Časopis pro pěstování matematiky, roč. 105 (1980), Praha

l_∞-NORM OF ITERATES AND THE SPECTRAL RADIUS OF MATRICES

ZDENĚK DOSTÁL, Ostrava

(Received October 18, 1977)

Let B be a finite dimensional Banach space. Let $L(B)$ denote the algebra of all linear operators on B and let the operator norm and the spectral radius of $A \in L(B)$ be denoted by $|A|$ and $|A|_\sigma$, respectively.

If $A \in L(B)$ and $|A| = 1$, then the spectral radius formula suggests the conjecture that for some natural number m, nontrivial bounds for $|A^m|$ in terms of $|A|_\sigma$ and vice versa may be given.

The first positive result of the kind was presented by V. PTÁK and J. MAŘÍK [1], who have computed the critical exponent of the l_∞-space. If we denote the complex n-dimensional vector space by $B_{n,\infty}$, the norm $|x|_\infty$ of the vector $x = (x_1, \ldots, x_n)$ being defined by the formula

$$|x|_\infty = \max_{i=1,\ldots,n} |x_i| ,$$

then their theorem says that the spectral radius of $A \in L(B_{n,\infty})$, $|A|_\infty = |A^{n^2-n+1}|_\infty = 1$, is equal to one.

Later, V. Pták [2] introduced for $0 < r < 1$ the quantity

$$C(B, r, m) = \sup \{|A^m| : A \in L(B),\ |A| \leqq 1,\ |A|_\sigma \leqq r\}$$

and found, for an n-dimensional Hilbert space H_n, a certain operator $A \in L(H_n)$ such that $|A| = 1$, $|A|_\sigma = r$ and $|A^n| = C(H_n, r, n)$. Recently, the present author [3] has proved that this extremal operator is unique up to multiplication by a complex unit and similarity by a unitary mapping. For further references see [2].

The purpose of this note was originally to find the extremal operators in $L(B_{n,\infty})$. We have not succeeded in general, nevertheless, we have found for each r, $0 \leqq r \leqq 2^{1/n} - 1$, an operator $A \in L(B_{n,\infty})$ such that $|A|_\infty = 1$, $|A|_\sigma = r$ and $|A^m|_\infty = C(B_{n,\infty}, r, m)$ for all natural m.

Let n be a fixed natural number and let M_n denote the algebra of all $n \times n$ complex valued matrices.

Regarding a matrix $A = (a_{ij})$ as an operator on $B_{n,\infty}$, we can write

$$|A|_\infty = \max_i \sum_{j=1}^n |a_{ij}| .$$

Let $\alpha_1, \ldots, \alpha_n$ be given complex numbers. Consider the recursive relation

$$x_{k+n} = \alpha_1 x_k + \ldots + \alpha_n x_{k+n-1}\,. \tag{1}$$

For each i, $1 \leqq i \leqq n$, we denote by $w_i(\alpha_1, \ldots, \alpha_n)$ the solution $(w_{i0}, w_{i1}, w_{i2}, \ldots)$ of this relation with initial conditions

$$w_{ik}(\alpha_1, \ldots, \alpha_n) = \delta_{i,k+1}\,, \quad 0 \leqq k \leqq n-1\,. \tag{2}$$

In the following lemma we shall learn the meaning of w_{ik}:

Lemma 1. *Let $A \in M_n$ and*

$$A^n = \alpha_1 E + \alpha_2 A + \ldots + \alpha_n A^{n-1}\,. \tag{3}$$

Then for all $k \geqq 0$,

$$A^k = w_{1k}E + w_{2k}A + \ldots + w_{nk}A^{n-1}\,. \tag{4}$$

Proof. The statement is obvious for $k \leqq n$. To prove the lemma for $k > n$ by induction, suppose that $s > n$ and that (4) holds for $k = 0, 1, \ldots, s-1$. Put $q = s - n$. If we multiply (3) by A^q and use the induction hypothesis, we successively get

$$A^s = \sum_{i=1}^{n} \alpha_i A^{q+i-1} = \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{n} w_{j,q+i-1} A^{j-1} =$$

$$= \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} \alpha_i w_{j,q+i-1} \Big) A^{j-1} = \sum_{j=1}^{n} w_{js} A^{j-1}\,.$$

Let us denote now the companion matrix of the equation

$$x^n = \alpha_1 + \alpha_2 x + \ldots + \alpha_n x^{n-1} \tag{5}$$

by $T(\alpha_1, \ldots, \alpha_n)$, that is

$$T = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ . & . & . & \ldots & . \\ 0 & 0 & 0 & \ldots & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_n \end{bmatrix},$$

and observe that (5) is the characteristic equation of T. Thus by Cayley-Hamilton's theorem T satisfies the assumptions of Lemma 1 and we can write for each $k = 0, 1, 2, \ldots$

$$T^k = w_{1k}E + w_{2k}T + \ldots + w_{nk}T^{n-1}\,. \tag{6}$$

This equation enables us to solve the special maximum problem:

Lemma 2. *Let $A \in M_n$, $|A|_\infty \leqq 1$. If the characteristic equation* (5) *of the matrix A fulfils*

$$\sum_{i=1}^{n} |\alpha_i| \leqq 1\,, \tag{7}$$

then for all $k \geqq 0$,

$$|A^k|_\infty \leqq T(\alpha_1, \ldots, \alpha_n)^k = \sum_{i=1}^{n} |w_{ik}|\,.$$

Proof. We may apply Lemma 1 to get

$$|A^k|_\infty = \Big|\sum_{i=1}^{n} w_{ik} A^{i-1}\Big|_\infty \leqq \sum_{i=1}^{n} |w_{ik}|\, |A^{i-1}|_\infty \leqq \sum_{i=1}^{n} |w_{ik}|$$

for each A under the assumptions. Note that, in particular, T satisfies the assumptions. The first row of T^k being $(w_{1k}, w_{2k}, \ldots, w_{nk})$ (see (6)), we get

$$|T^k|_\infty = \sum_{i=1}^{n} |w_{ik}|\,.$$

Now we shall denote, for $1 \leqq i \leqq n$, by E_i the polynomial

$$E_i(x_1, \ldots, x_n) = \sum_{\substack{e_j \in \{0,1\} \\ e_1 + \ldots + e_n = i}} x_1^{e_1} x_2^{e_2} \ldots x_n^{e_n}\,. \tag{8}$$

For any complex numbers $\varrho_1, \ldots, \varrho_n$ and $i = 1, 2, \ldots, n$, we put

$$\alpha_i(\varrho_1, \ldots, \varrho_n) = (-1)^{n-i}\, E_{n-i+1}(\varrho_1, \ldots, \varrho_n)\,,$$

so that the roots of the equation (5) with coefficients $\alpha_i = \alpha_i(\varrho_1, \ldots, \varrho_n)$ are exactly $\varrho_1, \ldots, \varrho_n$.

Let us compute an upper bound for such r's that $|\varrho_i| \leqq r$ implies

$$\sum_{i=1}^{n} |\alpha_i(\varrho_1, \ldots, \varrho_n)| \leqq 1\,. \tag{9}$$

Lemma 3. *Let $\varrho_1, \ldots, \varrho_n$ be any complex numbers. If $|\varrho_i| \leqq 2^{1/n} - 1$ for all $i = 1, \ldots, n$, then the inequality* (9) *holds true.*

Proof. Let $0 < r < 1$ and note that

$$\alpha_i(r, r, \ldots, r) = (-1)^{n-i} \binom{n}{n-i+1} r^{n-i+1}\,,$$

$i = 1, \ldots, n$. If $|\varrho_i| \leqq r$ holds for all $i = 1, \ldots, n$, then $|\alpha_i(\varrho_1, \ldots, \varrho_n)| \leqq |\alpha_i(r, r, \ldots, r)|$. Thus the supremum r_0 of the set of all r's we are interested in is the only positive root of the equation

$$1 - \sum_{i=1}^{n} \binom{n}{i} x^i = 0\,.$$

Easy computation shows that $r_0 = 2^{1/n} - 1$.

To compute $C(B_{n,\infty}, r, k)$ for $r \leqq 2^{1/n} - 1$ and given k, it is enough to find

$$\max_{|\varrho_1| \leqq r, \ldots, |\varrho_n| \leqq r} \sum_{i=1}^{n} |w_{ik}(\varrho_1, \ldots, \varrho_n)| .$$

The fact that the maximum is attained for all natural k if $\varrho_i = r$ is an easy consequence of the following lemma, which was proved by V. Knichal ([2], Lemma 7).

Lemma 4. *For each $i = 1, 2, \ldots, n$ and each $k \geqq n$,*

$$w_{ik}(\varrho_1, \ldots, \varrho_n) = \varepsilon_i Q_{ik}(\varrho_1, \ldots, \varrho_n) ,$$

where $\varepsilon_i = (-1)^{n-i}$ and

$$Q_{ik}(\varrho_1, \ldots, \varrho_n) = \sum_{\substack{e_j \geqq 0 \\ e_1 + \ldots + e_n = k-i+1}} c_{ik}(e_1, \ldots, e_n)\, \varrho_1^{e_1} \ldots \varrho_n^{e_n} ,$$

where all $c_{ik}(e_1, \ldots, e_n) \geqq 0$.

The point of the lemma is that for $k \geqq n$ and i fixed, all the coefficients of w_{ik} are of the same sign. Thus if $|\varrho_i| \leqq r$ for $i = 1, \ldots, n$, then

$$|w_{ik}(\varrho_1, \ldots, \varrho_n)| = |Q_{ik}(\varrho_1, \ldots, \varrho_n)| \leqq$$
$$\leqq |Q_{ik}(r, \ldots, r)| = |w_{ik}(r, \ldots, r)| , \quad i = 1, \ldots, n .$$

We can sum up our results into the following theorem:

Theorem 1. *Let $0 < r \leqq 2^{1/n} - 1$, let*

$$\alpha_i = (-1)^{n-i} \binom{n}{n-i+1} r^{n-i+1}$$

for $i = 1, \ldots, n$ and let

$$T = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ . & . & . & \ldots & . \\ 0 & 0 & 0 & \ldots & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_n \end{bmatrix} .$$

Then $|T|_\infty = 1$, $|T|_\sigma = r$ and for each natural k,

$$|T^k|_\infty = \sum_{i=1}^{n} |w_{ik}| = C(B_{n,\infty}, r, k) ,$$

where w_{ik} are the solutions of the recurrent relation

$$x_{s+n} = \alpha_1 x_s + \alpha_2 x_{s+1} + \ldots + \alpha_n x_{s+n-1}$$

with initial conditions $w_{ij} = \delta_{i,j+1}$, $i = 1, \ldots, n$, $j = 0, 1, \ldots, n-1$.