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INTEGRAL REPRESENTATION OF ORTHOGONAL EXPONENTIAL POLYNOMIALS

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An integral representation is derived for the orthogonal exponential polynomials which have been used in engineering and science.

INTRODUCTION

The coefficients d_{nk} in the linear combination of exponential functions $\sum_{k=1}^{k=n} d_{nk}e^{-kt} = \psi_n(t)$ can be chosen so that, for $n = 1, 2, 3, \ldots$, a system of orthogonal functions over $[0, +\infty)$ is obtained.

Definition. *Let t be a real variable and $n = 1, 2, 3, \ldots$. The functions*

$$(1) \qquad \mathrm{oep}_n(t) = \sum_{k=1}^{n} b_{nk}e^{-kt}, \quad b_{nk} = (-1)^{n+k}\binom{n}{k}\binom{n+k-1}{k-1}$$

will be called orthogonal exponential polynomials. [5]

The following equations are valid for arbitrary $m, n = 1, 2, 3, \ldots$: $\mathrm{oep}_n(0) = 1$, $\mathrm{oep}_n(+\infty) = \lim_{t\to+\infty} \mathrm{oep}_n(t) = 0$ and $\int_0^\infty \mathrm{oep}_m(t)\,\mathrm{oep}_n(t)\,\mathrm{d}t = \delta_{mn}(2n)^{-1}$, where δ_{mn} is the Kronecker delta function. Orthogonal exponential polynomials have been used in Automatic Control Theory and in Electrical Circuits Theory. One of the more recent publications is Dmitriyev's book on the applications of orthogonal exponential polynomials in Hydrometeorology [4]. However, not much detailed information is available on the properties of orthogonal exponential polynomials, which might, under certain circumstances, be treated as special functions *sui generis*. Orthogonal exponential polynomials are entire transcendental functions if t in Definition (1) is considered complex.

Orthogonal exponential polynomials result by orthogonalization from the system of exponential functions $e^{-t}, e^{-2t}, e^{-3t}, \ldots$ over $(0, +\infty)$ with the weight function $w(t) = 1$, that is to say in $L_2(0, +\infty)$. The notation $\varphi_n(t)$ or $\psi_n(t)$ was used for orthogonal exponential polynomials in [5], [6], [8]. In this paper the acronym oep_n is used to denote orthogonal exponential polynomials so that a specific meaning need not be ascribed to Greek letters φ and ψ. The standardization $\operatorname{oep}_n(0) = 1$ for all natural n is implied.

INTEGRAL REPRESENTATION

First and Second Integral Representations of orthogonal exponential polynomials will be derived. The terminology is analogous to that used by Lavrentyev and Shabat to describe integral representations of Legendre polynomials [9].

First Integral Representation. Theorem I. *Let $\mathscr{C}$ be a closed Jordan's curve of finite length and assume that the point $z_0 = e^{-t}$ lies inside $\mathscr{C}$. Let $\operatorname{oep}_n(t)$ be an orthogonal exponential polynomial in accordance with Definition (1). Then, for a real t,*

$$\text{(2)} \qquad \operatorname{oep}_n(t) = (2\pi i)^{-1} \oint_{\mathscr{C}} z^n (z-1)^{n-1} (z - e^{-t})^{-n} \, dz \,, \quad n = 1, 2, 3, \ldots .$$

Proof. The connection between the orthogonal exponential polynomials and the Jacobi polynomials as shown in [5],

$$\text{(3)} \qquad \operatorname{oep}_n(t) = (-1)^{n-1} n e^{-t} G_{n-1}(2, 2, e^{-t}) \,, \quad n = 1, 2, 3, \ldots ,$$

can be conveniently used to prove (2). The Jacobi polynomials $G_n(p, q, x)$ are orthogonal over $(0, 1)$ with respect to the weight function $x^{q-1}(1-x)^{p-q}$, $q > 0$, $p - q > -1$, and their standardization is $G_n(p, q, 0) = 1$ for all n (Courant-Hilbert [3]). Rodrigues' formula will be used in the form

$$\text{(4)} \qquad G_{n-1}(2, 2, x) = (x n!)^{-1} D^{n-1} x^n (1-x)^{n-1} \,, \quad n = 1, 2, 3, \ldots ,$$

where the m-th derivative is denoted by D^m. The function $z \mapsto z^n(1-z)^{n-1}$ is an entire function of the complex variable z so that Cauchy's formula can be employed to express the derivative in (4), and for arbitrary z_0 we find

$$\text{(5)} \qquad G_{n-1}(2, 2, z_0) = (2n\pi i z_0)^{-1} \oint_{\mathscr{C}} z^n (1-z)^{n-1} (z - z_0)^{-n} \, dz \,,$$

where $\mathscr{C}$ is a closed Jordan's curve of finite length and the point z_0 lies inside $\mathscr{C}$. Equation (2) follows from (3) and (5) if z_0 is replaced by e^{-t}. □

Remark. First Integral Representation is an analogue to the Schläfli integral for Legendre polynomials [1], [9]. The proof of Theorem I is valid for any complex $t \neq \infty$. First Integral Representation yields

$$\operatorname{oep}_n(0) = (2\pi i)^{-1} \oint_{\mathscr{C}} z^n (z-1)^{-1} \, dz = 1$$

for all natural n and this agrees with the standardization as accepted for oep_n.

The integrand in (2) has an n-th order pole in $z = e^{-t}$ as long as $t \neq 0$. The numerator is

$$z^n(z-1)^{n-1} = \sum_{k=1}^{n} (-1)^{n+k} \binom{n-1}{k-1} z^{n+k-1}.$$

If the residue theorem is used we find

$$\mathrm{oep}_n(t) = \sum_{k=1}^{n} (-1)^{n+k} \binom{n-1}{k-1} \operatorname*{Res}_{z=e^{-t}} \left[z^{n+k-1}(z-e^{-t})^{-n}\right] = \sum_{k=1}^{n} b_{nk} e^{-kt},$$

where

$$b_{nk} = (-1)^{n+k} \binom{n}{k} \binom{n+k-1}{k-1}, \quad \sum_{k=1}^{n} b_{nk} = 1,$$

and this again agrees with Definition (1). Another expression for orthogonal exponential polynomials is found if the numerator in (2) is written as

$$\sum_{k=0}^{n} \binom{n}{k} (z-1)^{n+k-1}.$$

We have

$$\mathrm{oep}_n(t) = \sum_{k=0}^{n} g_{nk}(1-e^{-t})^k; \quad g_{nk} = (-1)^k \binom{n}{k} \binom{n+k-1}{k}; \tag{6}$$

$$\sum_{k=0}^{n} g_{nk} = 0.$$

If no changes are made in the numerator, the result is

$$\mathrm{oep}_n(t) = \sum_{k=1}^{n} h_{nk} e^{-kt}(1-e^{-t})^{n-k}; \quad h_{nk} = (-1)^{n+k} \binom{n}{k} \binom{n-1}{k-1}; \tag{7}$$

$$\sum_{k=1}^{n} h_{nk} = 2^n \, \mathrm{oep}_n(\ln 2).$$

Clearly, equations (1), (6), (7) follow directly from (4) if analogous changes are made in Rodrigues' formula.

Second Integral Representation. Theorem II. *Let* $\mathrm{oep}_n(t)$ *be an orthogonal exponential polynomial in accordance with Definition* (1). *Then, for a nonnegative* t *and* $n = 1, 2, 3, \ldots,$ *we have*

$$\mathrm{oep}_n(t) = \pi^{-1} \int_0^{\pi} \left[e^{-t} + i(e^{-t} - e^{-2t})^{1/2} \cos\theta\right] . \tag{8}$$
$$. \left[2e^{-t} - 1 + 2i(e^{-t} - e^{-2t})^{1/2} \cos\theta\right]^{n-1} d\theta.$$

Proof. Equation (8) is evidently valid for $t = 0$ as $\mathrm{oep}_n(0) = \pi^{-1} \int_0^{\pi} d\theta = 1$ for all n. Thus, only $t \in (0, +\infty)$ needs to be considered. Suppose that in (2) a circle with a radius $(e^{-t} - e^{-2t})^{1/2}$ and its centre at $z_0 = e^{-t}$ is chosen as $\mathscr{C}$, that is to say $z = e^{-t} + e^{i\theta}(e^{-t} - e^{-2t})^{1/2}$, $\theta \in I$, and I is an arbitrary real interval the length of which is 2π. We find $dz = i(z - e^{-t})\, d\theta$ and if the corresponding substitution is made in (2) the result is

$$\mathrm{oep}_n(t) = (2\pi)^{-1} \int_I [e^{-t} + (e^{-t} - e^{-2t})^{1/2} e^{i\theta}] \,.$$
$$. [2e^{-t} - 1 + 2i(e^{-t} - e^{-2t})^{1/2} \sin \theta]^{n-1} \, d\theta \,.$$

Replacing θ by $\theta + \pi/2$ and choosing $I = (-\pi, \pi)$ we find

$$\mathrm{oep}_n(t) = (2\pi)^{-1} \int_{-\pi}^{+\pi} [e^{-t} + i(e^{-t} - e^{-2t})^{1/2} e^{i\theta}] \,.$$
$$. [2e^{-t} - 1 + 2i(e^{-t} - e^{-2t})^{1/2} \cos \theta]^{n-1} \, d\theta \,,$$

and this is clearly equivalent to (8). □

Remark. The integrand in Second Integral Representation (8) is a complex function of a real variable. Despite of this the result is evidently real. As a matter of fact, Second Integral Representation can be rewritten as

(9) $$\mathrm{oep}_n(t) = (2/\pi) \operatorname{Re} \int_0^{\pi/2} [e^{-t} + i(e^{-t} - e^{-2t})^{1/2} \cos \theta] \,.$$
$$. [2e^{-t} - 1 + 2i(e^{-t} - e^{-2t})^{1/2} \cos \theta]^{n-1} \, d\theta \,.$$

Second Integral Representation (8) is an analogue to the Laplace integral for the Legendre polynomials [1], [9], [11]:

(10) $$P_n(u) = \pi^{-1} \int_0^{\pi} [u + i(1 - u^2)^{1/2} \cos \theta]^n \, d\theta \,.$$

Comparing (8) and (10) we find an important connection between the orthogonal exponential polynomials and the Legendre polynomials:

(11) $$\mathrm{oep}_n(t) = \tfrac{1}{2}[P_n(2e^{-t} - 1) + P_{n-1}(2e^{-t} - 1)] \,; \quad n = 1, 2, 3, \ldots \,.$$

For $0 \leqq \theta \leqq \pi$ and $t \geqq 0$ the following inequalities hold: $|e^{-t} + i(e^{-t} - e^{-2t})^{1/2} . \cos \theta| \leqq 1$ and $|2e^{-t} - 1 + 2i(e^{-t} - e^{-2t})^{1/2} \cos \theta| \leqq 1$. As a result $|\mathrm{oep}_n(t)| \leqq 1$ for all $t \geqq 0$ and $n = 1, 2, 3, \ldots$. Finally, Second Integral Representation may be used for the calculation of values of orthogonal exponential polynomials. E.g.

$$\mathrm{oep}_n(\ln 2) = (2\pi)^{-1} \int_0^{\pi} (1 + i \cos \theta)\, (i \cos \theta)^{n-1} \, d\theta \,.$$

The result is $\mathrm{oep}_1(\ln 2) = \frac{1}{2}$ and, for $k = 1, 2, 3, \ldots$:

(12) $$\mathrm{oep}_{2k}(\ln 2) = \mathrm{oep}_{2k+1}(\ln 2) = (-1)^k \tfrac{1}{2}(2k - 1)!!/(2k)!!$$