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ON THE EXISTENCE OF A 3-FACTOR IN THE FOURTH POWER OF A GRAPH

LADISLAV NEBESKÝ, Praha

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Let G be a graph in the sense of [1] or [2]. We denote by $V(G)$ and $E(G)$ its vertex set and edge set, respectively. The cardinality $|V(G)|$ of $V(G)$ is referred to as the order of G. If W is a nonempty subset of $V(G)$, then we denote by $\langle W \rangle_G$ the subgraph of G induced by W. A regular graph of degree m which is a spanning subgraph of G is called an m-factor of G. It is well-known if G has an m-factor for some odd m, then the order of G is even. If n is a positive integer, then by the n-th power G^n of G we mean the graph G' with the properties that $V(G') = V(G)$ and

$$E(G') = \{uv;\, u, v \in V(G) \text{ such that } 1 \leqq d(u, v) \leqq n\},$$

where $d(w_1, w_2)$ denotes the distance of vertices w_1 and w_2 in G.

CHARTRAND, POLIMENI and STEWART [2], and SUMNER [5] proved that if G is a connected graph of even order, then G^2 has a 1-factor.

The second power of none of the connected graphs in Fig. 1 has a 2-factor. But if G is a connected graph of an order $p \geqq 3$, then G^3 has a 2-factor; this follows from a theorem due to SEKANINA [4], which asserts that the third power of any connected graph is hamiltonian connected.

The third power of none of the connected graphs of even order which are given in Fig. 2 has a 3-factor. But for the fourth power the situation is different:

Theorem. *Let G be a connected graph of an even order $p \geqq 4$. Then G^4 has a 3-factor, each component of which is either K_4 or $K_2 \times K_3$.*

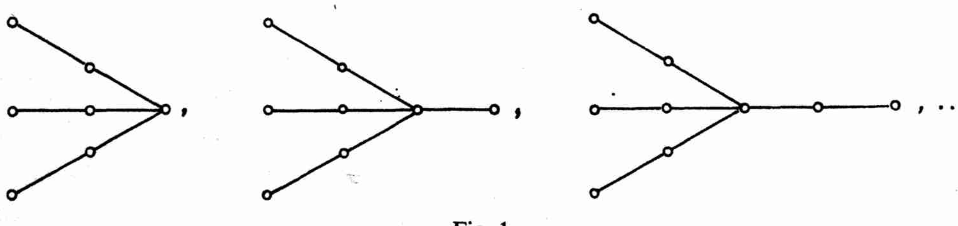

Fig. 1.

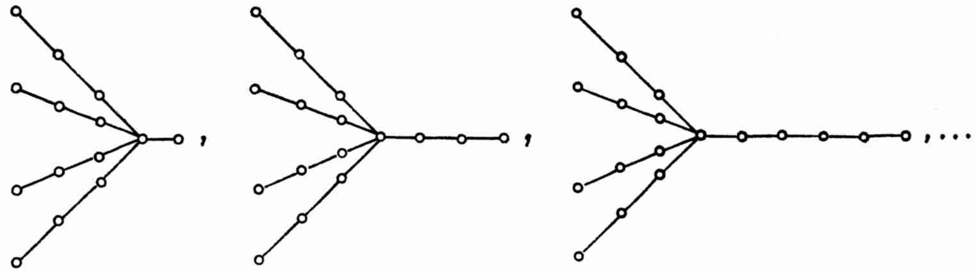

Fig. 2.

Note that K_n denotes the complete graph of order n, and $K_2 \times K_3$ denotes the product of K_2 and K_3 (see Fig. 3).

Before proving the theorem we establish one lemma. Let T be a nontrivial tree. Consider adjacent vertices u and v. Obviously, $T - uv$ has exactly two components, say T_1 and T_2. Without loss of generality we assume that $u \in V(T_1)$ and $v \in V(T_2)$. Denote $V(T, u, v) = V(T_1)$ and $V(T, v, u) = V(T_2)$.

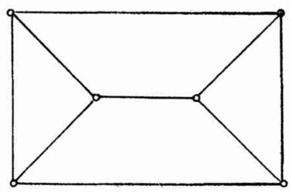

Fig. 3.

Lemma. *Let T be a tree of an order $p \geqq 5$. Then there exist adjacent vertices u and v such that*

(i) $|V(T, u, v)| \geqq 4$ *and*

(ii) $|V(T, w, u)| \leqq 3$ *for every vertex $w \neq v$ such that $uw \in E(T)$.*

Proof of the lemma. Assume that to every pair of adjacent vertices u and v such that $|V(T, u, v)| \geqq 4$, there exists a vertex $w \neq v$ such that $uw \in E(T)$ and $|V(T, w, u)| \geqq 4$. Since $p \geqq 5$, it is possible to find an infinite sequence of vertices $v_0, v_1, v_2, \ldots$ in T such that

(a) v_0 has degree one;

(b) $v_0v_1, v_1v_2, v_2v_3, \ldots \in E(T)$;

(c) $v_2 \neq v_0$, $v_3 \neq v_1$, $v_4 \neq v_2, \ldots$; and

(d) $|V(T, v_1, v_0)| \geqq 4$, $|V(T, v_2, v_1)| \geqq 4$, $|V(T, v_3, v_2)| \geqq 4, \ldots$

Since T is a tree, (b) and (c) imply that the vertices $v_0, v_1, v_2, \ldots$ are mutually different, which is a contradiction. Hence the lemma follows.

Proof of the theorem. Since G is connected, it contains a spanning tree, say T. First, let $p = 4$, 6, or 8. If $p = 4$, then $G^4 = T^4 = K_4$.

Let $p = 6$. Then T is isomorphic to one of the six trees of order six (see the list in [3], p. 233). It is easy to see that T^4 and therefore G^4 contains a 3-factor isomorphic to $K_2 \times K_3$.

Let $p = 8$. By Lemma there exist adjacent vertices u and v of T such that (i) and (ii) hold. If $|V(T, u, v)| = 4$, then T^4 (and therefore G^4) contains a 3-factor which consists of two disjoint copies of K_4. Let $|V(T, u, v)| \geqq 5$. Since $p = 8$, we have $|V(T, w, u)| \leqq 3$ for every vertex w adjacent to u, $w \neq v$. Then there exists a set R of two, three, or four vertices adjacent to u such that

$$\langle \bigcup_{r \in R} V(T, r, u) \rangle_T$$

is isomorphic to one of the graphs $F_1 - F_4$ in Fig. 4. Denote

$$V_R = \bigcup_{r \in R} V(T, r, u) .$$

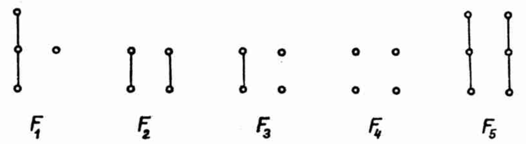

Fig .4.

It is clear that $\langle V_R \rangle_{T^4} = K_4$. Since $T - V_R$ is a tree of order four, we conclude that G^4 has a 3-factor which consists of two disjoint copies of K_4.

Next, let $p \geqq 10$. Assume that for every connected graph G' of order $p - 6$ or $p - 4$ we have proved that $(G')^4$ has a 3-factor, each component of which is either K_4 or $K_2 \times K_3$. By Lemma there exist adjacent vertices u and v of T such that (i) and (ii) hold. Let $|V(T, u, v)| = 4$ or 6; then $\langle V(T, u, v) \rangle_{T^4}$ contains a 3-factor isomorphic to either K_4 or $K_2 \times K_3$; since $G - V(T, u, v)$ is connected, by the induction assumption $(G - V(T, u, v))^4$ has a 3-factor, each component of which is either K_4 or $K_2 \times K_3$; hence G^4 has a 3-factor with the required property. Now, let either $|V(T, u, v)| = 5$ or $|V(T, u, v)| \geqq 7$. Then there exists a set S of two, three, or four vertices adjacent to u such that

$$\langle \bigcup_{s \in S} V(T, s, u) \rangle_T$$

is isomorphic to one of the graphs $F_1 - F_5$ in Fig. 4. Denote

$$V_S = \bigcup_{s \in S} V(T, s, u) .$$

Since $T - V_S$ is a tree, we conclude that $G - V_S$ is a connected graph. According to the induction assumption $(G - V_S)^4$ has a 3-factor, each component of which is