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CONCERNING THE CHARACTERIZATION OF GENERATORS OF DISTRIBUTION SEMIGROUPS

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In the first part of this paper we prove a new characteristic property of generators of distribution semigroups of operators using only the behavior of their revolvents on the real halfaxis. It is similar to that of OHARU [1] but does not involve the graph spaces of powers of the generator (Theorem 1.2).

In the second part, we prove the necessity of the above mentioned property directly from Chazarain's condition [2] on the behavior of the resolvent in a logarithmic domain of the complex plane (Theorem 2.5 and 2.7).

In the sequel, E will be an arbitrary Banach space over the real R or complex C field (real in the first and complex in the second part).

If A is an arbitrary linear operator from E into E, we define formally $A^0 = I$, I being the identity operator.

1. SEMIGROUPS AND RESOLVENTS

1.1. Lemma. *For every $\lambda > 1$, $r \in \{1, 2, \ldots\}$ and $p \in \{0, 1, \ldots\}$ we have*

$$\left| \frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} \frac{1}{\lambda^r} \right| \leqq \frac{p!}{(\lambda - 1)^{p+1}} .$$

Proof is easy.

1.2. Theorem. *Let A be a linear operator from E into E. Then the following two statements are equivalent:*

(O) *there exists a constant $\varkappa \geqq 0$ such that*

(I) $(\varkappa, \infty) \subseteq \varrho(A)$,

(II) *for every* $T > 0$ *there exist* $k \in \{0, 1, \ldots\}$ *and* $K \geqq 0$ *such that*

$$\|(\lambda I - A)^{-n} x\| \leqq \frac{K}{(\lambda - \varkappa)^n} \sum_{j=0}^{k} \|A^j x\| \quad \textit{for every} \quad n \in \{1, 2, \ldots\},\ \lambda > \frac{n}{T} + \varkappa$$

and $x \in \mathsf{D}(A^k)$,

(C) *there exists a constant* $\omega \geqq 0$ *such that*

(I) $(\omega, \infty) \subseteq \varrho(A)$,

(II) *for every* $T > 0$ *there exist* $l \in \{0, 1, \ldots\}$ *and* $M \geqq 0$ *such that*

$$\left\| \frac{d^p}{d\lambda^p} \frac{1}{\lambda^l} (\lambda I - A)^{-1} \right\| \leqq \frac{Mp!}{(\lambda - \omega)^{p+1}} \quad \textit{for every} \quad p \in \{0, 1, \ldots\} \quad \textit{and}$$

$\lambda > \dfrac{p+1}{T} + \omega.$

Proof. For the sake of simplicity we shall write $R(\lambda) = (\lambda I - A)^{-1}$ for every $\lambda \in \varrho(A)$.

It is well-known that

(1) $A\, R(\mu) = \mu\, R(\mu) - I$ for every $\mu \in \varrho(A)$,

(2) $\dfrac{\mathrm{d}^p}{\mathrm{d}\mu^p} R(\mu) = (-1)^p\, p!\, R(\mu)^{p+1}$ for every $\mu \in \varrho(A)$ and $p \in \{0, 1, \ldots\}$.

Now we begin by proving (O) $\Rightarrow$ (C).

Using (1) we easily obtain by induction on s that

(3) $\dfrac{1}{\lambda^s} R(\lambda) = R(\lambda)\, R(\alpha)^s - \alpha\, R(\lambda) \displaystyle\sum_{r=1}^{s} \frac{1}{\lambda^r} R(\alpha)^{s+1-r} - \sum_{r=1}^{s} \frac{1}{\lambda^r} R(\alpha)^{s+1-r}$ for every

$\lambda \in \varrho(A)$, $\lambda \neq 0$, $\alpha \in \varrho(A)$ and $s \in \{1, 2, \ldots\}$.

Let us now choose $\varkappa \geqq 0$ so that (O) holds.

Let $T > 0$ be fixed.

For this $T > 0$ we can find $k \in \{0, 1, \ldots\}$ and $K \geqq 0$ such that (O) (II) holds.

The case $k = 0$ is trivial according to (2) and therefore we suppose $k \in \{1, 2, \ldots\}$.

We write $|||x||| = \sum_{j=0}^{k} \|A^j x\|$ for $x \in \mathsf{D}(A^k)$.

Then we have according to (2) that

(4) $\left\| \dfrac{\mathrm{d}^p}{\mathrm{d}\lambda^p} R(\lambda)(x) \right\| \leqq \dfrac{Kp!}{(\lambda - \varkappa)^{p+1}} |||x|||$ for every $p \in \{0, 1, \ldots\}$, $\lambda > \dfrac{p+1}{T} + \varkappa$ and

$x \in \mathsf{D}(A^k)$.

Using (4) we get by means of Lemma 1.1 that

(5) $\left\| \dfrac{\mathrm{d}^p}{\mathrm{d}\lambda^p} \dfrac{1}{\lambda^r} R(\lambda)(x) \right\| = \left\| \displaystyle\sum_{q=0}^{p} \binom{p}{q} \frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} \frac{1}{\lambda^r} \frac{\mathrm{d}^{p-q}}{\mathrm{d}\lambda^{p-q}} R(\lambda)(x) \right\| \leqq$

$$\leqq \sum_{q=0}^{p} \binom{p}{q} \frac{q!}{(\lambda - 1)^{q+1}} \frac{K(p-q)!}{(\lambda - \varkappa)^{p-q+1}} \vert\vert\vert x \vert\vert\vert =$$

$$= K \sum_{q=0}^{p} \binom{p}{q} \frac{q!}{(\lambda - \varkappa - 1)^{q+1}} \frac{(p-q)!}{(\lambda - \varkappa - 1)^{p-q+1}} \vert\vert\vert x \vert\vert\vert =$$

$$= (-1)^p K \sum_{q=0}^{p} \binom{p}{q} \frac{\mathrm{d}^q}{\mathrm{d}\lambda^q} \frac{1}{\lambda - \varkappa - 1} \frac{\mathrm{d}^{p-q}}{\mathrm{d}\lambda^{p-q}} \frac{1}{\lambda - \varkappa - 1} \vert\vert\vert x \vert\vert\vert =$$

$$= (-1)^p K \frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} \frac{1}{(\lambda - \varkappa - 1)^2} \vert\vert\vert x \vert\vert\vert = K \frac{p!}{(\lambda - \varkappa - 2)^{p+1}} \vert\vert\vert x \vert\vert\vert$$

for every $x \in \mathsf{D}(A^k)$, $r \in \{1, 2, \ldots\}$, $p \in \{0, 1, \ldots\}$ and $\lambda > \dfrac{p+1}{T} + \varkappa + 2$.

Let us now fix an arbitrary $\alpha \in \varrho(A)$.
It follows from (1) that there exists a $K_0 \geqq 0$ such that

(7) $\vert\vert\vert R(\alpha)^j x \vert\vert\vert \leqq K_0 \|x\|$ for every $x \in E$ and $j \in \{0, 1, \ldots, k\}$.
It is clear that (7) also implies

(8) $\|R(\alpha)^j x\| \leqq K_0 \|x\|$ for every $x \in E$ and $j \in \{0, 1, \ldots, k\}$.
Now by (3), (4), (5) and (7), (8) and by Lemma 1.1

$$\text{(9)} \quad \left\| \frac{\mathrm{d}\lambda^p}{\mathrm{d}^p} \frac{1}{\lambda^k} R(\lambda)\, x \right\| \leqq \frac{Kp!}{(\lambda - \varkappa)^{p+1}} K_0 \|x\| + \alpha k \frac{Kp!}{(\lambda - \varkappa - 2)^{p+1}} K_0 \|x\| +$$

$$+ k \frac{p!}{(\lambda - 1)^{p+1}} K_0 \|x\| \leqq \frac{(K + \alpha k K + k)\, K_0 p!}{(\lambda - \varkappa - 2)^{p+1}} \|x\|$$

for every $x \in E$, $p \in \{0, 1, \ldots\}$ and $\lambda > \dfrac{p+1}{T} + \varkappa + 2$.

Taking $\omega = \varkappa + 2$, $l = k$, $M = (K + \alpha k K + k) K_0$ we see that (9) proves (C) because $T > 0$ was arbitrary.

We return to the verification of (C) $\Rightarrow$ (O).

It follows from (1) that

(10) $R(\lambda) \supseteq \dfrac{1}{\lambda} R(\lambda)\, A - \dfrac{1}{\lambda} I$ for every $\lambda \in \varrho(A)$, $\lambda \neq 0$.

Using (10) we prove easily by induction on s that

(11) $R(\lambda) \supseteq \dfrac{1}{\lambda^s} R(\lambda)\, A^s - \sum\limits_{r=1}^{s} \dfrac{1}{\lambda^r} A^{r-1}$ for every $\lambda \in \varrho(A)$, $\lambda \neq 0$ and $s \in \{1, 2, \ldots\}$.

Let $\omega \geqq 0$ be such that (C) holds.
Now we fix a $T > 0$.

For this $T > 0$ we can find $l \in \{0, 1, \ldots\}$ and $M \geqq 0$ such that (C) (II) holds. We omit the case $l = 0$ which is trivial according to (2) and therefore we suppose $l \in \{1, 2, \ldots\}$.

It follows from (1) and (11) by means of Lemma 1.1 that

$$(12)\quad \|R(\lambda)^n x\| = \left\| (-1)^{n-1} \frac{1}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}\lambda^{n-1}} R(\lambda)\, x \right\| =$$

$$= \frac{1}{(n-1)!} \left\| \frac{\mathrm{d}^{n-1}}{\mathrm{d}\lambda^{n-1}} R(\lambda)\, x \right\| \leqq \frac{1}{(n-1)!} \left[\frac{M(n-1)!}{(\lambda-\omega)^n} \|A^l x\| + \right.$$

$$\left. + \frac{(n-1)!}{(\lambda-1)^n} \sum_{r=1}^{l} \|A^{r-1} x\| \right] = \frac{M}{(\lambda-\omega-1)^n} \sum_{j=0}^{l} \|A^j x\| \quad \text{for every} \quad n \in \{1, 2, \ldots\},$$

$$\lambda > \frac{n}{T} + \omega + 1 \quad \text{and} \quad x \in \mathrm{D}(A^l).$$

Taking $\varkappa = \omega + 1$, $k = l$ and $K = M$ we see that (12) proves (O) because $T > 0$ was arbitrary.

1.3. First Characterization Theorem. *Let A be a linear operator from E into E. Then the operator A is the generator of a regular distribution semigroup if and only if it is densely defined and possesses the property (C) from the preceding theorem.*

Proof. Immediate consequence of Theorem 1.2 and Oharu's results from [1].

1.4. Remark. The advantage of the property (C) from the above theorems consists in the possibility to extend it to a characteristic property of the correctness of the Cauchy problem for abstract higher order equations – see Part 2 and compare [2]. The Oharu method in this case brings about certain hardly surmountable dificutlies.

2. RESOLVENTS IN COMPLEX AND REAL DOMAINS

2.1. Sublemma. $1/(1-\xi) \leqq \mathrm{e}^{2\xi}$ *for every* $0 \leqq \xi \leqq \frac{1}{2}$.

Proof. The function $\mathrm{e}^{-2\xi}(1/(1-\xi))$ has the value **1** at the point $\xi = 0$ and the value $2\mathrm{e}^{-1} < 1$ at the point $\xi = \frac{1}{2}$. Hence it suffices to prove that it is nondecreasing in the interval $\langle 0, \frac{1}{2} \rangle$. But this is clear because its derivative is nonnegative.

2.2. Sublemma. $\dfrac{\exp\left(-2\xi \log \dfrac{1+c}{c}\right)}{1 - 2\xi + (1+c)\,\xi^2} \leqq 1$ *for every* $c > 0$ *and* $\xi \geqq \frac{1}{2}$.

Proof. Let $c > 0$ be fixed. The roots of the polynomial $1 - 2\xi + (1 + c)\,\xi^2$ are

$$\frac{1}{1+c} \pm \sqrt{\left(\left(\frac{1}{1+c}\right)^2 - \frac{1}{1+c}\right)} = \frac{1}{1+c} \pm \mathrm{i}\sqrt{\left(\frac{1}{1+c} - \frac{1}{(1+c)^2}\right)} = \frac{1}{1+c} \pm$$

$$\pm\,\mathrm{i}\,\frac{\sqrt{c}}{1+c}\,.$$

Consequently, the function $1 - 2\xi + (1 + c)\,\xi^2$ is positive on R and

$$1 - 2\xi + (1+c)\,\xi^2 = \left|(1+c)\left(\xi - \frac{1}{1+c} - \mathrm{i}\,\frac{\sqrt{c}}{1+c}\right)\left(-\frac{1}{1+c} + \mathrm{i}\,\frac{\sqrt{c}}{1+c}\right)\right| \geqq$$

$$\geqq (1+c)\frac{\sqrt{c}}{1+c}\,\frac{\sqrt{c}}{1+c} = \frac{c}{1+c} \quad \text{for every} \quad \xi \in \mathrm{R}\,.$$

Thus we have

$$(1) \quad \frac{1}{1 - 2\xi + (1+c)\,\xi^2} \leqq \frac{1+c}{c} \text{ for every } \xi \in \mathrm{R}.$$

On the other hand,

$$(2) \ \exp\left(-2\xi \log\frac{1+c}{c}\right) = \left(\frac{c}{1+c}\right)^{2\xi} = \left(\frac{c}{1+c}\right)^{2\xi-1}\frac{c}{1+c} \leqq \frac{c}{1+c} \text{ for every}$$

$\xi \geqq \frac{1}{2}$.

Now (1) and (2) give the desired estimate.

2.3. Lemma.

$$\left|\frac{1}{1 - \dfrac{z}{p+1}}\right|^{p+1} \leqq e^{(2 + \log(1+a^2))\mathrm{Re}\, z}$$

for every $a > 0$ *and* $\mathrm{Re}\, z \geqq 0$ *such that*

$$|\mathrm{Im}\, z| \geqq \frac{1}{a}\,\mathrm{Re}\, z$$

and for every $p \in \{0, 1, \ldots\}$.

Proof. Let $a > 0$.

We can write

$$(1) \left|\frac{1}{1 - \dfrac{\alpha + \mathrm{i}\beta}{p+1}}\right|^{p+1} = \left(\frac{1}{\left(1 - \dfrac{\alpha}{p+1}\right)^2 + \left(\dfrac{\beta}{p+1}\right)^2}\right)^{(p+1)/2}$$

for every $\alpha, \beta \in R$ and $p \in \{0, 1, \ldots\}$.

Taking $\xi = \alpha/(p + 1)$ in Sublemma 2.1 we see from (1) that

$$(2) \quad \left|\frac{1}{1 - \dfrac{\alpha + i\beta}{p + 1}}\right|^{p+1} \leqq \left(\frac{1}{1 - \dfrac{\alpha}{p + 1}}\right)^{p+1} \leqq e^{2\alpha} \text{ for every } 0 \leqq \alpha \leqq \frac{p + 1}{2}, \quad \beta \in \mathrm{R}$$

and $p \in \{0, 1, \ldots\}$.
On the other hand, we have

$$(3) \quad \left(1 - \frac{\alpha}{p + 1}\right)^2 + \left(\frac{\beta}{p + 1}\right)^2 \geqq \left(1 - \frac{\alpha}{p + 1}\right)^2 + \frac{1}{a^2}\left(\frac{\alpha}{p + 1}\right)^2 = 1 - 2\frac{\alpha}{p + 1} +$$
$$+ \left(\frac{\alpha}{p + 1}\right)^2 + \frac{1}{a^2}\left(\frac{\alpha}{p + 1}\right)^2 = 1 - 2\,\frac{\alpha}{p + 1} + \left(1 + \frac{1}{a^2}\right)\left(\frac{\alpha}{p + 1}\right)^2 \text{ for every}$$

$\alpha \geqq 0$ and $\beta \in \mathrm{R}$ such that $|\beta| \geqq \dfrac{1}{a}\alpha$ and for every $p \in \{0, 1, \ldots\}$.

It follows from (1) and (3) that

$$(4) \quad \left|\frac{1}{1 - \dfrac{\alpha + i\beta}{p + 1}}\right|^{p+1} \leqq \left(\frac{1}{1 - 2\dfrac{\alpha}{p + 1} + \left(1 + \dfrac{1}{a^2}\right)\left(\dfrac{\alpha}{p + 1^2}\right)}\right)^{(p+1)/2} \text{for every } \alpha \geqq 0,$$

$\beta \in \mathrm{R}$ such that $|\beta| \geqq \dfrac{1}{a}\alpha$ and for every $p \in \{0, 1, \ldots\}$.

Now we obtain from (4), taking $\xi = \alpha/(p + 1)$ and $c = 1/a^2$ in Sublemma 2.2, that

$$(5) \quad \left|\frac{1}{1 - \dfrac{\alpha + i\beta}{p + 1}}\right|^{p+1} \leqq e^{\alpha\log(1 + a^2)} \text{ for every } \alpha \geqq \frac{p + 1}{2},\ |\beta| \geqq \frac{1}{a}\alpha \text{ and } p \in \{0, 1, \ldots\}.$$

Summing up (2) and (5) we get at once

$$\left|\frac{1}{1 - \dfrac{\alpha + i\beta}{p + 1}}\right|^{p+1} \leqq e^{(2 + \log(1 + a^2))\alpha} \text{ for every } \alpha \geqq 0,\ |\beta| \geqq \frac{1}{a}\alpha \text{ and } p \in \{0, 1, \ldots\},$$

which is the desired result if we take $\mathrm{Re}\, z = \alpha$, $\mathrm{Im}\, z = \beta$.

2.4. Proposition. *Let Λ be an open subset of* C *and R a mapping of Λ into E. If the function R is analytic in Λ and if there exist $a \geqq 0$, $b \geqq 0$, $K \geqq 0$ and $v \geqq 0$ so that*

(α) $\{z : z \in \mathrm{C}, \mathrm{Re}\, z > a \log(1 + |\mathrm{Im}\, z|) + b\} \subseteq \Lambda$, (β) $\|R(z)\| \leqq K(1 + |z|)^{\nu}$ *for every* $z \in \mathrm{C}$ *such that* $\operatorname{Re} z > a \log(1 + |\operatorname{Im} z|) + b$, *then there exist* $M \geqq 0$, $\omega \geqq 0$, $\chi \in \{0, 1, \ldots\}$ *and* $m \in \{0, 1, \ldots\}$ *such that*

(a) $(\omega, \infty) \subseteq \Lambda$,

(b) $\left\| \frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} \frac{1}{\lambda^{\chi T + m}} R(\lambda) \right\| \leqq \frac{M p!}{(\lambda - \omega)^{p+1}}$ *for every* $T \in \{1, 2, \ldots\}$, $\lambda > \omega$ *and* $p \in \{0, 1, \ldots\}$

such that $\lambda \geqq \frac{p+1}{T} + \omega$.

Note. If the constants $a \geqq 0$, $b \geqq 0$, $K \geqq 0$ and $\nu \geqq 0$ are given so that the assumptions (α), (β) hold, then the constants $M \geqq 0$, $\omega \geqq 0$, $\chi \in \{0, 1, \ldots\}$ and $m \in \{0, 1, \ldots\}$ can be chosen, for example, in the following way:

$$M = \frac{2^{\nu} K(1 + a)}{2}, \quad \omega = b + 2, \quad \chi - 1 < 4a + 2 \log(1 + a^2) \leqq \chi,$$

$$m - 1 < \nu + 2 \leqq m.$$

Proof. Let us first fix constants $a \geqq 0$, $b \geqq 0$, $K \geqq 0$ and $\nu \geqq 0$ so that the assumptions (α) and (β) hold.

For the sake of simplicity we shall denote

(1) $\Omega = \{z : \operatorname{Re} z > a \log(1 + |\operatorname{Im} z|) + b\}$,

(2) $\Gamma = \{z : \operatorname{Re} z = a \log(1 + |\operatorname{Im} z|) + b + 2\}$,

(3) $\omega = b + 2$.

Further, we need the function

(4) $z(\xi) = a \log(1 + |\xi|) + b + 2 + \mathrm{i}\xi$ for $\xi \in \mathrm{R}$.

It is clear from (1), (2) and (4) that

(5) $\Gamma = \{z(\xi) : \xi \in \mathrm{R}\} \subseteq \Omega$,

(6) $z'(\xi) = \frac{a}{1 + |\xi|} + \mathrm{i}$ for every $\xi \in \mathrm{R}$.

Regarding (1) we can rewrite the assumptions of Proposition 2.4 in the form

(7) the function R is analytic in Ω,

(8) $\|R(z)\| \leqq K(1 + |z|)^{\nu}$ for every $z \in \Omega$.

It is easy to prove from (1)–(3), (7) and (8) by means of Cauchy's integral theorem that

$$\frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} \frac{1}{\lambda^{l+\nu+2}} R(\lambda) = \frac{p!}{2\pi\mathrm{i}} \int_{\Gamma} \frac{R(z)}{z^{l+\nu+2}(z - \lambda)^{p+1}} \mathrm{d}z$$

for every $\lambda > \omega$, $l \in \{0, 1, \ldots\}$ and $p \in \{0, 1, \ldots\}$.

This identity can by written in the following form used below:

$$(9)\quad \frac{(\lambda-\omega)^{p+1}}{p!}\frac{\mathrm{d}^p}{\mathrm{d}\lambda^p}\frac{1}{\lambda^{l+\nu+2}}R(\lambda)=\frac{1}{2\pi\mathrm{i}}\int_\Gamma\frac{R(z)}{z^{l+\nu+2}}\left(\frac{\lambda-\omega}{z-\lambda}\right)^{p+1}\mathrm{d}z$$

for every $\lambda > \omega$, $l \in \{0, 1, \ldots\}$ and $p \in \{0, 1, \ldots\}$.

Let us recall that, as is well known,

$$(10)\quad \int_{-\infty}^{\infty}\frac{1}{1+\eta^2}\,\mathrm{d}\eta=\pi\,.$$

Further, (4) and (6) immediately imply

$$(11)\quad \frac{1+|z(\xi)|}{z(\xi)}\leqq 2\,,\quad |z(\xi)|\geqq\sqrt{(2+\xi^2)}\,,\quad |z'(\xi)|\leqq 1+a\quad\text{for every}\quad \xi\in\mathrm{R}.$$

Let us now consider the case $a = 0$.

In this case we have

$$\left|\frac{\lambda-\omega}{z(\xi)-\omega}\right|=\left|\frac{\lambda-\omega}{z(\xi)-\omega-(\lambda-\omega)}\right|=\left|\frac{\lambda-\omega}{\mathrm{i}\xi-(\lambda-\omega)}\right|=\frac{\lambda-\omega}{[(\lambda-\omega)^2+\xi^2]^{1/2}}\leqq 1$$

for every $\lambda > \omega$ and $\xi \in \mathrm{R}$ and consequently

$$(12)\quad \left|\frac{\lambda-\omega}{z(\xi)-\omega}\right|^{p+1}\leqq 1\ \text{for every}\ \lambda>\omega\ \text{and}\ \xi\in\mathrm{R}.$$

Let us denote by m an integer such that

$$(13)\quad m-1<\nu+2\leqq m.$$

It is clear from (11) and (13) that

$$(14)\quad \frac{1}{|z(\xi)|^{m-\nu-2}}\leqq 1\ \text{for every}\ \xi\in\mathrm{R}.$$

It follows from (4), (5) and (8)–(14) that

$$(15)\quad \left\|\frac{(\lambda-\omega)^{p+1}}{p!}\frac{\mathrm{d}^p}{\mathrm{d}\lambda^p}\frac{1}{\lambda^m}R(\lambda)\right\|=\left\|\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{R(z(\xi))}{z(\xi)^m}\left(\frac{\lambda-\omega}{z(\xi)-\lambda}\right)^{p+1}z'(\xi)\,\mathrm{d}\xi\right\|\leqq$$

$$\leqq\frac{K}{2\pi}\int_{-\infty}^{\infty}\frac{(1+|z(\xi)|)^\nu}{|z(\xi)|^{\nu+2}}\,\mathrm{d}\xi\leqq\frac{K}{2\pi}\int_{-\infty}^{\infty}\frac{2^\nu}{2+\xi^2}\,\mathrm{d}\xi\leqq$$

$$\leqq\frac{2^\nu K}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1+\xi^2}\,\mathrm{d}\xi\leqq\frac{2^\nu K}{2}\ \text{for every}\ \lambda>\omega\ \text{and}\ p\in\{0,1,\ldots\}.$$

Now we suppose $a > 0$.

We easily see from (3), (4) and (6) that under this hypothesis

(16) $\operatorname{Re} z(\xi) - \omega \geqq 0$ and $|\operatorname{Im}(z(\xi) - \omega)| \geqq \frac{1}{a} \operatorname{Re}(z(\xi) - \omega)$ for every $\xi \in \mathrm{R}$.

Using Lemma 2.3 we obtain from (16) that

(17) $$\left|\frac{\lambda - \omega}{z(\xi) - \lambda}\right|^{p+1} = \left|\frac{\lambda - \omega}{z(\xi) - \omega - (\lambda - \omega)}\right|^{p+1} = \left|\frac{1}{1 - \dfrac{z(\xi) - \omega}{\lambda - \omega}}\right|^{p+1} =$$

$$= \left|\frac{1}{1 - \dfrac{(p+1)\dfrac{z(\xi) - \omega}{\lambda - \omega}}{p+1}}\right|^{p+1} \leqq \exp\left[(2 + \log(1 + a^2))(p+1)\frac{\operatorname{Re}(z(\xi) - \omega)}{\lambda - \omega}\right]$$

for every $\xi \in \mathrm{R}$, $\lambda > \omega$ and $p \in \{0, 1, \ldots\}$.

For the sake of brevity let us now denote by χ an integer such that

(18) $\chi - 1 < 2\{[2 + \log(1 + a^2)]\, a\} \leqq \chi$.

It follows from (4), (6), (17 and (18) that

(19) $$\left|\frac{\lambda - \omega}{z(\xi) - \lambda}\right|^{p+1} \leqq \exp\left[(2 + \log(1 + a^2))(p+1)\frac{a \log(1 + |\xi|)}{\lambda - \omega}\right] =$$

$= (1 + |\xi|)^{(\chi/2)((p+1)/(\lambda-\omega))}$ for every $\xi \in \mathrm{R}$, $\lambda > \omega$ and $p \in \{0, 1, \ldots\}$.

Finally, let us recall two elementary facts:

(20) $\dfrac{1 + |\xi|}{2 + \xi^2} \leqq 1$ for every $\xi \in \mathrm{R}$;

(21) $\dfrac{p + 1}{T(\lambda - \omega)} \leqq 1$ for every $T > 0$, $\lambda > \omega$ and $p \in \{0, 1, \ldots\}$ such that

$\lambda > \dfrac{p+1}{T} + \omega$.

Using (4), (5), (8)–(11), (13), (14) and (18)–(21) we obtain

(22) $$\left\|\frac{(\lambda - \omega)^{p+1}}{p!} \frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} \frac{1}{\lambda^{\chi T + m}} R(\lambda)\right\| =$$

$$= \left\|\frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{R(z(\xi))}{z(\xi)^{\chi T + m}} \left(\frac{\lambda - \omega}{z(\xi) - \lambda}\right)^{p+1} z'(\xi)\,\mathrm{d}\xi\right\| \leqq$$

$$\leqq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\|R(z(\xi))\|}{|z(\xi)|^{\chi T + \nu + 2}\, |z(\xi)|^{m - \nu - 2}} \left|\frac{\lambda - \omega}{z(\xi) - \lambda}\right|^{p+1} |z'(\xi)|\,\mathrm{d}\xi \leqq$$

$$\leqq \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{K(1+|z(\xi)|)^{\nu}}{(\sqrt{(2+\xi^2)})^{\chi T+2}\,|z(\xi)|^{\nu}} (1+|\xi|)^{(\chi/2)((p+1)/(\lambda-\omega))}(1+a)\,d\xi =$$

$$= \frac{K(1+a)}{2\pi}\int_{-\infty}^{\infty} \left(\frac{1+|z(\xi)|}{|z(\xi)|}\right)^{\nu} \frac{(1+|\xi|)^{(\chi/2)((p+1)/(\lambda-\omega))}}{(\sqrt{(2+\xi^2)})^{\chi T}} \frac{1}{2+\xi^2}\,d\xi \leqq$$

$$\leqq \frac{K(1+a)}{2\pi}\int_{-\infty}^{\infty} 2^{\nu} \frac{(1+|\xi|)^{(\chi T/2)((p+1)/(T(\lambda-\omega)))}}{(2+\xi^2)^{\chi T/2}} \frac{1}{1+\xi^2}\,d\xi \leqq$$

$$\leqq \frac{2^{\nu}K(1+a)}{2\pi}\int_{-\infty}^{\infty} \frac{(1+|\xi|)^{\chi T/2}}{(2+\xi^2)^{\chi T/2}} \frac{1}{1+\xi^2}\,d\xi \leqq$$

$$\leqq \frac{2^{\nu}K(1+a)}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1+\xi^2}\,d\xi \leqq \frac{2^{\nu}K(1+a)}{2}$$

for every $T \in \{1, 2, \ldots\}$, $\lambda > \omega$ and $p \in \{0, 1, \ldots\}$ such that $\lambda \geqq (p+1)/T + \omega$.

The statement of our proposition follows from (1), (2), (6), (13), (14), (17) and (20) if we take $M = 2^{\nu} K(1+a)/2$.

2.5. Theorem. *Let A be a linear operator from E into E. If there exist $a \geqq 0$, $b \geqq 0$, $K \geqq 0$ and $\nu \geqq 0$ such that*

(α) $\{z : z \in C, \operatorname{Re} z > a \log(1 + |\operatorname{Im} z|) + b\} \subseteq \varrho(A)$,

(β) $\|(zI - A)^{-1}\| \leqq K(1 + |z|)^{\nu}$ *for every* $z \in C$ *such that* $\operatorname{Re} z > a \log(1 + |\operatorname{Im} z|) + b$,

then the following condition is fulfilled:

(D) *there exist $M \geqq 0$, $\omega \geqq 0$, $\chi \in \{0, 1, \ldots\}$ and $m \in \{0, 1, \ldots\}$ such that*

(a) $(\omega, \infty) \subseteq \varrho(A)$,

(b) $\left\|\frac{d^p}{d\lambda^p} \frac{1}{\lambda^{\chi T+m}} (\lambda I - A)^{-1}\right\| \leqq \frac{Mp!}{(\lambda - \omega)^{p+1}}$ *for every $T \in \{1, 2, \ldots\}$, $\lambda > \omega$ and $p \in \{0, 1, \ldots\}$ such that* $\lambda \geqq \frac{p+1}{T} + \omega$.

Proof. Let us denote $\Lambda = \varrho(A)$ = the resolvent set of the operator A and $R(z) = (zI - A)^{-1}$ for $z \in \varrho(A)$. It is well known that the set Λ is open and the funclion R is in this case analytic on Λ. Thus our theorem immediately follows from Proposition 2.4.

2.6. Remark. The converse of Theorem 2.5, provided the operator A is densely defined, follows from Oharu's results in [1] and from Theorem 1.2. A direct proof of this converse is not known to the author. It would be desirable to construct a proof not involving sufficiently "smoth" elements, i.e., elements of higher powers of the operator A, which is not convenient in particular if we consider the Cauchy problem for equations of higher degrees (cf. [2] and the following Theorem 2.9) because in this case great difficulties arise connected with the use of "smooth" elements produced by many noncommutative unbounded operators.

2.7. Second Characterisation Theorem. *Let A be a linear operator from E into E. Then the operator A is the generator of regular distribution semigroup if and only if it is densely defined and possesses the property* (D) *from the preceding theorem.*

Proof. Immediate consequence of Theorems 1.3 and 2.5 and of Oharu's results from [1].

2.8. Remark. For regular distribution semigroups it is possible to prove the following growth property (D′), similar and closely related to the property (D) from Theorem 2.5.

Let $\mathfrak{D}(\mathrm{R})$ be the linear space of infinitely differentiable real-valued functions on R with compact support.

If $\mathcal{T}$, as a mapping of $\mathfrak{D}(\mathrm{R})$ into $\mathsf{L}(E)$, is a regular distribution semigroup, then (D′) *there exist $M \geqq 0$, $\omega \geqq 0$, $\chi \in \{0, 1, \ldots\}$ and $m \in \{0, 1, \ldots\}$ such that*

$$\|\mathcal{T}(\varphi)\| \leqq M e^{\omega T} \sup_{\xi \in R} \left(\left|\varphi^{(\chi T + m)}(\xi)\right|\right)$$

for every $T \in \{1, 2, \ldots\}$ and $\varphi \in \mathfrak{D}(\mathrm{R})$ satisfying support $(\varphi) \subseteq (-\infty, T]$.

2.9. Theorem. *Let $A_1, A_2, \ldots, A_n$, $n \in \{1, 2, \ldots\}$, be linear operators from E into E. If the operators $A_1, A_2, \ldots, A_n$ are closed and if there exist $a \geqq 0$, $b \geqq 0$, $K \geqq 0$ and $\nu \geqq 0$ such that*

(α) *$z^n I + z^{n-1} A_1 + \ldots + A_n$ is a one-to-one operator and its inverse is everywhere defined and bounded for every* $\operatorname{Re} z > a \log (1 + |\operatorname{Im} z|) + b$,

(β) $\|A_i(z^n I + z^{n-1} A_1 + \ldots + A_n)^{-1}\| \leqq K(1 + |z|)^\nu$ *for every $i \in \{1, 2, \ldots, n\}$ and for every $z \in \mathrm{C}$ such that* $\operatorname{Re} z > a \log (1 + |\operatorname{Im} z|) + b$,

then there exist $M \geqq 0$, $\omega \geqq 0$, $\chi \in \{0, 1, \ldots\}$ and $m \in \{0, 1, \ldots\}$ such that

(a) *$\lambda^n I + \lambda^{n-1} A_1 + \ldots + A_n$ is a one-to-one operator and its inverse is everywhere defined and bounded for every $\lambda > \omega$,*

(b) $\left\|\frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} \frac{1}{\lambda^{\chi T + m}} A_i(\lambda^n I + \lambda^{n-1} A_1 + \ldots + A_n)^{-1}\right\| \leqq \frac{Mp!}{(\lambda - \omega)^{p+1}}$ *for every $T \in \{1, 2, \ldots\}$, $\lambda > \omega$ and $p \in \{0, 1, \ldots\}$ such that $\lambda > (p + 1)/T + \omega$ and for every $i \in \{1, 2, \ldots, n\}$.*

Proof. Let us denote by Λ the set of all $z \in \mathrm{C}$ such that $z^n I + z^{n-1} A_1 + \ldots + A_n$