geqq 4 \sin^2 \frac{\pi}{n} \sum_{i=1}^{n} x_i^2 . \tag{1.2}$$

The equality in (1.2) *holds if and only if*

$$x_i = A \cos \frac{2\pi i}{n} + B \sin \frac{2\pi i}{n}, \quad i = 1, \dots, n, \quad A, B = \text{const.} \tag{1.3}$$

Theorem 1.2. *If* $x_1, \dots, x_n$ *are n real numbers and* $x_1 = 0$, *then*

$$\sum_{i=1}^{n-1} (x_i - x_{i+1})^2 \geqq 4 \sin^2 \frac{\pi}{2(2n-1)} \sum_{i=2}^{n} x_i^2 . \tag{1.4}$$

The equality in (1.4) *holds if and only if*

$$x_i = A \sin \frac{(i-1)\pi}{n}, \quad i = 1, \dots, n, \quad A = \text{const.} \tag{1.5}$$

Theorem 1.3. *If* $x_1, \dots, x_n$ *are n real numbers, then*

$$\sum_{i=0}^{n} (x_i - x_{i+1})^2 \geqq 4 \sin^2 \frac{\pi}{2(n+1)} \sum_{i=0}^{n} x_i^2 , \tag{1.6}$$

where $x_0 = x_{n+1} = 0$. *The equality in* (1.6) *holds if and only if*

$$x_i = A \sin \frac{i\pi}{n+1}, \quad i = 1, \dots, n, \quad A = \text{const.} \tag{1.7}$$

Theorem 1.4. *Let* $x_1, \dots, x_n$ *be n real numbers satisfying* (1.1). *Then*

$$\sum_{i=1}^{n-1} (x_i - x_{i+1})^2 \geqq 4 \sin^2 \frac{\pi}{2n} \sum_{i=1}^{n} x_i^2 . \tag{1.8}$$

The equality in (1.8) *holds if and only if*

$$x_i = A \cos \frac{(2i-1)\pi}{2n}, \quad i = 1, \dots, n, \quad A = \text{const.} \tag{1.9}$$

2. SYMMETRICAL CASE

Notation. To simplify the form of inequalities, we shall write $D^2 x_{ij}$ instead of $(x_{ij} - x_{i+1,j})^2 + (x_{ij} - x_{i,j+1})^2$.

The basic theorem in this article is Theorem 2.1, the two-dimensional analogue of Theorem 1.1. Theorems 2.2 through 2.4 are analogues of Theorems 1.2 through 1.4. Theorem 2.5 is a sharpening of Theorem 2.1 and Theorem 2.6 is a sharpening of Theorem 2.4.

Theorem 2.1. *Let* $\{x_{ij}\}_{i,j=1}^{n}$ *be* n^2 *real numbers such that*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} = 0 . \tag{2.1}$$

Let us define $x_{i,n+1} = x_{i1}$, $x_{n+1,i} = x_{1i}$, $i = 1, \ldots, n$. *Then*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} D^2 x_{ij} \geqq 4 \sin^2 \frac{\pi}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij}^2 . \tag{2.2}$$

The equality in (2.2) *holds if and only if*

$$x_{ij} = A \cos \frac{2\pi i}{n} + B \sin \frac{2\pi i}{n} + C \cos \frac{2\pi j}{n} + D \sin \frac{2\pi j}{n}, \tag{2.3}$$

$$i, j = 1, \ldots, n, \quad A, B, C, D = \text{const.}$$

The proof of Theorem 2.1 will be given in Section 4.

Theorem 2.2. *Let* $\{x_{ij}\}_{i,j=1}^{n}$ *be* n^2 *real numbers such that* $x_{i1} = x_{1i} = 0$, $i = 1, \ldots, n$. *Then (putting* $x_{n+1,j} = x_{nj}$, $x_{j,n+1} = x_{jn}$)

$$\sum_{i=1}^{n} \sum_{j=1}^{n} D^2 x_{ij} \geqq 4 \sin^2 \frac{\pi}{2(2n-1)} \sum_{i=2}^{n} \sum_{j=2}^{n} x_{ij}^2 . \tag{2.4}$$

The equality in (2.4) *holds if and only if*

$$x_{ij} = A \sin \frac{\pi(i-1)}{2n-1} + B \sin \frac{\pi(j-1)}{2n-1}, \tag{2.5}$$

$$i, j = 1, \ldots, n, \quad A, B = \text{const.}$$

Proof. We apply Theorem 2.1 to a new array $\{y_{kl}\}_{k,l=1}^{2(2n-1)}$ (analogously to the proof of Theorem 2 in [3]) defined as follows (schematically written in the form of a matrix):

$$\begin{array}{cccccccc}
x_{11}, \ldots, & x_{1n}, & x_{1n}, \ldots, & x_{12}, & -x_{11}, \ldots, & -x_{1n}, & -x_{1n}, \ldots, & -x_{12} \\
\vdots & & & & & & & \\
x_{n1}, \ldots, & x_{nn}, & x_{nn}, \ldots, & x_{n2}, & -x_{n1}, \ldots, & -x_{nn}, & -x_{nn}, \ldots, & -x_{n2} \\
x_{n1}, \ldots, & x_{nn}, & x_{nn}, \ldots, & x_{n2}, & -x_{n1}, \ldots, & -x_{nn}, & -x_{nn}, \ldots, & -x_{n2} \\
\vdots & & & & & & & \\
x_{21}, \ldots, & x_{2n}, & x_{2n}, \ldots, & x_{22}, & -x_{21}, \ldots, & -x_{2n}, & -x_{2n}, \ldots, & -x_{22} \\
-x_{11}, \ldots, & -x_{1n}, & -x_{1n}, \ldots, & -x_{12}, & x_{11}, \ldots, & x_{1n}, & x_{1n}, \ldots, & x_{12} \\
\vdots & & & & & & & \\
-x_{n1}, \ldots, & -x_{nn}, & -x_{nn}, \ldots, & -x_{n2}, & x_{n1}, \ldots, & x_{nn}, & x_{nn}, \ldots, & x_{n2} \\
-x_{n1}, \ldots, & -x_{nn}, & -x_{nn}, \ldots, & -x_{n2}, & x_{n1}, \ldots, & x_{nn}, & x_{nn}, \ldots, & x_{n2} \\
\vdots & & & & & & & \\
-x_{21}, \ldots, & -x_{2n}, & -x_{2n}, \ldots, & -x_{22}, & x_{21}, \ldots, & x_{2n}, & x_{2n}, \ldots, & x_{22},
\end{array}$$

$$y_{4n-1,l} = y_{l,4n-1} = 0 .$$

(2.5) follows from (2.3) for y_{kl} and from the equalities

$$y_{11} = y_{2n,1}, \quad y_{11} = y_{1,2n} .$$

Theorem 2.3. *Let* $\{x_{ij}\}_{i,j=1}^{n}$ *be* n^2 *real numbers such that* $x_{0j} = x_{n+1,j} = x_{j0} = x_{j,n+1} = 0, j = 1, \ldots, n$. *Then*

$$\sum_{i=0}^{n} \sum_{j=0}^{n} D^2 x_{ij} \geqq 8 \sin^2 \frac{\pi}{2(n+1)} \sum_{i=0}^{n} \sum_{j=0}^{n} x_{ij}^2 . \tag{2.6}$$

The equality in (2.6) *holds if and only if*

$$x_{ij} = A \sin \frac{i\pi}{n+1} \sin \frac{j\pi}{n+1}, \quad i, j = 1, \ldots, n, \quad A = \text{const.} \tag{2.7}$$

Remark. 1. (2.6) is a discrete analogue of (*) for a special case $D = (0, \pi) \times (0, \pi)$, $\Lambda^2 = 2$. This inequality can be derived from (2.6).

2. Using the method of the proof of Theorem 2.2 with $\{y_{kl}\}_{k,l=1}^{2(n+1)}$ defined as follows (analogously to the proof of Theorem 3 in [3]):

$$\begin{matrix} 0, & 0, & \ldots, 0, & 0, & 0, & \ldots, & 0 \\ 0, & x_{11}, & \ldots, x_{1n}, & 0, & 0, & \ldots, & 0 \\ \vdots & & & & & & \\ 0, & x_{n1}, & \ldots, x_{nn}, & 0, & 0, & \ldots, & 0 \\ 0, & 0, & \ldots, 0, & 0, & 0, & \ldots, & 0 \\ 0, & 0, & \ldots, 0, & 0, & -x_{11}, & \ldots, & -x_{1n} \\ \vdots & & & & & & \\ 0, & 0, & \ldots, 0, & 0, & -x_{n1}, & \ldots, & -x_{nn}, \end{matrix}$$

$y_{2n+3,l} = y_{l,2n+3} = 0$, we could derive an inequality similar to (2.6) with the constant 4 instead of 8 at the right hand side and with the equality achieved only for $x_{ij} = 0$, $i, j = 1, \ldots, n$.

Proof. Choosing i fix, $1 \leqq i \leqq n$, we can apply Theorem 1.3 to the numbers x_{ij}, $j = 1, \ldots, n$. Adding these inequalities for i, $1 \leqq i \leqq n$, and applying similarly Theorem 1.3 to the numbers x_{ij}, $i = 1, \ldots, n$, for j fix, $1 \leqq j \leqq n$, we obtain (2.6), (2.7).

Theorem 2.4. Let $\{x_{ij}\}_{i,j=1}^{n}$ *be* n^2 *real numbers satisfying* (2.1). *Then* (*putting* $x_{n+1,j} = x_{nj}$, $x_{j,n+1} = x_{jn}$)

$$\sum_{i=1}^{n} \sum_{j=1}^{n} D^2 x_{ij} \geqq 4 \sin^2 \frac{\pi}{2n} \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij}^2 . \tag{2.8}$$

The equality in (2.8) *holds if and only if*

$$x_{ij} = A \cos \frac{(2i-1)\pi}{2n} + B \cos \frac{(2j-1)\pi}{2n}, \quad i, j = 1, \ldots, n, \quad A, B = \text{const.} \tag{2.9}$$

Proof. Let us apply Theorem 2.1 to a new array $\{y_{kl}\}_{k,l=1}^{2n}$ defined as follows (analogously to the proof of Theorem 4 in [3]):

$$\begin{array}{l} x_{11}, \ldots, x_{1n}, x_{1n}, \ldots, x_{11} \\ \vdots \\ x_{n1}, \ldots, x_{nn}, x_{nn}, \ldots, x_{n1} \\ x_{n1}, \ldots, x_{nn}, x_{nn}, \ldots, x_{n1} \\ \vdots \\ x_{11}, \ldots, x_{1n}, x_{1n}, \ldots, x_{11}\,, \end{array}$$

$y_{2n+1,l} = y_{l,2n+1} = y_{l1}$, which also satisfies (2.1). Then (2.8), (2.9) follow from (2.2), (2.3).

Theorem 2.5 (sharpening of Theorem 2.1 for n even). *Let $n = 2m$, $n \geqq 4$. Let $\{x_{ij}\}_{i,j=1}^{n}$ be n^2 real numbers satisfying* (2.1). *Let us define $x_{n+i,n+j} = x_{ij}$, $i, j = 1, \ldots, m$. Then*

$$(2.10)\quad \sum_{i=1}^{n} \sum_{j=1}^{n} D^2 x_{ij} \geqq \tfrac{1}{4} \sin^2 \frac{\pi}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} (x_{ij} + x_{i+m,j+m})^2 + 4 \sin^2 \frac{\pi}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij}^2\,.$$

The equality in (2.10) *holds if and only if x_{ij} satisfy* (2.3).

The proof of Theorem 2.5 will be given in Section 4.

3. ASYMMETRICAL CASE

Here we shall study inequalities involving $\sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij}^2$ and $\sum_{i=1}^{n} \sum_{j=1}^{n} (x_{ij} - x_{i+1,j})^2$. To simplify the form of inequalities, we shall denote $A^2 x_{ij} = (x_{ij} - x_{i+1,j})^2$. To derive these inequalities we shall use Theorems 1.1 through 1.4.

Theorem 3.1. *Let $n = 2m$. Let $\{x_{ij}\}_{i,j=1}^{n}$ be n^2 real numbers such that $x_{i1} = x_{i,m+1} = c$, $i = 1, \ldots, n$, and*

$$(3.1)\qquad \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} = 0\,.$$

Let us define $x_{n+1,j} = x_{1j}$, $j = 1, \ldots, n$. Then

$$(3.2)\qquad \sum_{i=1}^{n} \sum_{j=1}^{n} A^2 x_{ij} \geqq 4 \sin^2 \frac{\pi}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij}^2 + 4n^2 c^2 \sin^2 \frac{\pi}{n}\,.$$

The equality in (3.2) *holds if and only if*

$$(3.3)\quad x_{ij} = \begin{cases} c + A_i \sin \dfrac{(j-1)\pi}{m}\,, & j = 1, \ldots, m\,, \quad i = 1, \ldots, n\,, \\[2ex] c + B_i \sin \dfrac{(j-m-1)\pi}{m}\,, & j = m+1, \ldots, n\,, \quad i = 1, \ldots, n\,, \end{cases}$$

where the numbers A_i, B_i do not depend on j and satisfy the relation

$$(3.4)\qquad n^2c + \operatorname{cotg}\frac{\pi}{n}\sum_{i=1}^{n}(A_i + B_i) = 0\,.$$

Proof. Take i fix, $1 \leqq i \leqq n$. Let us define one-dimensional arrays $\{y_k\}_{k=0}^{m}$, $\{z_k\}_{k=0}^{m}$ as follows: $y_k = x_{i,k+1} - c$, $z_k = x_{i,m+k+1} - c$. Then $y_0 = y_m = z_0 = z_m = 0$; applying Theorem 1.3 to the arrays $\{y_k\}_{k=1}^{m-1}$, $\{z_k\}_{k=1}^{m-1}$ and adding the obtained relations for i, $1 \leqq i \leqq n$, we obtain the statement of Theorem 3.1.

Theorem 3.2. *Let $\{x_{ij}\}_{i,j=1}^{n}$ be n^2 real numbers satisfying (3.1) and such that $x_{i1} = c$, $i = 1, \ldots, n$. Then*

$$(3.5)\qquad \sum_{i=1}^{n}\sum_{j=1}^{n-1} A^2 x_{ij} \geqq 4\sin^2\frac{\pi}{2(2n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}^2 + 4n^2c^2\sin^2\frac{\pi}{2(2n-1)}\,.$$

The equality in (3.5) holds if and only if

$$(3.6)\qquad x_{ij} = c + A_i \sin\frac{(j-1)\pi}{2n-1}\,,$$

where the numbers A_i do not depend on j and satisfy the relation

$$(3.7)\qquad 2n^2c + \operatorname{cotg}\frac{\pi}{2(2n-1)}\sum_{i=1}^{n} A_i = 0\,.$$

Proof is similar to the previous one, but we apply Theorem 1.2 to the one-dimensional array $\{y_k\}_{k=1}^{n}$, $y_k = x_{ik} - c$, i fixed.

Theorem 3.3. *Let $\{x_{ij}\}_{i,j=1}^{n}$ satisfy the assumption of Theorem 3.2. Let us define $x_{n+1,j} = x_{1j} = c$, $j = 1, \ldots, n$. Then*

$$(3.8)\qquad \sum_{i=1}^{n}\sum_{j=1}^{n} A^2 x_{ij} \geqq 4\sin^2\frac{\pi}{2n}\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}^2 + 4n^2c^2\sin^2\frac{\pi}{2n}\,.$$

The equality in (3.8) holds if and only if

$$(3.9)\qquad x_{ij} = c + A_i \sin\frac{(j-1)\pi}{n}\,, \quad i, j = 1, \ldots, n\,,$$

where the numbers A_i do not depend on j and satisfy the relation

$$(3.10)\qquad n^2c + \operatorname{cotg}\frac{\pi}{2n}\sum_{i=1}^{n} A_i = 0\,.$$

Proof. Theorem 3.3 follows from Theorem 1.3 in a similar way as the previous two theorems or from Theorem 3.1 when defining the two-dimensional array $\{y_{kl}\}_{k,l=1}^{2n}$ as follows:

$$
\begin{array}{l}
x_{11}, \ldots, x_{1n}, \; x_{11}, \ldots, x_{1n} \\
\vdots \\
x_{n1}, \ldots, x_{nn}, \; x_{n1}, \ldots, x_{nn} \\
x_{11}, \ldots, x_{1n}, \; x_{11}, \ldots, x_{1n} \\
\vdots \\
x_{n1}, \ldots, x_{nn}, \; x_{n1}, \ldots, x_{nn} .
\end{array}
$$

In the previous three theorems the assumption (3.1) was very important. Now we shall show two more theorems without using this assumption. However, we have to assume that the constant $c = 0$. Theorems follow from Thorem 3.1 in a way analogous to the proofs in Section 2. We shall only define new arrays in the schematic form of a matrix.

Theorem 3.4. *Let* $\{x_{ij}\}_{i,j=1}^{n}$ *be* n^2 *real numbers such that* $x_{i1} = 0$, $i = 1, \ldots, n$. *Then*

$$
\sum_{i=1}^{n} \sum_{j=1}^{n-1} A^2 x_{ij} \geqq 4 \sin^2 \frac{\pi}{2(2n-1)} \sum_{i=1}^{n} \sum_{j=2}^{n} x_{ij}^2 . \tag{3.11}
$$

The equality in (3.11) *holds if and only if*

$$
x_{ij} = A_i \sin \frac{(j-1)\pi}{2n-1} , \quad i, j = 1, \ldots, n , \quad A_i \text{ do not depend on } j . \tag{3.12}
$$

Proof. $\{y_{kl}\}_{k,l=1}^{2(2n-1)}$:

$$
\begin{array}{l}
x_{11}, \ldots, x_{1n}, x_{1n}, \ldots, x_{12}, \; -x_{11}, \ldots, -x_{1n}, \; -x_{1n}, \ldots, -x_{12} \\
\vdots \\
x_{n1}, \ldots, x_{nn}, x_{nn}, \ldots, x_{n2}, \; -x_{n1}, \ldots, -x_{nn}, \; -x_{nn}, \ldots, -x_{n2} \\
\qquad\qquad\qquad\qquad \mathbf{0},
\end{array}
$$

$y_{4n-1,l} = y_{1l}$; then $c = 0$, $n_1 = 2(2n-1)$.

Theorem 3.5. *Let* $\{x_{ij}\}_{i,j=1}^{n}$ *be* n^2 *real numbers such that* $x_{i0} = x_{i,n+1} = 0$, $i = 1, \ldots, n$. *Then*

$$
\sum_{i=1}^{n} \sum_{j=0}^{n} A^2 x_{ij} \geqq 4 \sin^2 \frac{\pi}{2(n+1)} \sum_{i=1}^{n} \sum_{j=0}^{n} x_{ij}^2 . \tag{3.13}
$$

The equality in (3.13) *holds if and only if*

$$
x_{ij} = A_i \sin \frac{j\pi}{n+1} , \quad i, j = 1, \ldots, n , \quad A_i \text{ do not depend on } j . \tag{3.14}
$$

Proof. $\{y_{kl}\}_{k,l=1}^{2(n+1)}$:

$$
\begin{array}{l}
0, \; x_{11} \ldots, x_{1n}, \; 0, \; -x_{11}, \ldots, -x_{1n} \\
\vdots \\
0, \; x_{n1}, \ldots, x_{nn}, \; 0, \; -x_{n1}, \ldots, -x_{nn} \\
\qquad\qquad \mathbf{0},
\end{array}
$$

$y_{2n+3,l} = y_{1l}$; then $c = 0$, $n_1 = 2(n+1)$.

4. PROOFS OF THEOREMS 2.1 AND 2.5

In a way analogous to the introduction of trigonometric polynomials in [1] we can show that for any array $\{x_{ij}\}_{i,j=1}^{n}$ there exist such numbers $\xi_0, \xi_p, \xi_p^*, \eta_p, \eta_p^*, p = 1, \ldots, m, \vartheta_{st}, \vartheta_{st}^*, \mu_{st}, \mu_{st}^*, s, t = 1, \ldots, m$, that

$$x_{ij} = \xi_0 + \sum_{p=1}^{m} \left(\xi_p \cos pi \frac{2\pi}{n} + \xi_p^* \sin pi \frac{2\pi}{n} + \eta_p \cos pj \frac{2\pi}{n} + \eta_p^* \sin pj \frac{2\pi}{n} \right) + \sum_{s=1}^{m} \sum_{t=1}^{m} \left(\vartheta_{st} \cos si \frac{2\pi}{n} \sin tj \frac{2\pi}{n} + \vartheta_{st}^* \sin si \frac{2\pi}{n} \cos tj \frac{2\pi}{n} + \mu_{st} \cos si \frac{2\pi}{n} \cos tj \frac{2\pi}{n} + \mu_{st}^* \sin si \frac{2\pi}{n} \sin tj \frac{2\pi}{n} \right), \quad i, j = 1, \ldots, n, \tag{4.1}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij}^2 = n^2 \xi_0^2 + \frac{n^2}{2} \sum_{p=1}^{m} (\xi_p^2 + \xi_p^{*2} + \eta_p^2 + \eta_p^{*2}) + \frac{n^2}{4} \sum_{s=1}^{m} \sum_{t=1}^{m} (\vartheta_{st}^2 + \vartheta_{st}^{*2} + \mu_{st}^2 + \mu_{st}^{*2}), \tag{4.2}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} D^2 x_{ij} = 2n^2 \sum_{p=1}^{m} (\xi_p^2 + \xi_p^{*2} + \eta_p^2 + \eta_p^{*2}) \sin^2 p \frac{\pi}{n} + n^2 \sum_{s=1}^{m} \sum_{t=1}^{m} (\vartheta_{st}^2 + \vartheta_{st}^{*2} + \mu_{st}^2 + \mu_{st}^{*2}) \, . \left(\sin^2 s \frac{\pi}{n} + \sin^2 t \frac{\pi}{n} \right). \tag{4.3}$$

From (2.1) it follows that

$$\xi_0 = 0 \, . \tag{4.4}$$

Theorem 2.1 follows immediately from (4.1)–(4.4).

Using (4.1) and (4.2) we derive

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (x_{ij} + x_{i+m,j+m})^2 = \frac{n^2}{2} \sum_{p=1}^{m} (\xi_p^2 + \xi_p^{*2} + \eta_p^2 + \eta_p^{*2}) [1 + (-1)^p]^2 + \frac{n^2}{4} \sum_{s=1}^{m} \sum_{t=1}^{m} (\vartheta_{st}^2 + \vartheta_{st}^{*2} + \mu_{st}^2 + \mu_{st}^{*2}) \, . [1 + (-1)^s + (-1)^t + (-1)^{st}]^2 \, . \tag{4.5}$$

Theorem 2.5 is a consequence of (4.1)–(4.5) in an analogous way as in [3] (the proof of Theorem 2.5).