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AN EXAMPLE OF REMOVABLE SINGULARITIES FOR BOUNDED HOLOMORPHIC FUNCTIONS

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I. INTRODUCTION

The results concerning removable singularities for bounded holomorphic functions are usually described in terms of analytic capacity which is a set function introduced by L. AHLFORS in [1] (and given name later by V. D. EROCHIN). Information about the definition of analytic capacity and its basic properties can be found e.g. in [14], [4], [6], [10]. The concept of analytic capacity will be used here for conciseness only and all we actually need are the following two propositions:

I. A set F (relatively) closed in a domain D of the complex plane $\mathbb{C}$ is a set of removable singularities for bounded functions holomorphic on $D \setminus F$ if and only if every compact $K \subset F$ is a set of removable singularities for bounded functions holomorphic on $\mathbb{C} \setminus K$.

II. Let K be a compact subset of $\mathbb{C}$. Then the following properties are equivalent:

(i) K has zero analytic capacity.

(ii) K is a set of removable singularities for bounded functions holomorphic on $\mathbb{C} \setminus K$.

(iii) Every bounded function holomorphic on $\mathbb{C} \setminus K$ is a constant function.

Much research has been done on the relations of the analytic capacity to various metric characterizations of the set. We shall be interested in its relation to the linear (Hausdorff) measure of the set only. The first steps in this direction were made by P. PAINLEVÉ at the end of the last century. The results of his treatise [11] imply, among other, that every compact of zero linear measure has also zero analytic capacity. About 1909 several communications appeared in Comptes Rendus, where A. DENJOY investigated the behaviour of Cauchy integrals on the support of the corresponding measure. It was shown there that, given a compact subset of a straight line with positive measure, one can construct a nonconstant function bounded and holomorphic on its complement in the complex plane (cf. [2]). Thus a compact situated on a straight line has zero analytic capacity if and only if it has zero linear measure. Let us note that CH. POMMERENKE in [12] and L. D. IVANOV in [7] obtained later a deeper result, namely that the analytic capacity of a compact on a straight line is one quarter of the corresponding linear measure.

We shall say briefly that a set lying in the complex plane has the Denjoy property if each of its compact subsets of zero analytic capacity has also zero linear measure. Thus every straight line has the Denjoy property, but a simple arc need not have this property at all. In fact, several examples of compact sets with zero analytic capacity and positive linear measure were constructed by A. G. VITUŠKIN, L. D. IVANOV and J. GARNETT (cf. [17], [9], [5]) and any such compact is a perfect discontinuum, which lies on a simple arc by a result of Denjoy from 1910 (cf. [2]). The assertion that every rectifiable curve has the Denjoy property is usually called the Denjoy conjecture. This assertion has been generally proved only recently*). Earlier some results were obtained concerning comparability of the linear measure and the analytic capacity of sets with additional conditions on the curves or continua on which they are situated (cf. L. D. IVANOV [8], N. A. ŠIROKOV [13], J. FUKA and J. KRÁL [3]).

According to the Denjoy conjecture, the rectifiability is sufficient for an arc to have the Denjoy property, and one may ask to what extent it is necessary. A common feature of all mentioned examples is that the corresponding sets with zero analytic capacity and positive linear measure are in a certain sense very "dispersed" and it is therefore not clear à priori whether such a set can be situated on a "nice" curve. Modifying a method of Vituškin it is possible to construct a real function satisfying the Hölder condition such that its graph carries a compact of positive linear measure and zero analytic capacity. In the following we construct such a function satisfying the Hölder condition for every exponent $\alpha \in (0, 1)$. The graph of this function has not the Denjoy property although the function is only "a little worse" than a Lipschitz function, the graph of which is rectifiable.

II. ESSENTIAL CONSTRUCTIONS NEEDED IN THE SEQUEL

1. As usual, we shall denote by $\mathbb{C}$ the set of all complex numbers which will be identified with the Euclidean plane $\mathbb{R}^2$. For $E \subset \mathbb{C}$ we shall denote by int E and diam E the interior and the diameter of E, respectively. If $A, B \subset \mathbb{C}$, then

$$\text{dist}(A, B) = \inf\{|a - b| : a \in A,\ b \in B\}. \tag{1}$$

Given $E \subset \mathbb{C}$ and $\varepsilon > 0$ we put

$$\mathscr{H}_\varepsilon(E) = \inf\{\sum_{n=1}^{\infty} \operatorname{diam} E_n : E \subset \bigcup_{n=1}^{\infty} E_n,\ \operatorname{diam} E_n \leqq \varepsilon\}. \tag{2}$$

*) The validity of the Denjoy conjecture is a consequence of the recent result of Calderón on the boundedness of Cauchy's operator. It is necessary to combine Calderón's result from [16] with earlier results of DAVIE, HAVIN and HAVINSON (cf. [17], [18], [19]).

The linear (Hausdorff) measure of E is defined by

(3) $$\mathscr{H}(E) = \lim_{\varepsilon \to 0+} \mathscr{H}_\varepsilon(E) = \sup_{\varepsilon > 0} \mathscr{H}_\varepsilon(E) .$$

It can be easily shown that the linear measure of the orthogonal projection of E into an arbitrary straight line is not greater than the linear measure of E.

2. Let $\Delta \subset \mathbb{R}$ be a compact interval. Given positive integers k, j such that $1 \leqq j \leqq k$ we denote by $(j \mid k)\,\Delta$ the j-th of the compact intervals which arise by dividing Δ into k non-overlapping parts of equal length. Moreover, if $q \in (0, 1)$ we denote by $(j \mid k; q)\,\Delta$ the compact interval concentric with $(j \mid k)\,\Delta$, the length of which is a $(1 - q)$ – multiple of the length of $(j \mid k)\,\Delta$. Thus we put

(4) $$(j \mid k)\,\Delta = \left\langle a + (j - 1)\frac{b - a}{k},\ a + j\frac{b - a}{k} \right\rangle,$$

$$(j \mid k; q)\,\Delta = \left\langle a + (j - 1 + \tfrac{1}{2}q)\frac{b - a}{k},\quad a + (j - \tfrac{1}{2}q)\frac{b - a}{k} \right\rangle$$

in the case $\Delta = \langle a, b \rangle$.

Given a compact interval $\Delta \subset \mathbb{R}$ and numbers k, q, v such that k is a positive integer, $0 < q < 1$, $v > 0$, there exists a unique function $\varphi : \mathbb{R} \to \mathbb{R}$ with the following properties:

(5_1) φ is continuous in $\mathbb{R}$;

(5_2) $\varphi(x) = 0$ for $x \notin \Delta$;

(5_3) $\varphi(x) = (-1)^{j-1}\, v$ for $x \in (j \mid k; q)\,\Delta$, $j = 1, 2, \ldots, k$;

(5_4) φ is affine on each interval $J \subset \Delta \setminus \bigcup_{j=1}^{k} ((j \mid k; q)\,\Delta)$.

In what follows we shall use

(6) $$v = v(\Delta; k) = (2k)^{-1}\, \mathscr{H}(\Delta)$$

and the corresponding function φ will be denoted more explicitly by $\varphi_{\Delta;k,q}$ (if necessary). Then obviously

(7) $$\max \left|\varphi_{\Delta;k,q}(x)\right| = v(\Delta; k) .$$

If we put, in addition,

(8) $$\sigma(\Delta; k, q) = q\, v(\Delta; k) = (2k)^{-1}\, q\, \mathscr{H}(\Delta) ,$$

then σ is the length of intervals contiguous to $(j \mid k; q)\,\Delta$ in $(j \mid k)\,\Delta$.

3. The following constructions will depend on the choice of two sequences of numbers, one of them being a sequence of positive integers k_n and the other a sequence of real numbers $q_n \in (0, 1)$, $n = 1, 2, \ldots$ We shall restrict this choice by some additional conditions later, but we shall assume in the following that the numbers k_n and q_n are fixed and we shall not designate the dependence on this choice any more.

We shall denote by $\mathscr{W}$ the set of all sequences of positive integers j_n such that $1 \leqq j_n \leqq k_n$, $n = 1, 2, \ldots$ For $\{j_n\} \in \mathscr{W}$ we define by induction

$$
\begin{aligned}
(9) \qquad \tilde{\Delta}_{j_1} &= (j_1 \mid k_1) \langle 0, 1 \rangle\,, \quad \Delta_{j_1} = (j_1 \mid k_1; q_1) \langle 0, 1 \rangle\,; \\
\tilde{\Delta}_{j_1,\ldots,j_{n+1}} &= (j_{n+1} \mid k_{n+1}) \, \Delta_{j_1,\ldots,j_n}\,, \\
\Delta_{j_1,\ldots,j_{n+1}} &= (j_{n+1} \mid k_{n+1}; q_{n+1}) \, \Delta_{j_1,\ldots,j_n}\,.
\end{aligned}
$$

The intervals $\tilde{\Delta}_{j_1,\ldots,j_n}$ will be termed the unreduced intervals of order n and the intervals $\Delta_{j_1,\ldots,j_n}$ will be termed the reduced intervals of order n; moreover, the interval $\langle 0, 1 \rangle$ will be termed the (reduced) interval of order 0.

It is easily shown by induction that the following assertions hold:

(a) any two different unreduced intervals of the same order n do not overlap and have the same length $\tilde{h}_n$;

(b) all reduced intervals of the same order n are mutually disjoint and have the same length h_n;

(c) we have $h_0 = 1$, $\tilde{h}_n = k_n^{-1} h_{n-1}$, $h_n = (1 - q_n)\, \tilde{h}_n$ for $n = 1, 2, \ldots$; this implies

$$(10) \qquad h_n = \prod_{m=1}^{n} k_m^{-1}(1 - q_m)\,, \quad n = 0, 1, 2, \ldots\,;$$

(d) the number of all reduced intervals of order n is

$$(11) \qquad p_n = \prod_{m=1}^{n} k_m\,, \quad n = 0, 1, 2, \ldots$$

Furthermore, it is seen that the quantities $v(\Delta_{j_1,\ldots,j_{n-1}}; k_n)$ and $\sigma(\Delta_{j_1,\ldots,j_{n-1}}; k_n, q_n)$ depend on n only and we shall denote them simply by v_n and σ_n, respectively; thus

$$(12) \qquad v_n = (2(1 - q_n))^{-1}\, h_n = (2k_n)^{-1}\, h_{n-1}\,, \quad \sigma_n = q_n v_n\,.$$

Finally, let us notice that the distance of two neighboring reduced intervals of order n, which lie in the same interval of order $n - 1$, is exactly $2\sigma_n$ and that the distance of any two reduced intervals of order n in any other position is greater than $2\sigma_n$. Therefore the distance of two unreduced intervals of order $n + 1$, which are not included in the same interval of order n, is at least $2\sigma_n$.

4. Using the ordering of the real line we shall introduce a new indexing for intervals of the same order. We shall denote by $\Delta_i^{(n)}$ the reduced interval of order n, which lies on the l-th place $(1 \leqq l \leqq p_n)$ in the ordering from the left to the right; the analogous notation will be used for unreduced intervals etc. This means that we set

$$(13) \qquad \Delta_1^{(0)} = \langle 0, 1 \rangle\,, \quad \tilde{\Delta}_l^{(n)} = (j \mid k_n)\, \Delta_{l'}^{(n-1)}\,, \quad \Delta_l^{(n)} = (j \mid k_n; q_n)\, \Delta_{l'}^{(n-1)}$$

with positive integers l', j uniquely determined by conditions

$$(13') \qquad l = (l' - 1)\, k_n + j\,, \quad 1 \leqq j \leqq k_n\,.$$

Let us notice that $1 \leqq l' \leqq p_{n-1}$, since $1 \leqq l \leqq p_n$.

Both the intervals $\Delta_l^{(n)} = \Delta_{j_1,\ldots,j_n}$ and $\tilde{\Delta}_l^{(n)} = \tilde{\Delta}_{j_1,\ldots,j_n}$ have a common center, which we shall denote by $s_l^{(n)}$ or $s_{j_1,\ldots,j_n}$. Hence

$$(14) \qquad \tilde{\Delta}_l^{(n)} = \langle s_l^{(n)} - v_n, s_l^{(n)} + v_n \rangle\,,$$

$$\Delta_l^{(n)} = \langle s_l^{(n)} - v_n + \sigma_n, s_l^{(n)} + v_n - \sigma_n \rangle = \langle s_l^{(n)} - \tfrac{1}{2}h_n, s_l^{(n)} + \tfrac{1}{2}h_n \rangle\,.$$

5. For every positive integer n we define

$$(15) \qquad \psi_n = \sum_{l=1}^{p_{n-1}} \omega_l^{(n-1)} \varphi_{\Delta_l^{(n-1)}; k_n, q_n}\,,$$

$$(16) \qquad \chi_n = \sum_{m=1}^{n} \psi_m\,,$$

where $\varphi \ldots$ are the functions described in Section 2 restricted to the interval $\langle 0, 1 \rangle$ and where each $\omega_l^{(n-1)}$ is either 1 or -1. The choice of these signs is fully arbitrary and we shall not restrict it anywhere.

Since the reduced intervals of the same order are disjoint, it follows from (15) that the function ψ_n takes the constant value v_n or $-v_n$ on each reduced interval of order n and that

$$(17) \qquad \max_{0 \leqq x \leqq 1} |\psi_n(x)| = v_n\,.$$

Furthermore, it follows from (16) that the function χ_n is also constant on each reduced interval of order n with

$$(18) \qquad \chi_{n+1}(s_{j_1,\ldots,j_{n+1}}) = \chi_n(s_{j_1,\ldots,j_n}) \pm v_n\,,$$

if $1 \leqq j_1 \leqq k_1, \ldots, 1 \leqq j_{n+1} \leqq k_{n+1}$, and that

$$(19) \qquad \max_{0 \leqq x \leqq 1} |\chi_n(x)| \leqq \sum_{m=1}^{n} v_m\,.$$

6. Further we put

$$(20) \qquad L_l^{(n)} = \tilde{\Delta}_l^{(n)} \times \langle \chi_n(s_l^{(n)}) - \sigma_n, \chi_n(s_l^{(n)}) + \sigma_n \rangle\,.$$

If we denote by $\Gamma_l^{(n)}$ the boundaries of these rectangles (with negative orientation if needed), then we have

(21) $$\mathscr{H}(\Gamma_l^{(n)}) = 4(v_n + \sigma_n) = 4(1 + q_n)\, v_n = 2h_n(1 - q_n)^{-1}\,(1 + q_n)\,.$$

Similarly to Section 4 we shall also use the notation $L_{j_1,\dots,j_n}$ and $\Gamma_{j_1,\dots,j_n}$.

If $|l' - l''| \geqq 2$, then obviously $\operatorname{dist}(L_{l'}^{(n)}, L_{l''}^{(n)}) \geqq 2v_n = (1 - q_n)^{-1}\, h_n$. Let us consider two rectangles $L_l^{(n)}, L_{l+1}^{(n)}$ the projections $\tilde{\Delta}_l^{(n)}, \tilde{\Delta}_{l+1}^{(n)}$ of which are subsets of the same interval $\Delta_{l_*}^{(n-1)}$. Then $|\chi_n(s_l^{(n)}) - \chi_n(s_{l+1}^{(n)})| = 2v_n$ by the definition of the function $\varphi_{\Delta_{l_*}^{(n-1)};k_n,q_n}$ and hence $\operatorname{dist}(L_l^{(n)}, L_{l+1}^{(n)}) = 2(v_n - \sigma_n) = 2(1 - q_n)\, v_n = h_n$. This means that

(22) $$\operatorname{dist}(L_{j_1,\dots,j_{n-1},j_n'}\,,\ L_{j_1,\dots,j_{n-1},j_n''}) \geqq h_n$$

if $1 \leqq j_1 \leqq k_1, \dots, 1 \leqq j_{n-1} \leqq k_{n-1}, 1 \leqq j_n' \leqq k_n, 1 \leqq j_n'' \leqq k_n, j_n' \neq j_n''$.

7. We desire to find a condition for the validity of the inclusion

(23) $$L_{j_1,\dots,j_n,j_{n+1}} \subset \operatorname{int}(L_{j_1,\dots,j_n})\,.$$

In virtue of the inclusion $\tilde{\Delta}_{j_1,\dots,j_n,j_{n+1}} \subset \Delta_{j_1,\dots,j_n}$ it suffices by (18) when

(24) $$v_{n+1} + \sigma_{n+1} < \sigma_n\,.$$

Since $v_{n+1} + \sigma_{n+1} = (2k_{n+1})^{-1}\,(1 + q_{n+1})\, h_n$ and $\sigma_n = \frac{1}{2}q_n(1 - q_n)^{-1}\, h_n$, the condition (24) is equivalent to $k_{n+1}^{-1}(1 + q_{n+1}) < (1 - q_n)^{-1}\, q_n$ or, which is the same, to the condition

(P1) $$k_{n+1} > (1 + q_{n+1})\,(q_n^{-1} - 1)\,.$$

The choice of k_1 is not restricted by this. But if we choose $k_1 \geqq 2$, then $v_1 + \sigma_1 = (2k_1)^{-1}\,(1 + q_1) < \frac{1}{2}$ and any rectangle $L_l^{(1)}$ is contained in the rectangle

(25) $$L_1^{(0)} = \langle 0, 1\rangle \times \langle -\tfrac{1}{2}, \tfrac{1}{2}\rangle\,.$$

In what follows we shall always suppose that the numbers k_n, q_n are chosen so that $k_1 \geqq 2$ and that the condition (P1) is fulfilled for $n = 1, 2, \dots$. We have then (23) for every sequence $\{j_n\} \in \mathscr{W}$ and all rectangles $L_l^{(n)}$ are contained in the rectangle $L_1^{(0)}$. From (22) we then obtain easily by induction that any two different rectangles of order n satisfy

(26) $$\operatorname{dist}(L_{l'}^{(n)}, L_{l''}^{(n)}) \geqq h_n\,(l' \neq l'')$$

(even $\operatorname{dist}(L_{l'}^{(n)}, L_{l''}^{(n)}) > h_{n-1}$, if both rectangles lie in different rectangles of order $n - 1$).

8. The assumptions made about the choice of the numbers k_n, q_n guarantee uniform convergence of the functions χ_n. Indeed, from (24) we obtain by induction $v_{n+2} + \dots + v_{n+p} + \sigma_{n+p} < \sigma_{n+1}$ for any integer $p \geqq 2$ and hence

$$(27) \qquad \sum_{p=n+1}^{\infty} v_p \leqq v_{n+1} + \sigma_{n+1} < \sigma_n .$$

According to (17) this means exactly that the series $\sum_{n=1}^{\infty} \psi_n(x)$ is uniformly convergent on $\langle 0, 1\rangle$ and we put

$$(28) \qquad \chi(x) = \lim_{n\to\infty} \chi_n(x) = \sum_{n=1}^{\infty} \psi_n(x) \quad \text{for} \quad x \in \langle 0, 1\rangle .$$

We have thus defined a continuous function the graph of which we denote by G, i.e.

$$(29) \qquad G = \{x + \mathrm{i}\chi(x) \in \mathbb{C} : x \in \langle 0, 1\rangle\} .$$

According to (27), $|\chi(x) - \chi_n(x)| \leqq \sum_{p=n+1}^{\infty} v_p < \sigma_n$ and hence $(\Delta_l^{(n)} \times \mathbb{R}) \cap G \subset \subset \operatorname{int} L_l^{(n)}$.

9. The rectrictions of the function χ to the intervals $\Delta_l^{(n)}$ are to a certain degree analogous to the function χ itself. Let us introduce some transformations of the complex plane to describe this analogy which is important in the sequel.

Given positive integers $n, l \leqq p_n$ we put

$$(30) \qquad T_l^{(n)}(w) = h_n w + (s_l^{(n)} - \tfrac{1}{2}h_n) + \mathrm{i}\chi(s_l^{(n)} - \tfrac{1}{2}h_n)$$

for any $w \in \mathbb{C}$, so that $T_l^{(n)}$ is a composition of a homothety and a translation. The transformation $T_l^{(n)}$ maps the interval $\langle 0, 1\rangle$ of the real line on the line segment $\Delta_l^{(n)} \times \chi_n(\Delta_l^{(n)})$ and the inverse transfotmation maps the graph of the function χ on the graph of the function $\chi_l^{(n)}$, which is defined by

$$(31) \qquad \chi_l^{(n)}(t) = h_n^{-1}(\chi(s_l^{(n)} + h_n(t - \tfrac{1}{2})) - \chi(s_l^{(n)} - \tfrac{1}{2}h_n)) .$$

The restriction of the function $\chi_l^{(n)}$ to the interval $\langle 0, 1\rangle$ results by an analogous construction as the function χ, if we use the chosen sequences of numbers k_n, q_n beginning from the $(n + 1)$-th member. It is namely

$$(32) \qquad \chi_l^{(n)}(t) = h_n^{-1} \sum_{p=n+1}^{\infty} \psi_p(s_l^{(n)} + h_n(t - \tfrac{1}{2})) .$$

The inequality (27) implies then the estimate

$$(33) \qquad \max_{0\leqq t\leqq 1} |\chi_l^{(n)}(t)| \leqq h_n^{-1}(v_{n+1} + \sigma_{n+1}) < k_{n+1}^{-1}.$$

10. Definition of compacts H, K. We put

$$(34) \qquad H = \bigcap_{n=0}^{\infty} \bigcup_{l=1}^{p_n} \Delta_l^{(n)} = \bigcup_{\{j_n\}\in\mathscr{W}} \bigcap_{n=1}^{\infty} \Delta_{j_1,\ldots,j_n},$$

$$(35) \qquad K = \bigcap_{n=0}^{\infty} \bigcup_{l=1}^{p_n} L_l^{(n)} = \bigcup_{\{j_n\}\in\mathscr{W}} \bigcap_{n=1}^{\infty} L_{j_1,\ldots,j_n},$$

and further

(36) $$H_l^{(n)} = H \cap \Delta_l^{(n)}, \quad K_l^{(n)} = K \cap L_l^{(n)}$$

for $n = 0, 1, 2, \ldots; \ l = 1, 2, \ldots, p_n$.

We will prove that

(37) $$K = G \cap (H \times \mathbb{R});$$

this means that K is the graph of the restriction of the function χ to the set H. If $x \in H$, then there exists a sequence $\{j_n'\} \in \mathscr{W}$ such that $x \in \bigcap_{n=1}^{\infty} \Delta_{j_1',\ldots,j_n}$ (even $\{x\} = \bigcap_{n=1}^{\infty} \Delta_{j_1',\ldots,j_n'}$). According to Section 8 it follows that $x + i\chi(x) \in \bigcap_{n=1}^{\infty} L_{j_1',\ldots,j_n'} \subset K$. This means that $G \cap (H \times \mathbb{R}) \subset K$. On the other hand, if $x + iy \in K$, then there exists a sequence $\{j_n''\} \in \mathscr{W}$ such that $x + iy \in \bigcap_{n=1}^{\infty} L_{j_1'',\ldots,j_n''}$. Hence $x \in \bigcap_{n=1}^{\infty} \tilde{\Delta}_{j_1'',\ldots,j_n''}$. Since $\tilde{\Delta}_{j_1'',\ldots,j_{n+1}''} \subset \Delta_{j_1'',\ldots,j_n''}$, it follows that $x \in H$. Further, it holds $|y - \chi_n(x)| = |y - \chi_n(s_{j_1'',\ldots,j_n''})| \leqq \sigma_n$, which implies $y = \lim_{n\to\infty} \chi_n(x) = \chi(x)$. This means that $x + iy \in G$. Hence $K \subset G \cap (H \times \mathbb{R})$.

It can be shown similarly that

(38) $$K_l^{(n)} = G \cap (H_l^{(n)} \times \mathbb{R})$$

for any $n, l \leqq p_n$.

III. PROPERTIES OF THE OBJECTS CONSTRUCTED

1. Linear measures of H and K. Since the reduced intervals of order n are disjoint, it holds

$$\mathscr{H}\Big(\bigcup_{l=1}^{p_n} \Delta_l^{(n)}\Big) = p_n h_n = \prod_{m=1}^{n} (1 - q_m),$$

and hence

(39) $$\mathscr{H}(H) = \prod_{m=1}^{\infty} (1 - q_m).$$

Now we shall compute the linear measure of K. Let $\varepsilon > 0$. Since $\operatorname{diam} L_l^{(n)} = 2(v_n^2 + \sigma_n^2)^{1/2} = 2v_n(1 + q_n^2)^{1/2} = k_n^{-1} h_{n-1}(1 + q_n^2)^{1/2}$,

(40) $$\mathscr{H}_\varepsilon(K) \leqq (1 + q_n^2)^{1/2} \prod_{m=1}^{n-1} (1 - q_m)$$

holds for any n large enough. Hence $\prod_{m=1}^{\infty}(1 - q_m) = 0$ implies $\mathscr{H}_\varepsilon(K) = 0$, because $q_n \in (0, 1)$. If $\prod_{m=1}^{\infty}(1 - q_m) > 0$, then $\lim_{n\to\infty} q_n = 0$ and (40) implies $\mathscr{H}_\varepsilon(K) \leqq \leqq \prod_{m=1}^{\infty}(1 - q_m)$. In each case by making $\varepsilon \to 0+$ we obtain $\mathscr{H}(K) = \prod_{m=1}^{\infty}(1 - q_m)$. According to (39) we have thus $\mathscr{H}(K) \leqq \mathscr{H}(H)$. The opposite inequality is a consequence of (37), since the linear measure does not increase by the orthogonal projection. Combining these facts we get

$$\mathscr{H}(K) = \mathscr{H}(H) = \prod_{u=1}^{\infty}(1 - q_m)\,. \tag{41}$$

2. Decomposition of a function holomorphic on $\mathbb{C} \setminus K$. Given a complex function f holomorphic on $\mathbb{C} \setminus K$ such that $\lim_{z\to\infty} f(z) = 0$ we define functions $f_l^{(n)}$ $n = 0, 1, \ldots;$ $l = 1, 2, \ldots, p_n$, holomorphic on $\mathbb{C} \setminus K_l^{(n)}$ in the following way: If $z \in \mathbb{C} \setminus L_l^{(n)}$, we put

$$f_l^{(n)}(z) = (2\pi\mathrm{i})^{-1} \int_{\Gamma_l{}^{(n)}} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta\,, \tag{42}$$

and then we continue this function holomorphically to $\mathbb{C} \setminus K_l^{(n)}$. The functions $f_l^{(n)}$ vanish at ∞ and

$$f(z) = \sum_{l=1}^{p_n} f_l^{(n)}(z) \tag{43}$$

on $\mathbb{C} \setminus K$.

3. Uniform boundedness of $f_l^{(n)}$. We shall assume that the function f established in the previous section is bounded and we set $\|f\| = \sup_{z\in\mathbb{C}\setminus K} |f(z)|$. If $\prod_{m=1}^{\infty}(1 - q_m) = 0$, then K has zero linear measure and f vanishes everywhere by the result of Painlevé cited in Introduction. Thus we shall assume that $\prod_{m=1}^{\infty}(1 - q_m) > 0$. Then $\lim_{n\to\infty} q_n = 0$ and there exists $q^* = \max_{n=1,2,\ldots} q_n < 1$.

Let us fix positive integers n and $l \leqq p_n$. If $l' \neq l$ and $z \in L_{l'}^{(n)}$, then (21), (26) and (12) imply the estimates

$$\begin{aligned} |f_l^{(n)}(z)| &\leqq (2\pi)^{-1} \|f\| \,(\operatorname{dist}(L_{l'}^{(n)}, L_l^{(n)}))^{-1} \mathscr{H}(\Gamma_l^{(n)}) \leqq \\ &\leqq (2\pi)^{-1} \|f\|\, h_n^{-1}(1 - q_n)^{-1}\, 2h_n(1 + q_n) \leqq 2\pi^{-1}\|f\|\,(1 - q^*)^{-1}\,. \end{aligned} \tag{44}$$

Let us set further

$$f_l^{(n)*}(z) = f(z) - f_l^{(n)}(z) \tag{45}$$

for $z \in \mathbb{C} \setminus K$ and let us continue this function holomorphically to $(\mathbb{C} \setminus K) \cup K_l^{(n)}$. On $\Gamma_l^{(n)*} = \bigcap_{\substack{l'=1 \\ l' \neq l}}^{p_n} \Gamma_{l'}^{(n)}$ we have by (44)

$$|f_l^{(n)*}(z)| \leqq \|f\| \left(1 + 2\pi^{-1}(1 - q^*)^{-1}\right). \tag{46}$$

Since $\lim_{z \to \infty} f_l^{(n)*}(z) = 0$, the same estimate holds on $\mathbb{C} \setminus L_l^{(n)*}$ with $L_l^{(n)*} = \bigcup_{\substack{l'=1 \\ l' \neq l}}^{p_n} L_{l'}^{(n)}$.

This implies

$$|f_l^{(n)}(z)| \leqq 2\|f\| \left(1 + \pi^{-1}(1 - q^*)^{-1}\right) \tag{47}$$

on $\mathbb{C} \setminus L_l^{(n)*} \setminus K_l^{(n)}$, since $f_l^{(n)}(z) = f(z) - f_l^{(n)*}(z)$ on this set. On $L_l^{(n)*}$ we have the estimate (44), hence (47) holds on the whole set $\mathbb{C} \setminus K_1^{(n)}$.

4. New integral representation of $f_l^{(n)}$. Retaining our assumptions and notation we put

$$\lambda(H_l^{(n)}) = (2\pi \mathrm{i})^{-1} \int_{\Gamma_l^{(n)}} f(\zeta)\, \mathrm{d}\zeta \tag{48}$$

for every admissible n, l. Let us show that we can extend the set function λ to a complex measure with support in H.

For $x \in \langle 0, 1 \rangle \setminus H$ let n_x be the smallest n such that $x \notin \bigcup_{l=1}^{p_n} \Delta_l^{(n)}$. Let us further denote by $J_x^{(n)}$ the set of all l's such that the interval $\Delta_l^{(n)}$ precedes x. Since

$$\lambda(H_{j_1,\ldots,j_n}) = \sum_{j=1}^{k_{n+1}} \lambda(H_{j_1,\ldots,j_n,j}) \tag{49}$$

the value of $\sum_{l \in J_x^{(n)}} \lambda(H_l^{(n)})$ is independent of n for every $x \in \langle 0, 1 \rangle \setminus H$ and $n \geqq n_x$.

We can thus define a function P on $\langle 0, 1 \rangle \setminus H$ putting

$$P(x) = \sum_{l \in J_x^{(n)}} \lambda(H_l^{(n)}), \tag{50}$$

where we choose $n \geqq n_x$.

Let $x, y \in \langle 0, 1 \rangle \setminus H$, $x < y$. If we choose $n \geqq \max(n_x, n_y)$, then

$$P(y) - P(x) = \sum_{l \in J_{x,y}^{(n)}} \lambda(H_l^{(n)}) \quad \text{with} \quad J_{x,y}^{(n)} = J_y^{(n)} \setminus J_x^{(n)},$$

and hence

$$|P(y) - P(x)| \leqq \sum_{l \in J_{x,y}^{(n)}} |\lambda(H_l^{(n)})| \leqq (2\pi)^{-1} \|f\| \sum_{l \in J_{x,y}^{(n)}} \mathscr{H}(\Gamma_l^{(n)}).$$

But $\mathscr{H}(\Gamma_l^{(n)}) = 2(1 - q_n)^{-1}(1 + q_n)\,\mathscr{H}(\Delta_l^{(n)})$ by (21) and, consequently,

$$|P(y) - P(x)| \leqq \pi^{-1}\|f\|(1 - q_n)^{-1}(1 + q_n) \sum_{l \in J_{x,y}^{(n)}} \mathscr{H}(\Delta_l^{(n)}) \leqq$$

$$\leqq 2\pi^{-1}\|f\|(1 - q^*)^{-1}(y - x).$$

We can thus extend the function P to $\langle 0, 1 \rangle$ in such a way that

$$|P(y) - P(x)| \leqq 2\pi^{-1}\|f\|(1 - q^*)^{-1}|y - x| \tag{51}$$

holds for every $x, y \in \langle 0, 1 \rangle$.

The function P is then the distribution function of a complex measure λ, which is absolutely continuous with respect to the Lebesgue measure and the density of which is, moreover, bounded. This means that there exists a bounded complex function p on the interval $\langle 0, 1 \rangle$ such that

$$P(x) = \int_0^x p(t)\,\mathrm{d}t \tag{52}$$

for every $x \in \langle 0, 1 \rangle$. Since P is constant on each interval contiguous to the set H, we can suppose that $p(x) = 0$ for every $x \in \mathbb{R} \setminus H$. Then

$$\lambda(H_l^{(n)}) = \int_{\Delta_l^{(n)}} p(x)\,\mathrm{d}x = (2\pi\mathrm{i})^{-1} \int_{\Gamma_l^{(n)}} f(\zeta)\,\mathrm{d}\zeta \tag{53}$$

for every admissible n, l.

For fixed n, l and $z \in \mathbb{C} \setminus L_l^{(n)}$ let us examine the sums $\sum \lambda(H_{l'}^{(m)})(s_{l'}^{(m)} + i\chi(s_{l'}^{(m)}) - z)^{-1}$, where m is large enough and l' are indexes such that $H_{l'}^{(m)} \subset \Delta_l^{(n)}$. We find out that

$$f_l^{(n)}(z) = \int_{\Delta_l^{(n)}} \frac{p(x)}{x + \mathrm{i}\chi(x) - z}\,\mathrm{d}x\,. \tag{54}$$

5. Removability of singularities on K. Now let us prove that K is a set of removable singularities for bounded holomorphic functions. If K has zero linear measure, then this is true by a result of Painlevé as we have mentioned above. If K has a positive linear measure, then it suffices to show that the bounded function f examined in the previous sections vanishes everywhere. Since we have

$$f(z) = \int_0^1 \frac{p(x)}{x + \mathrm{i}\chi(x) - z}\,\mathrm{d}x \tag{55}$$

as a special case of (54), it suffices to show that the function p vanishes almost everywhere on $\mathbb{R}$.

For n, l fixed let us change the variable in the integral (54) setting

$$x = S_l^{(n)}(t) = h_n t + s_l^{(n)} - \tfrac{1}{2}h_n\,. \tag{56}$$

The function $S_l^{(n)}$ is the real part of the function $T_l^{(n)}$ from (30) and it maps the interval $\langle 0, 1\rangle$ onto $\Delta_l^{(n)}$. Every function $f_l^{(n)}$ will thus be expressed by an integral over the interval $\langle 0, 1\rangle$. Using the substitution (56) we get

$$f_l^{(n)}(z) = \int_0^1 \frac{p(S_l^{(n)}(t))}{t + \mathrm{i}\,\chi_l^{(n)}(t) - w}\,\mathrm{d}t = F_l^{(n)}(w)\,, \tag{57}$$

where $\chi_l^{(n)}$ is defined by (31) and $z = T_l^{(n)}(w)$. From Section 3 it follows that the functions $F_l^{(n)}$, $n = 0, 1, \ldots$; $l = 1, 2, \ldots, p_n$ are uniformly bounded on $\mathbb{C} \setminus L_1^{(0)}$.

Let $x_0 \in H$ be a Lebesgue point of the function p. For every positive integer n we denote by l_n the index for which $x_0 \in \Delta_{l_n}^{(n)}$. Then

$$\left| F_{l_n}^{(n)}(w) - p(x_0)\int_0^1 \frac{\mathrm{d}t}{t - w}\right| \leqq$$

$$\leqq (\operatorname{dist}(w, L_1^{(0)}))^{-1} \int_0^1 |pS_{l_n}^{(n)}(t) - p(x_0)|\,\mathrm{d}t +$$

$$+ |p(x_0)| \left| \int_0^1 \left(\frac{1}{t + \mathrm{i}\chi_{l_n}^{(n)}(t) - w} - \frac{1}{t - w}\right)\mathrm{d}t\right| \leqq$$

$$\leqq (\operatorname{dist}(w, L_1^{(0)}))^{-1}\, h_n^{-1} \int_{\Delta_{l_n}(n)} |p(x) - p(x_0)|\,\mathrm{d}x +$$

$$+ |p(x_0)|\,(\operatorname{dist}(w, L_1^{(0)}))^{-2} \max_{0 \leqq t \leqq 1} |\chi_{l_n}^{(n)}|t)$$

for every $w \in \mathbb{C} \setminus L_1^{(0)}$. According to (33) we have thus

$$\left| F_{l_n}^{(n)}(w) - p(x_0)\int_0^1 \frac{\mathrm{d}t}{t - w}\right| \leqq \tag{58}$$

$$\leqq (\operatorname{dist}(w, L_1^{(0)}))^{-1}\, h_n^{-1} \int_{\Delta_{l_n}(n)} |p(x) - p(x_0)|\,\mathrm{d}x +$$

$$+ k_{n+1}^{-1} |p(x_0)|\,(\operatorname{dist}(w, L_1^{(0)}))^{-2}\,.$$

Since $\lim_{n\to\infty} q_n = 0$, (P1) implies $\lim_{n\to\infty} k_n = \infty$. Using further the fact that x_0 is a Lebesgue point of p we infer from (58) that

$$\lim_{n\to\infty} F_{l_n}^{(n)}(w) = p(x_0)\int_0^1 \frac{\mathrm{d}t}{t - w}\,. \tag{59}$$

Since the functions $F_l^{(n)}$ are uniformly bounded on $\mathbb{C} \setminus L_1^{(0)}$, the function $p(x_0)\,.$ $.\int_0^1 (t - w)^{-1}\,\mathrm{d}t$ must be bounded there, too. But $\int_0^1 (t - w)^{-1}\,\mathrm{d}t = \ln(1 - w^{-1})$

for every $w < 0$ and this function is not bounded on $(-\infty, 0)$. This means that $p(x_0) = 0$.

Since p vanishes on $\mathbb{R} \setminus H$ and $p(x) = 0$ at every Lebesgue point $x \in H$, p vanishes almost everywhere, q.e.d.

6. Modulus of continuity of χ. Finally, let us prove the following assertion:

Let ω be a positive function nondecreasing on $(0, 1\rangle$ such that

$$\lim_{\delta \to 0+} \frac{\delta}{\omega(\delta)} = 0 \,, \quad \omega(0) = \omega(0+) \,. \tag{60}$$

Then for every choice of numbers $q_n \in (0, 1)$ there exist positive integers k_n satisfying (P1) such that

$$|\chi(x) - \chi(y)| \leqq \omega(|x - y|) \tag{61}$$

for the corresponding function χ.

For this purpose let us choose the numbers $c_n > 0$, $n = 1, 2, \ldots$ such that $\sum_{n=1}^{\infty} c_n < 1$. According to (60) we may then fix numbers $\delta_n > 0$ such that

$$\delta \leqq q_n c_n \, \omega(\delta) \tag{62}$$

if $0 < \delta \leqq \delta_n$. Finally, we choose by induction integers k_n such that (P1) holds and $2 \, \sigma_n \leqq \delta_n$, that is

$$k_n \geqq \delta_n^{-1} q_n h_{n-1} \,. \tag{P2}$$

Then $|\psi_n(x) - \psi_n(y)| \leqq q_n^{-1}|x - y| \leqq c_n \, \omega(|x - y|)$ if $|x - y| < 2\sigma_n$, and $|\psi_n(x) - \psi_n(y)| \leqq 2v_n = q_n^{-1} 2\sigma_n \leqq c_n \, \omega(2\sigma_n) \leqq c_n \, \omega(|x - y|)$ if $|x - y| \geqq 2\sigma_n$. This means that

$$|\psi_n(x) - \psi_n(y)| \leqq c_n \, \omega(|x - y|) \tag{63}$$

for every $x, y \in \langle 0, 1\rangle$, $n = 1, 2, \ldots$. By adding up we get

$$|\chi(x) - \chi(y)| \leqq \omega(|x - y|) \tag{64}$$

for every $x, y \in \langle 0, 1\rangle$.

7. Notes. a) Choosing $\omega(\delta) = \delta(\ln(\delta^{-1}) + 1)$ in the previous section, we obtain a function the modulus of continuity of which is of order $o(\delta^\alpha)$ for every $\alpha \in (0, 1)$.

b) An essential role is played by the set K. The function χ can be modified in such a way that it may be affine on the intervals contiguous to the set H. Its modulus of continuity will not "become worse" by this modification.