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THE HEAT AND ADJOINT HEAT POTENTIALS

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Let G stand for the fundamental solution of the heat equation in R^{n+1}, i.e.

$$G(x, t) = (4\pi t)^{-n/2} \exp\left(-\frac{|x|^2}{4t}\right) \quad \text{for} \quad x \in R^n, \quad t > 0,$$

$$G(x, t) = 0 \quad \text{for} \quad x \in R^n, \quad t \leqq 0.$$

By the term measure we mean a finite Borel measure with compact support in R^m. If μ is a measure in R^{n+1}, the heat potential G is defined by the equality

$$G_\mu(x, t) = \int_{R^{n+1}} G(x - \xi, t - \tau) \, d\mu(\xi, \tau).$$

Similarly one can define the adjoint heat potential G_μ^* by

$$G_\mu^*(x, t) = \int_{R^{n+1}} G^*(x - \xi, t - \tau) \, d\mu(\xi, \tau),$$

where G^* is the fundamental solution of the adjoint heat equation; $G^*(x, t) = G(x, -t)$.

Let μ be a measure in R^{n+1}. It is known (see [1], [3], [4]) that for $\alpha \in (0, 1)$ the condition

$$\text{(1)} \qquad \sup \{|G_\mu(x_1, t_1) - G_\mu(x_2, t_2)| ; \; x_1, x_2 \in R^n, |x_1 - x_2| \leqq \varepsilon, |t_1 - t_2| \leqq \varepsilon^2\} \leqq K\varepsilon^\alpha$$

(i.e. G_μ is a Hölder-continuous function with the coefficient α in the variable x and the coefficient $\frac{1}{2}\alpha$ in the variable t) is fulfilled if and only if the condition

$$\text{(2)} \quad \sup \{\mu(\{(x, t) \in R^{n+1}; |x - \xi| \leqq \varepsilon, |\tau - t| \leqq \varepsilon^2\}); (\xi, \tau) \in R^{n+1}\} \leqq M\varepsilon^{n+\alpha}$$

holds. As the condition (2) is "symmetric in the variable t", an analogous condition to (1) is fulfilled for the adjoint heat potential G_μ^* if and only if (2) holds. It is seen from this that the potential G_μ^* is a Hölder-continuous function with the coefficient α in the variable x and with the coefficient $\frac{1}{2}\alpha$ in the variable t if and only if the potential G_μ possesses the same property. We will show that the assumption $\alpha > 0$ is essential. It holds (see [3], [4]) that the potential G_μ is uniformly continuous on R^{n+1} if and only if the condition

$$(3)\qquad \lim_{a\to\infty}\left(\sup\left\{\int_a^\infty \mu(A(x,t,c))\,dc;\ (x,t)\in R^{n+1}\right\}\right) = 0$$

is fulfilled, where

$$A(x,t,c) = \{(\xi,\tau)\in R^{n+1};\ G(x-\xi,\ t-\tau) > c\}\quad (c > 0)\,.$$

For the uniform continuity of the adjoint heat potential G_μ^* we have an analogous condition under which G_μ^* is uniformly continuous:

$$(4)\qquad \lim_{a\to\infty}\left(\sup\left\{\int_a^\infty \mu(A^*(x,t,c))\,dc;\ (x,t)\in R^{n+1}\right\}\right) = 0\,,$$

where

$$A^*(x,t,c) = \{(\xi,\tau)\,R^{n+1};\ G^*(x-\xi,\ t-\tau) > c\}\quad (c > 0)\,.$$

However, the conditions (3), (4) are not "symmetric in the variable t" which raises the following question: are the conditions (3), (4) equivalent to each other, or in other words, is it right that the potential G_μ^* is uniformly continuous if and only if the potential G_μ is? The following example shows that the answer to that question is negative.

If a measure μ in R^{n+1} is of the form $\mu = \delta_{x_0}\otimes\lambda$, where $x_0\in R^n$ (δ_{x_0} is a Dirac measure in R^n), λ is a measure on R^1, then the conditions (3), (4) are reduced to the conditions

$$(3')\qquad \lim_{a\to\infty}\left(\sup\left\{\int_a^\infty \lambda\left(\left\langle t - \frac{1}{4\pi}c^{-2/n},\ t\right\rangle\right)dc;\ t\in R^1\right\}\right) = 0\,,$$

$$(4')\qquad \lim_{a\to\infty}\left(\sup\left\{\int_a^\infty \lambda\left(\left\langle t,\ t + \frac{1}{4\pi}c^{-2/n}\right\rangle\right)dc;\ t\in R^1\right\}\right) = 0$$

(cf. [4]).

Let us now consider the case $n = 1$. Let λ be a measure on R^1 with its support $\operatorname{supp}\lambda = \langle 0, e^{-1}\rangle$, which is defined by the density h (with respect to the Lebesgue measure):

$$h(t) = \frac{-1}{\sqrt{(t)}\,\ln t}\,,\qquad t\in(0, e^{-1})\,,$$

$h(t) = 0$ for $t\in R^1 - (0, e^{-1})$. First let us show that for each $a > 0$

$$\int_a^\infty \lambda\left(\left\langle 0, \frac{1}{4\pi}c^{-2}\right\rangle\right)dc = +\infty\,,$$

i.e. the condition (4′) (for $n = 1$) is not fulfilled. Let $a > \frac{1}{2}\sqrt{(e/\pi)}$. Then

$$\int_a^\infty \lambda\left(\left\langle 0, \frac{1}{4\pi} c^{-2}\right\rangle\right) dc = -\int_0^{(1/4\pi)c^{-2}} \left(\frac{dt}{\sqrt{(t)}\ln t}\right) dc =$$

$$= -\int_0^{(1/4\pi)a^{-2}} dt \int_a^{(1/2)(\pi t)^{-1/2}} \frac{dc}{\sqrt{(t)}\ln t} = \int_0^{(1/4\pi)a^{-2}} \left(\frac{a}{\sqrt{(t)}\ln t} - \right.$$

$$\left. - \frac{1}{2\sqrt{(\pi)}\, t \ln t}\right) dt = +\infty$$

since

$$\left|\int_0^{(1/4\pi)a^{-2}} \frac{a}{\sqrt{(t)}\ln t}\, dt\right| < +\infty\,, \quad -\int_0^{(1/4\pi)a^{-2}} \frac{dt}{2\sqrt{(\pi)}\, t \ln t} = +\infty\,.$$

Note that if μ is a measure in R^2 which is, for instance, of the form $\mu = \delta_0 \otimes \lambda$ (δ_0 is the Dirac measure in R^1 supported by the point 0), then one can even calculate the value

$$G^*_\mu(0,0) = \int_{R^2} G^*(-\xi, -\tau)\, d\mu(\xi, \tau) = \int_0^{e^{-1}} G^*(0, -\tau)\, h(\tau)\, d\tau =$$

$$= -\int_0^{e^{-1}} \frac{1}{2\sqrt{(\pi\tau)}} \frac{1}{\sqrt{(\tau)}\ln \tau}\, d\tau = +\infty\,.$$

Now let us prove that the condition (3′) (for $n = 1$) is fulfilled, i.e. for $\mu = \delta_0 \otimes \lambda$ the potential G_μ is uniformly continuous on R^2. It is obvious that it suffices to show that

$$\text{(3'')} \qquad \lim_{a\to\infty}\left(\sup\left\{\int_a^\infty \lambda\left(\left\langle t - \frac{1}{4\pi}c^{-2}, t\right\rangle\right) dc;\ t \in \langle 0, e^{-1}\rangle\right\}\right) = 0$$

as (for any $c > 0$)

$$\lambda\left(\left\langle t - \frac{1}{4\pi}c^{-2}, t\right\rangle\right) = 0 \quad \text{for} \quad t \leqq 0\,,$$

$$\lambda\left(\left\langle t - \frac{1}{4\pi}c^{-2}, t\right\rangle\right) \leqq \lambda\left(\left\langle e^{-1} - \frac{1}{4\pi}c^{-2}, e^{-1}\right\rangle\right) \quad \text{for} \quad t \geqq e^{-1}\,.$$

Let $t \in (0, e^{-1}\rangle$. In order to calculate the value $\lambda(\langle t - (1/4\pi)\, c^{-2}, t\rangle)$ let us distinguish the following two cases:

a) $t - \dfrac{1}{4\pi} c^{-2} < 0$ (i.e. $c < \frac{1}{2}(\pi t)^{-1/2}$),

b) $t - \dfrac{1}{4\pi} c^{-2} \geqq 0$ (i.e. $c \geqq \frac{1}{2}(\pi t)^{-1/2}$).

In the case a) we have

$$\lambda\left(\left\langle t - \frac{1}{4\pi} c^{-2}, t\right\rangle\right) = - \int_0^t \frac{d\tau}{\sqrt{(\tau)} \ln \tau}$$

and in the case b)

$$\lambda\left(\left\langle t - \frac{1}{4\pi} c^{-2}, t\right\rangle\right) = - \int_{t-(1/4\pi)c^{-2}}^t \frac{d\tau}{\sqrt{(\tau)} \ln \tau}.$$

Thus

$$(5) \qquad \int_a^\infty \lambda\left(\left\langle t - \frac{1}{4\pi} c^{-2}, t\right\rangle\right) \mathrm{d}c = - \int_a^{(1/2)(\pi t)^{-1/2}} \mathrm{d}c \int_0^t \frac{d\tau}{\sqrt{(\tau)} \ln \tau} -$$

$$- \int_{(1/2)(\pi t)^{-1/2}}^\infty \mathrm{d}c \int_{t-(1/4\pi)c^{-2}}^t \frac{d\tau}{\sqrt{(\tau)} \ln \tau} = I_1 + I_2$$

for $a < \frac{1}{2}(\pi t)^{-1/2}$. In the case $a \geqq \frac{1}{2}(\pi t)^{-1/2}$ we have

$$(6) \qquad \int_a^\infty \lambda\left(\left\langle t - \frac{1}{4\pi} c^{-2}, t\right\rangle\right) \mathrm{d}c = - \int_a^\infty \mathrm{d}c \int_{t-(1/4\pi)c^{-2}}^t \frac{d\tau}{\sqrt{(\tau)} \ln \tau} = I_3 .$$

The integral I_1 is evidently finite. The integrals I_2, I_3 are also finite, since

$$(7) \qquad \left| \int_{t-(1/4\pi)c^{-2}}^t \frac{d\tau}{\sqrt{(\tau)} \ln \tau} \right| \leqq \frac{1}{|\ln t|} \int_{t-(1/4\pi)c^{-2}}^t \frac{d\tau}{\sqrt{\tau}} =$$

$$= \frac{2}{|\ln t|} \left(\sqrt{t} - \sqrt{\left(t - \frac{1}{4\pi} c^{-2}\right)} \right) =$$

$$= \frac{1}{2\pi c^2} \frac{1}{|\ln t| \left(\sqrt{(t)} + \sqrt{\left(t - \frac{1}{4\pi} c^{-2}\right)} \right)} \leqq \frac{1}{c^2} \frac{1}{2\pi |\ln t| \sqrt{t}} .$$

It is easily seen that for a fixed $a > 0$ the function

$$f_a(t) = \int_a^\infty \lambda\left(\left\langle t - \frac{1}{4\pi} c^{-2}, t\right\rangle\right) \mathrm{d}c$$

is continuous on the interval $(0, e^{-1}\rangle$. Since the integral I_3 is finite, it holds for each $t \in (0, e^{-1}\rangle$ that

$$(8) \qquad f_a(t) \to 0 \quad \text{for} \quad a \to +\infty$$

monotonically. Let us show that for each $a > 0$

$$(9) \qquad \lim_{t \to 0+} f_a(t) = 0 .$$