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GENERALIZED CONTINUITY AND GENERALIZED CLOSED GRAPHS

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1. Introduction. In [13], some sufficient conditions for a weakly-continuous function to be continuous are investigated. In particular, Corollary 2 [13] states that if Y is a Hausdorff space such that every closed subset is N-closed, then a weakly-continuous map $f : X \to Y$ is continuous. As we show below, a Hausdorff space such that every closed subset is N-closed is compact. Consequently, this corollary is not a particularly significant result.

The major purpose for this present investigation is to use tH-monad theory and to discuss, for an arbitrary map $f : X \to Y$, some relations between (tH, sK)-continuity, (tH, sK)-closed graphs and, if X, Y are topological spaces, topological continuity. In the process, we are able to improve upon most of the results in [13]. For example, applying our results to topological spaces X and Y, it is shown that if $A \subset X$ is compact [resp. N-closed, αA-compact, completely-compact, SA-compact] and the graph, $G(f)$, of $f : X \to Y$ is closed [resp. has property (P), is strongly closed, is (I_X, w)-closed, is (I_X, S)-closed], then $f^{-1}[A]$ is closed in X. If Y is Hausdorff [resp. completely-Hausdorff] and each closed subset is θ-compact [resp. w-compact] and $f : X \to Y$ is almost-continuous [resp. a c-map], then f is continuous. If (Y, T) is rim-θ [resp. α]-compact, $f : (X, \tau) \to (Y, T)$ is weakly-continuous and $G(f)$ is strongly closed [resp. has property (P)], then f is continuous. Finally, we show that every rim-θ-compact, Urysohn [resp. rim-α-compact, Hausdorff; rim-S-compact, weakly-Hausdorff, extremally disconnected] space is regular.

2. Preliminaries. In the interest of brevity, we shall rely heavily upon the definitions and results which appear in the references [6], [7], [8], [9], [12]. Recall that $f : X \to Y$ is (tH, sK)-continuous at $p \in X$ if $^*f[\mu_t H(p)] \subset \mu_s K(f(p))$, where $\mu_t H(p)$ and $\mu_s K(q)$ are the tH and sK-monads on X and Y, respectively [8]. For the monad of ROBINSON [16] $\mu(p)$ [resp. α-monad $\mu_\alpha(p)$, θ-monad $\mu_\theta(p)$, w-monad $\mu_w(p)$], we have that a map $f : (X, \tau) \to (Y, T)$ is almost-continuous [19] [resp. θ-continuous

[2], weakly-continuous [13], a c-map [3]] at $p \in X$ iff it is (I_X, α) [resp. (θ, θ), (I_X, θ), (I_X, w)]-continuous at $p \in X$. We note that a weakly-continuous map is also known as a weakly-θ-continuous map. $^*\mathcal{M}$ is a highly saturated enlargement.

Definition 2.1. A map $f : X \to Y$ has a *(tH, sK)-closed graph* $G(f)$ if for each $(p, q) \notin G(f)$, $\mu_\pi((p, q)) \cap {}^*(G(f)) = \emptyset$, where π is generated by the tH and sK-monads (denoted by $\pi = tH \times sK$).

Let (X, τ) and (Y, T) denote topological spaces.

Example 2.1. (i) For $f : (X, \tau) \to (Y, T)$, the graph $G(f)$ is (I_X, I_Y)-closed iff $\mu((p, q)) \cap {}^*(G(f)) = \emptyset$ for each $(p, q) \notin G(f)$ iff $G(f)$ is closed in $X \times Y$.

(ii) For $f : (X, \tau) \to (Y, T)$, $G(f)$ is (I_X, θ)-closed iff it is *strongly closed* in the sense of Herrington and Long [5].

(iii) For $f : (X, \tau) \to (Y, T)$, $G(f)$ is (I_X, α)-closed iff it has *property* (P) discussed in [11] and [13].

(iv) A map $f : X \to Y$ has a (tH, sK)-closed graph iff $X - G(f)$ is π-open, where $\pi = tH \times sK$. In general, if $t \in PTH(X)$, $s \in PSK(Y)$, then if $G(f)$ is (tH, sK)-closed, then it is π-closed.

Finally, we point out that many of the results in this paper also hold for the q-monad of Puritz [15]. However, since we are particularly interested in topological spaces and certain closedness properties it appears more useful to concentrate upon the tH-monad approach due to certain special filter base properties which often appear unavoidable and which are exhibited by such nonstandard objects.

3. Major results. As stated in [6] for (X, τ), a set $A \subset X$ is N-closed iff it is αA-compact iff ${}^*A \subset \bigcup\{\mu_\alpha(x) \mid x \in A\}$.

Theorem 3.1. *Let (X, τ) be Hausdorff and assume that each closed set $A \subset X$ is N-closed. Then X is compact.*

Proof. Since X is N-closed (i.e. nearly-compact [18]) then X is almost-regular [17] and Urysohn (i.e. Urysohn = distinct points are separated by closed neighborhoods). Thus every closed subset of X is θ-compact, since for each $p \in X$, $\mu_\alpha(p) = \mu_\theta(p)$. Consequently, (X, τ) is C-compact in the sense of Viglino [22]. Thus X is semiregular by application of Theorem A in [22]. Therefore, X is regular and this completes the proof.

We now give an important characterization for (tH, sK)-closed graphs. For $\emptyset \neq \mathcal{F} \subset \mathcal{P}(X)$, the power set of X, we let $\operatorname{Nuc} \mathcal{F} = \bigcap\{{}^*F \mid F \in \mathcal{F}\}$ and if $f : X \to Y$, then $f[\mathcal{F}] = \{f[F] \mid F \in \mathcal{F}\}$.

Theorem 3.2. *A map $f : X \to Y$ has a (tH, sK)-closed graph, $G(f)$, iff whenever $\emptyset \neq \operatorname{Nuc} \mathcal{F} \subset \mu_t H(p)$, $p \in X$, $\mathcal{F} \subset \mathcal{P}(X)$, and $\operatorname{Nuc} f[\mathcal{F}] \subset \mu_s K(q)$ for some $q \in Y$, then $f(p) = q$.*

Proof. Let $\mathscr{F} \subset \mathscr{P}(X)$, $\emptyset \neq \text{Nuc}\,\mathscr{F} \subset \mu_t H(p)$, $p \in X$, and $\text{Nuc}\,f[\mathscr{F}] \subset \mu_s K(q)$ for some $q \in Y$. Assume that $x \in \text{Nuc}\,\mathscr{F}$ and $y \in \text{Nuc}\,f[\mathscr{F}]$. Hence $^*(x, y) \in \in \mu_\pi((p, q))$, $\pi = tH \times sK$. Consequently, $^*(F \times f[F]) \cap \mu_\pi((p, q)) \neq \emptyset$ for each $F \in \mathscr{F}$. Since $^*(F \times f[F]) \subset {}^*(G(f))$, we have that $\mu_\pi((p, q)) \cap {}^*(G(f)) \neq \emptyset$. Assuming that $G(f)$ is a (tH, sK)-closed graph this yields that $f(p) = q$.

Conversely, assume that whenever $\mathscr{F} \subset \mathscr{P}(X)$, $\emptyset \neq \text{Nuc}\,\mathscr{F} \subset \mu_t H(p)$ and $\text{Nuc}\,f[\mathscr{F}] \subset \mu_s K(q)$, $q \in Y$, then $f(p) = q$. Let $(p, q) \in (X \times Y) - G(f)$. Thus there does not exist a $\mathscr{T} \subset \mathscr{P}(X)$ such that $\emptyset \neq \text{Nuc}\,\mathscr{T} \subset \mu_t H(p)$ and $\text{Nuc}\,f[\mathscr{T}] \subset \subset \mu_s K(q)$. Suppose that $\mu_\pi((p, q)) \cap {}^*(G(f)) \neq \emptyset$. Then there exists some $x \in \in \mu_t H(p)$ and $y \in \mu_s K(q)$ such that $^*(x, y) \in {}^*(G(f))$. Now the ultramonad $\text{Nuc Fil}\{x\} = \text{NF}\{x\} \subset \mu_t H(p)$ and $^*f[\text{NF}\{x\}] = \text{NF}\{^*f(x)\} = \text{NF}\{y\} \subset \mu_s K(q)$. This contradiction implies that $\mu_\pi((p, q)) \cap {}^*(G(f)) = \emptyset$ and the proof is complete.

Recall that a space (X, τ) is compact [resp. nearly-compact [18], quasi-H-closed [14], completely-closed [10], S-closed [21]] iff $^*X = \bigcup\{\mu(x) \mid x \in X\}$ [resp. $^*X = = \bigcup\{\mu_\alpha(x) \mid x \in X\}$, $^*X = \bigcup\{\mu_\theta(x) \mid x \in X\}$, $^*X = \bigcup\{\mu_w(x) \mid x \in X\}$, $^*X = = \bigcup\{\mu\, S(x) \mid x \in X\}$ [6, 7, 8, 9, 10]]. The w-monad at $p \in X$ is $\mu_w(p) = = \bigcap\{^*f^{-1}[\mu(f(p))] \mid f \in C(X)\}$ and the S-monad is $\mu\, S(p) = \bigcap\{^*(\text{cl}_X A) \mid p \in A \in \in \text{SO}(X)\}$, where $\text{SO}(X)$ is a set of all semiopen subsets of X [1]. Also, $W \subset {}^*Y$ is sKA — compact iff $W \subset \bigcup\{\mu_s K(x) \mid x \in A\}$.

Theorem 3.3. *If $f : X \to Y$ has a (tH, sK)-closed graph and Y is sKY-compact (i.e. sK-compact), then f is (tH, sK)-continuous.*

Proof. Assume that $f : X \to Y$ has a (tH, sK)-closed graph and consider $^*f[\mu_t H(p)]$. By sKY-compactness, $^*f[\mu_t H(p)] \subset \bigcup\{\mu_s K(y) \mid y \in Y\}$. Assume that $^*f[\mu_t H(p)] \cap \mu_s K(q) \neq \emptyset$. Then there exists $x \in \mu_t H(p)$ such that $^*f(x) \in \mu_s K(q)$. However, $\text{NF}\{x\} \subset \mu_t H(p)$ and $^*f[\text{NF}\{x\}] = \text{NF}\{^*f(x)\}$ imply that $^*f[\text{NF}\{x\}] \subset \subset \mu_s K(q)$. Theorem 3.2 yields $f(p) = q$. Consequently, $^*f[\mu_t H(p)] \subset \mu_s K(f(p))$ and the proof is completed.

Corollary 3.3. *If $f : (X, \tau) \to (Y, T)$ has a (I_X, I_Y)- [resp. (I_X, α), (θ, I_Y), (θ, θ), (I_X, w), (I_X, S), (I_X, θ)]-closed graph, and Y is compact [resp. nearly-compact, compact, quasi-H-closed, completely-closed, S-closed, quasi-H-closed], then f is continuous [resp. almost-continuous [19], strongly-θ-continuous [8], θ-continuous [4], a c-map [3], (I_X, S)-continuous, weakly-continuous [13]].*

We now present a proposition which gives a strong converse to Theorem 3.3 and has numerous corollaries which improve upon Theorem 1 in [13]. A set Y is (sK, uV)-*separated* if for distinct $p, q \in Y$, $\mu_s K(p) \cap \mu_u V(q) = \emptyset$.

Theorem 3.4. *Let $f : X \to Y$ be (tH, sK)-continuous and Y be (sK, uV)-separated. Then f has a (tH, uV)-closed graph.*

Proof. Assume that $\emptyset \neq \text{Nuc}\,\mathscr{F} \subset \mu_t H(p)$, $p \in X$, $\mathscr{F} \subset \mathscr{P}(X)$, and $\text{Nuc}\,f[\mathscr{F}] \subset \subset \mu_u V(q)$, $q \in Y$. Then (tH, sK)-continuity implies that $\text{Nuc}\,f[\mathscr{F}] \subset \mu_s K(f(p))$.

Since $\operatorname{Nuc} f[\mathscr{F}] \neq \emptyset$, then (sK, uV)-separation implies that $f(p) = q$. Hence f has a (tH, uV)-closed graph.

Corollary 3.4.1. *If $f:(X, \tau) \to (Y, T)$ is continuous [resp. almost-continuous, strongly-θ-continuous, θ-continuous, weakly-continuous] and Y is Hausdroff, then f has a closed [resp. (I_X, θ)-closed, (θ, θ)-closed, (θ, α)-closed, (I_X, α)-closed] graph.*

Corollary 3.4.2. *If $f:(X, \tau) \to (Y, T)$ is weakly-continuous [resp. a c-map, (I_X, S)-continuous] Y is Urysohn [resp. completely-Hausdorff, weakly-Hausdorff], then f has a (I_X, θ) [resp. (I_X, w), (I_X, α)]-closed graph.*

Proof. The above results follow from Theorem 1.4 and 1.5 [6] and the result that if a space Y is completely-Hausdorff [resp. weakly-Hausdorff [20]], then for distinct $p, q \in Y$, $\mu_w(p) \cap \mu_w(q) = \emptyset$ [resp. $\mu_\alpha(p) \cap \mu S(q) = \emptyset$].

Remark 3.1. If $f: X \to Y$ has a (tH, sK)-closed graph and we have an rJ-monad system on X and a uV-monad system on Y such that for each $p \in X$ and $q \in Y$, $\mu_r J(p) \subset \mu_t H(p)$ and $\mu_u V(q) \subset \mu_s K(q)$, then f has an (rJ, uV)-closed graph. Hence each of the (tH, sK)-continuous maps in the hypothesis of Corollaries 3.4.1 and 3.4.2 has a closed graph.

Recall that for $W \subset {}^*X$, $St_t H(W) = \{x \mid [x \in X] \wedge [\mu_t H(p) \cap W \neq \emptyset]\}$.

Theorem 3.5. *Let $W \subset {}^*Y$ be sKA-compact. If $f: X \to Y$ has a (tH, sK)-closed graph, then*

$$St_t H({}^*f^{-1}[W]) \subset f^{-1}[A].$$

Proof. We know that $W \subset \bigcup\{\mu_s K(x) \mid x \in A\}$. Thus ${}^*f^{-1}[W] \subset \subset \bigcup\{{}^*f^{-1}[\mu_s K(x)] \mid x \in A\}$. Let $p \in St_t H({}^*f^{-1}[W])$. Then $\mu_t H(p) \cap {}^*f^{-1}[W] \neq \neq \emptyset$. Hence ${}^*f[\mu_t H(p)] \cap W \neq \emptyset$. Consequently, there exists $x \in A$ such that ${}^*f[\mu_t H(p)] \cap \mu_s K(x) \neq \emptyset$. Thus there exists $r \in \mu_t H(p)$ such that $\mathrm{NF}\{r\} \subset \mu_t H(p)$ and ${}^*f(r) \in \mu_s K(x)$. Therefore, $\mathrm{NF}\{{}^*f(r)\} \subset \mu_s K(p)$. Now (tH, sK)-closed graph implies by Theorem 3.2 that $f(p) = x$. (i.e. $p \in f^{-1}(x)$). Hence,

$$St_t H({}^*f^{-1}[W]) \subset f^{-1}[A].$$

Corollary 3.5.1. *Let $A \subset Y$ be sKA-compact and for each $p \in X$, let $t \in PTH(p)$. If $f: X \to Y$ has a (tH, sK)-closed graph, then $f^{-1}[A]$ is tH-closed.*

Corollary 3.5.2. *Let $A \subset Y$ be compact [resp. N-closed, SA-compact, completely-closed, SA-compact]. If $f:(X, \tau) \to (Y, T)$ has a (I_X, I_Y) [resp. (I_X, α), (I_X, θ), (I_X, w), (I_X, S)]-closed graph, then $f^{-1}[A]$ is closed in X.*

Corollary 3.5.3. *Let $A \subset Y$ be compact. If $f:(X, \tau) \to (Y, T)$ has a (θ, I_Y)-closed graph, then $f^{-1}[A]$ is closed in X.*

Example 2 in Viglino's paper [22] is that of a Hausdorff, non-Urysohn, non-compact space in which each closed set is θ-compact. He calls such a space *C-compact* and notes that a C-compact Urysohn space is compact. SOUNDARARAJAN [20] gives an example of a compact weakly-Hausdorff space which is not Hausdorff. The next result improves somewhat upon Corollary 2 in [13].

Theorem 3.6. *Let Y be Hausdorff [resp. completely-Hausdorff] and each closed subset of Y is θ-compact [resp. w-compact]. If $f:(X, \tau) \to (Y, T)$ is almost-continuous [resp. a c-map], then f is continuous.*

Remark 3.2. In Theorem 3.6, we have not included weakly-Hausdorff spaces in which every closed subset is S-closed. The reason for this is that a weakly-Hausdorff space which is S-closed is H-closed Urysohn and extremally disconnected. Such a space is thus N-closed and if a subset is S-closed, then it is N-closed. Consequently, Theorem 3.1 would imply that a weakly-Hausdorff space in which every closed subset if S-closed is a compact Hausdorff space.

As far as rim-compact spaces are concerned, we are able to extend or improve upon Theorems 3 and 4 in [13]. A space (X, τ) is *rim-tH-compact* if for each $p \in X$ and each neighborhood $V \in \dot{\tau}$ of p there exists some neighborhood $G_p \in \tau$ of p such that $\mathrm{Fr}(G_p) = \mathrm{cl}_X G - G$ is $tH(\mathrm{Fr}(G_p))$-compact and $G_p \subset V$. GROSS and VIGLINO [4] show than any C-compact Hausdorff space is rim-θ-compact. Viglino's example [22] is a C-compact Hausdorff, nonregular; hence, non-rim-compact but rim-θ-compact space.

We now modify the proof of Theorem 3 in [13] in order to obtain the following proposition.

Theorem 3.7. *If $(Y. T)$ is rim-sK-compact and $f:(X, \tau) \to (Y, T)$ is weakly-continuous with a (I_X, sK)-closed graph, then f is continuous.*

Proof. Let $p \in X$ and $f(p) \in V \in T$. Then there exists some $W \in T$ such that $f(p) \in W \subset V$ and $\mathrm{Fr}(W)$ is $sK(\mathrm{Fr}(W))$-compact. Clearly $f(p) \notin \mathrm{Fr}(W)$. Thus for each $y \in \mathrm{Fr}(W)$, $(p, y) \notin G(f)$. Since $G(f)$ is (I_X, sK)-closed, then $^*f[\mu(p)] \cap \mu_s K(y) = \emptyset$ for each $y \in \mathrm{Fr}(W)$. Consequently, $^*f[\mu(p)] \cap (\bigcup\{\mu_s K(y) \mid y \in \mathrm{Fr}(W)\}) = \emptyset$. Hence, $^*f[\mu(p)] \cap {}^*(\mathrm{Fr}(W)) = \emptyset$. Weak-continuity implies that $^*f[\mu(p)] \subset \mu_\theta(f(p)) \subset {}^*(\mathrm{cl}_Y W)$. Therefore,

$$^*f[\mu(p)] \cap {}^*(Y - W) = {}^*f[\mu(p)] \cap {}^*(\mathrm{Fr}(W)) = \emptyset\,.$$

Hence, $^*f[\mu(p)] \subset {}^*W \subset {}^*V$. Since V is an arbitrary open neighborhood of $f(p)$, then $^*f[\mu(p)] \subset \mu(f(p))$ and the proof is complete.

Corollary 3.7.1. *If (Y, T) is rim-θ-compact [resp. rim-α-compact] and $f:(X, \tau) \to (Y, T)$ is weakly-continuous where $G(f)$ is stronlgy closed [resp. has property (P)], then f is continuous.*