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ON FORMS AND CONNECTIONS ON FIBRE BUNDLES

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Let $\pi : E \to M$ be a fibre bundle. Let J^1E be the first prolongation of E, i.e. J^1E is the set of 1-jets of all local cross-sections of E. Let us recall (see for example [1], [4]) that a connection on E is a global cross-section $\Gamma : E \to J^1E$, that is a distribution of horizontal tangent subspaces Γ_u, where $T_uE = T_uE_x \oplus \Gamma_u$, $u \in E$, $\pi u = x$. In this paper we find some relations between forms and connections on E. Our considerations are in the category C^∞.

1. Let M be a differentiable manifold. Let $L(M)$ or $\Lambda(M)$ or $S(M)$ be the algebra of all forms or of all antisymmetric or of all symmetric forms, respectively, on M. Let $\psi : TM \to TM$ or $\varphi : \bigwedge^{r+1}TM \to TM$ be a vector bundle morphism or an antisymmetric vector bundle morphism, respectively. Let ω or ε be a form or an antisymmetric form, respectively, of degree p on M. Let f be a function on M. Put

$$D_\psi f = 0\,, \quad d_\varphi f = 0\,,$$

$$(D_\psi \omega)(X_1, \ldots, X_p) = \sum_{i=1}^{p} \omega(X_1, \ldots, \psi X_i, \ldots, X_p)\,,$$

$$(d_\varphi \varepsilon)(X_1, \ldots, X_{r+p}) = \sum_{\sigma \in S} \operatorname{sgn} \sigma \varepsilon[\varphi(X_{\sigma 1}, \ldots, X_{\sigma(r+1)}), \ldots, X_{\sigma(r+p)}]$$

where S is the set of all such permutations of the set $\{1, \ldots, r + p\}$ that $\sigma 1 < \ldots \ldots < \sigma(r + 1)$; $\sigma(r + 2) < \ldots < \sigma(r + p)$.

Let us recall the following properties.

Lemma 1. *The mapping $D_\psi : \omega \to D_\psi \omega$ is a differentiation of degree* 0 *on algebras* $L(M)$, $\Lambda(M)$, $S(M)$.

Lemma 2. *The mapping $d_\varphi : \omega \to d_\varphi \omega$ is a differentiation of degree r on $\Lambda(M)$, that is*

$$d_\varphi(\omega_1 \wedge \omega_2) = d_\varphi \omega_1 \wedge \omega_2 + (-1)^{pr} \omega_1 \wedge d_\varphi \omega_2\,,$$

where ω_1 is a p-form on M; i.e., if r is even or uneven, then d_φ is a differentiation or antidifferentiation of degree r on $\Lambda(M)$.

For $\varepsilon \in \Lambda(M)$ $d_\psi^{\gamma}\varepsilon = D_\psi \varepsilon$.

2. Let $\pi : E \to M$ be a fibre bundle. Let (x^i, y^α) or $(x^i, y^\alpha, y_i^\alpha)$, $i = 1, \dots, \dim M$, $\alpha = 1, \dots, \dim E_x$, be a local chart on E or on J^1E, respectively. Let a connection $\Gamma : E \to J^1E$ be locally given by $(x^i, y^\alpha) \to (x^i, y^\alpha, y_i^\alpha = a_i(x, y))$. Denote by Γ_u the horizontal tangent subspace determined by $\Gamma(u)$, $u \in E$. Then $T_uE = \Gamma_u \oplus T_uE_x$, $x = \pi u$. There are two canonical projections $v : T_uE \to T_uE_x$, $h : T_uE \to \Gamma_u$ and we have two canonical vector bundle morphisms $h : TE \to TE$ and $v : TE \to VTE$, where VTE denotes the fibre bundle of all vertical tangentvectors on E. Let ω be a form on E. Denote by $h^*\omega$ and $v^*\omega$ the forms ωh and ωv, respectively.

Proposition 1. *Let ω be a form of degree p on E. Then*

(1)
$$D_h\omega + D_v\omega = p\omega\,,$$
$$v^*D_v\omega = p(v^*\omega) = D_v(v^*\omega)\,,$$
$$h^*D_h\omega = p(h^*\omega) = D_h(h^*\omega)\,.$$

Proof.

$$\omega(X_1, \dots, X_p) = \omega(hX_1 + vX_1, X_2, \dots, X_p) = \omega(hX_1, \dots, X_p) + \omega(xX_1, \dots, X_p)$$

....

$$\omega(X_1, \dots, X_p) = \omega(X_1, \dots, X_{p-1}, hX_p + vX_p) = \omega(X_1, \dots, hX_p) + \omega(X_1, \dots, vX_p)\,.$$

By summation we get $D_h\omega + D_v\omega = p\omega$. Then $v^*D_v\omega = p(v^*\omega)$, $h^*D_h\omega = p(h^*\omega)$ and by the definitions of D_v, D_h we get $D_v(v^*\omega) = p(v^*\omega)$, $D_h(h^*\omega) = p(h^*\omega)$.

Since $v \,.\, h = h \,.\, v = 0$, the definitions of D_v and D_h immediately yield

Proposition 2. *The composition of D_v and D_h is commutative, i.e. $D_v \,.\, D_h = D_h \,.\, D_v$.*

A form ω of order p on E will be said to be Γ-vertical or total Γ-vertical, if $h^*\omega = 0$ or if $\omega(X_1, \dots, X_p) = 0$ when at least one vector of the set $\{X_1, \dots, X_p\}$ is horizontal. This implies

Proposition 3. *The form $v^*\omega$ or $D_v\omega$ is total Γ-vertical or Γ-vertical, respectively.*

Proposition 4. *If a form ω is total Γ-vertical then $D_h\omega = 0$ and $D_v\omega = p\omega$.*

It is easy to see that $\omega - h^*\omega$ is Γ-vertical.

Let us recall (see [2]) that a form ω is semi-basic if $\omega(X_1, \dots, X_p) = 0$ when $\exists i \in \{1, \dots, p\} : X_i \in VTE$. Therefore an antisymmetric p-form is semi-basic if and only if $i_Y\omega = 0$ for any vertical tangent vector Y, where $i_Y\omega$ denotes the contraction of ω by Y. Locally, a form ω is semi-basic if

$$\omega = a_{i_1 \dots i_p} dx^{i_1} \otimes \dots \otimes dx^{i_p}\,.$$

If ω is semi-basic then $D_v\omega = 0$ and $D_h\omega = p\omega$.

An antisymmetric p-form on E will be said to be quasi-semi-basic if $i_Y\omega$ is semi-basic for any $Y \in VTE$. Locally, ω is quasi-semi-basic if and only if

$$(3) \qquad \omega = a_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p} + a_{i_1 \ldots i_{p-1}\alpha} dx^{i_1} \wedge \ldots \wedge dx^{i_{p-1}} \wedge dy^\alpha .$$

By the definition of D_h, D_v we have $D_v(dx^i) = 0$, $D_v(dy^\alpha) = dy^\alpha - a_i^\alpha dx^i$, $D_h(dx^i) = dx^i$, $D_h(dy^\alpha) = a_i^\alpha dx^i$. This gives

Proposition 5. *If ω is quasi-semi-basic but not semi-basic then $D_v\omega$ and $D_h\omega$ are quasi-semi-basic but not semi-basic.*

Recall (see for example [1], [4]) that the curvature form of Γ is an antisymmetric 2-morphism

$$\Phi : TE \otimes TE \to TE$$
$$\Phi(X_u, Y_u) = v([hX, hY]) ,$$

where $[hX, hY]$ is the Lie bracket of such fields X, Y on E that $X_u \in X$, $Y_u \in Y$, $u \in E$. Locally

$$(4) \qquad \Phi = \frac{1}{2}\left[\left(\frac{\partial a_k^\alpha}{\partial y^\beta} a_j^\beta - \frac{\partial a_j^\alpha}{\partial y^\beta} a_k^\beta + \frac{\partial a_k^\alpha}{\partial x^j} - \frac{\partial a_j^\alpha}{\partial x^k}\right) dx^j \wedge dx^k\right] \otimes \frac{\partial}{\partial y^\alpha} =$$
$$= \tfrac{1}{2} A_{jk}^\alpha dx^j \wedge dx^k \otimes \frac{\partial}{\partial y^\alpha} .$$

The mapping d_Φ is an antidifferentiation of the first degree and

$$(5) \qquad d_\Phi(dx^i) = 0 , \quad d_\Phi(dy^\alpha) = \tfrac{1}{2} A_{jk}^\alpha dx^j \wedge dx^k .$$

Proposition 6. *Let Φ be the curvature form of the connection Γ. Then* $d_\Phi d_\Phi = 0$.

Proof. The mapping d_Φ being an antidifferentiation of $\Lambda(E)$ with the property $d_\Phi f = 0$ for any function f on E, it is determined by its action on $\Lambda^1(E)$. Using (5) we get our assertion.

Denote $H_u = \{\Phi(X, Y) : X, \ Y \in T_uE\}$.

Proposition 7. *Let ω be a $(p - 1)$-form on E. Let $i_Y\omega = 0$ for any vector tangent field, the value of which lie in the spaces H_u. Then $d_\Phi\omega = 0$.*

Proof. $d_\Phi\omega(X_1, \ldots, X_{p+1}) = \sum_{\sigma \in S} \operatorname{sgn} \sigma \omega(\Phi(X_{s_1}, X_{s_2}), X_{s_3}, \ldots, X_{s_{p+1}}) = \sum_{\sigma \in S} \operatorname{sgn} \sigma i_{\Phi(X_{s_1}, X_{s_2})} \omega(X_{s_3}, \ldots, X_{s_{p+1}})$. This completes our proof.

Quite analogously, if $i_Y\omega = 0$ for any horizontal tangent vector Y then $\omega \in \operatorname{Ker} D_h$.

Let d denote the exterior differentiation on $\Lambda(E)$. Then $\bar{d} = D_v d - d D_v$ is an antidifferentiation of degree 1 on $\Lambda(E)$. By Proposition 3 we get

$$(6) \qquad h^* \bar{d} = - h^* d D_v .$$

Proposition 8. *Let ω be a p-form on E. Then*

$$h^*(\bar{d}\omega) = -h^* d_\Phi \omega .$$

Proof. $h^* dD_v\omega(X_1, \ldots, X_{p+1}) = dD_v\omega(hX_1, \ldots, hX_{p+1}) =$

$$= -\sum_{i<j} (-1)^{i+j} D_v\omega([hX_i, hX_j], hX_1, \ldots, \widehat{hX_i}, \ldots, \widehat{hX_j}, \ldots, hX_{p+1}) =$$

$$= -\sum_{i<j} (-1)^{i+j} \omega(v[hX_i, hX_j], hX_1, \ldots, \widehat{hX_i}, \ldots, \widehat{hX_j}, \ldots, hX_{p+1}) =$$

$$= \sum_{i<j} (-1)^{i-1+j-2} \omega(\Phi(hX_i, hX_j), hX_1, \ldots, \widehat{hX_i}, \ldots, \widehat{hX_j}, \ldots, hX_{p+1}) =$$

$= d_\Phi\omega(hX_1, \ldots, hX_{p+1}) = h^* d_\Phi\omega(X_1, \ldots, X_{p+1})$, where the symbol $\wedge$ indicates that a vector $\hat{X}$ is dropped. The relation (6) completes our proof.

Proposition 9. *If the form $D_v\omega$ is closed then $\bar{d}\omega$ is Γ-vertical. If the form ω is closed then $D_v\omega$ is closed if and only if $\bar{d}\omega = 0$.*

Proof follows from the definition of $\bar{d}$.

3. In the sequel we are going to study in detail some relations between bilinear forms and connections on E. Let $\omega = a_{ij} dx^i \otimes dx^j + a_{\alpha i} dy^\alpha \otimes dx^i + a_{i\alpha} dx^i \otimes \otimes dy^\alpha + a_{\alpha\beta} dy^\alpha \otimes dy^\beta$ be a bilinear form on E. Then $D_h\omega$ is quasi-semi-basic. Let $Y = b^\alpha(\partial/\partial y^\alpha)$ be a vertical tangent field. Then

$$i_Y\omega = a_{\alpha i} b^\alpha dx^i + a_{\alpha\beta} b^\alpha dy^\beta , \quad h^*(i_Y\omega) = (a_{\alpha i} + a_{\alpha\beta} a_i^\beta)\, b^\alpha dx^i .$$

The form ω will be said to be associated with a connection Γ on E if $h^* i_Y\omega = 0$ for any vertical tangent vector Y. Locally, a bilinear form ω is associated with a connection Γ on E if and only if

$$(7) \qquad a_{\alpha i} + a_{\alpha\beta} a_i^\beta = 0 .$$

Let ${}^\omega T_u = \{X \in T_u E : i_Y\, \omega(X) = 0 \text{ for any } Y \in T_u E_m,\ \pi u = m\}$. The bilinear form ω on E will be called connecting if the distribution of the tangent subspaces ${}^\omega T_u$ determineds a connection on E. If ω is connecting then the connection of the tangent subspaces ${}^\omega T_u$ will be denoted by ${}^\omega\Gamma$.

As $\dim \{i_Y\omega : Y \in T_u E_m\} \leqq \dim E_m$, we have $\dim {}^\omega T_u \geqq \dim M$. Then the mapping $u \to {}^\omega T_u$ is a connection if and only if the assertion

$$(Z \in T_u E_m \wedge Z \in {}^\omega T_u) \Rightarrow Z = 0$$

is true for any $u \in E$. Locally, let $Z = c^\alpha(\partial/\partial y^\alpha)$. Then $Z \in {}^\omega T_u$ if and only if $i_Y\, \omega(Z) = 0$ for any $Y \in T_u E_m$, i.e. if and only if $a_{\alpha\beta} c^\alpha = 0$. Then ω is connecting if and only if $\det (a_{\alpha\beta}) \neq 0$, i.e. if and only if the restriction of ω to vertical tangent vectors is a regular form. This yields

Proposition 10. *Let ω be connecting. Then ω is associated with a connection Γ if and only if $\Gamma = {}^{\omega}\Gamma$.*

Let us recall that if ω is quasi-semi-basic then it is not connecting. If ω is a 2-form (i.e. antisymmetric of the second order) then it can be connecting only if $\dim E_x$ is even.

Proposition 11. *Let ω be a connecting 2-form on E. Then the connection ${}^{\omega}\Gamma$ is integrable if and only if*

$$h^*(L_Y\omega - i_Y d\omega) = 0$$

for any vertical tangent field Y.

Proof. By definition ${}^{\omega}\Gamma$ is integrable if and only if $h^*(di_Y\omega) = 0$ for any vertical tangent field Y. The known relation $L_Y = i_Y d + di_Y$ completes our proof.

Let ω or Γ be a bilinear form or a connection, respectively, on E. Denote by $\omega_{10}, \omega_{20}, \omega_{12}, \omega_{21}$ the following forms:

$$\omega_{10}(X, Y) = \omega(hX, Y), \quad \omega_{20}(X, Y) = \omega(vX, Y),$$
$$\omega_{01}(X, Y) = \omega(X, hY), \quad \omega_{02}(X, Y) = \omega(X, vY),$$
$$\omega_{12}(X, Y) = \omega(hX, vY), \quad \omega_{21}(X, Y) = \omega(vX, hY).$$

Lemma 3. *Let ω or Γ be a bilinear form or a connection, respectively, on E. Then*

$$\omega_{10} = h^*\omega + \omega_{12}, \qquad \omega_{20} = v^*\omega + \omega_{21},$$
$$\omega_{01} = h^*\omega + \omega_{21}, \qquad \omega_{02} = v^*\omega + \omega_{12},$$
$$D_h\omega = \omega_{10} + \omega_{01}, \qquad D_v\omega = \omega_{20} + \omega_{02},$$
$$D_h\omega - D_v\omega = 2(h^*\omega - v^*\omega), \qquad \omega = h^*\omega + D_vD_h\omega + v^*\omega,$$
$$D_vD_h\omega = \omega_{12} + \omega_{21}, \qquad D_vD_v\omega = D_v\omega + 2v^*\omega,$$
$$D_hD_h\omega = D_h\omega + 2h^*\omega. \tag{8}$$

Proof. $\omega_{10}(X, Y) = \omega(hX, hY + vY) = \omega(hX, hY) + \omega(hX, vY) = h^*\,\omega(X, Y) + \omega_{12}(X, Y)$. The other relations can be proved analogously.

Proposition 12. *A bilinear form ω is associated with a connection Γ if and only if $\omega_{21} = 0$.*

Proof. Let $\omega_{21} = 0$. Then $h^*i_Y\,\omega(X) = i_Y\,\omega(hX) = \omega(Y, hX) = \omega_{21}(Y, X) = 0$ for any vertical tangent vector Y. Let ω be associated with Γ. Then $\omega_{21}(Y, X) = \omega(vY, hX) = h^*i_{vY}\,\omega(X) = 0$.

Corollary. *The forms $\omega_{02}, \omega_{10}, \omega_{12}, h^*\omega, v^*\omega$ are associated with Γ.*

Lemma 4. *Let ω be either antisymmetric or symmetric. Then $\omega_{21} = 0$ if and only if $\omega_{12} = 0$.*

Proof is obvious.

Proposition 13. *Let ω be either antisymmetric or symmetric. Then ω is associated with a connection Γ if and only if $D_h\omega$ is semi-basic.*

Proof. $\omega_{21}(Y, X) = \omega(vY, hX) = D_h\omega(vY, X) = i_{vY}D_h\omega(X)$. Then the definition of the semi-basic form and Proposition 12 complete our proof.

Proposition 14. *Let ω be either antisymmetric or symmetric. Then ω is associated with Γ if and only if $i_Z\omega$ is semi-basic for any horizontal vector Z.*

Proof. $\omega_{12}(X, Y) = \omega(hX, vY) = i_{hX}\omega(vY)$. Proposition 12 and Lemma 4 complete the proof.

By the relation (8) we get

Proposition 15. *Let ω be either antisymmetric or symmetric and associated with Γ. Then*

$$D_hD_v\omega = 0\,, \quad D_v\omega = 2v^*\omega\,, \quad D_h\omega = 2h^*\omega\,, \quad \omega = h^*\omega + v^*\omega\,.$$

Corollary. *If ω is associated with Γ, Γ-vertical and either antisymmetric or symmetric then $D_v^n\omega = 2^n\omega$.*

Lemma 5. *Let ω or Γ be a bilinear form or a connection, respectively, on E. Then*

$$(\omega - h^*\omega)_{21} = (D_v\omega)_{21} = (D_h\omega)_{21} = (\omega_{20})_{21} = (\omega_{01})_{21} = \omega_{21}\,.$$

Proof. $(\omega - h^*\omega)_{21}(X, Y) = (\omega - h^*\omega)(vX, hY) = \omega(vX, hY) = \omega_{21}(X, Y)$. The other relations can be proved analogously.

Corollary of Lemma 5 and Proposition 12. *Let ω or Γ be a bilinear form or a connection respectively on E. Then the forms ω, $\omega - h^*\omega$, $D_v\omega$, $D_h\omega$, ω_{20}, ω_{01} are associated with Γ if and only if one of them is associated with Γ.*

Proposition 16. *Let ω be a bilinear connecting form on E. Let Γ be a connection on E. Then the forms $\omega - h^*\omega$, $D_v\omega$, ω_{20}, ω_{02}, $v^*\omega$ determined by Γ are connecting and $\Gamma = {}^{\omega_{02}}\Gamma = {}^{v^*\omega}\Gamma$.*

Proof. Let locally $\omega = a_{ij}dx^i \otimes dx^j + a_{\alpha i}dy^\alpha \otimes dx^i + a_{i\alpha}dx^i \otimes dy^\alpha + a_{\alpha\beta}dy^\alpha \otimes \otimes dy^\beta$. Let

$$\Omega \in \{D_v\omega, \omega_{20}, \omega_{02}, \omega - h^*\omega, v^*\omega\}\,.$$

Then $\Omega = C_{ij}dx^i \otimes dx^j + C_{\alpha i}dy^\alpha \otimes dx^i + C_{i\alpha}dx^i \otimes dy^\alpha + ca_{\alpha\beta}dy^\alpha \otimes dy^\beta$ where $c \neq 0$ is a constant. As $\det(ca_{\alpha\beta}) \neq 0$ we conclude that Ω is connecting. By Proposition 10 and Corollary of Proposition 12, $\Gamma = {}^{\omega_{20}}\Gamma = {}^{v^*\omega}\Gamma$.

Proposition 17. *Let ω be a bilinear connecting form on E. Let $\omega - h^*\omega$, $D_v\omega$, ω_{20} be determined by ${}^{\omega}\Gamma$. Then*

$$ {}^{\omega}\Gamma = {}^{\omega - h^*\omega}\Gamma = {}^{D_v\omega}\Gamma = {}^{\omega_{20}}\Gamma \, . $$

Proof. The form ω is associated with ${}^{\omega}\Gamma$. Therefore by Lemma 5 and Proposition 12 the forms $\omega - h^*\omega$, $D_v\omega$, ω_{20} are associated with ${}^{\omega}\Gamma$. Then Propositions 16 and 10 complete our proof.

Proposition 18. *Let ω be a connecting 2-form on E. Then a connection Γ on E is integrable if and only if $d_\Phi\omega$ is semi-basic.*

Proof. Let us recall that Γ is integrable if and only if the curvature form Φ of Γ vanishes, i.e. if $A_{jk} = 0$. Let $\omega = \frac{1}{2}a_{ij}dx^i \wedge dx^j + a_{\alpha i}dy^\alpha \wedge dx^i + \frac{1}{2}a_{\alpha\beta}dy^\alpha \wedge dy^\beta$. Then $d_\Phi\omega = a_{\alpha i}A^\alpha_{jk}dx^j \wedge dx^k \wedge dx^i + a_{\alpha\beta}A^\alpha_{jk}dx^j \wedge dx^k \wedge dy^\beta$ is semibasic if and only if $a_{\alpha\beta}A^\alpha_{jk} = 0$. As $\det(a_{\alpha\beta}) \neq 0$, it holds $a_{\alpha\beta}A^\alpha_{jk} = 0$ if and only if $A^\alpha_{jk} = 0$.

Remark. Using the local expresion of $d_\Phi\omega$ we obtain: If ω is a connecting 2-form and Γ is a connection on E then $d_\Phi\omega$ is semi-basic if and only if $d_\Phi\omega = 0$.

Let Ω be a ternary from on E. Let Γ be a connection on E. Denote by Ω_{112} the form determined by

$$ \Omega_{112}(X, Y, Z) = \Omega(hX, hY, vZ) \, . $$

Lemma 6. *Let ω be a connecting 2-form on E. Let Γ be a connection on E. Let Φ be the curvature form of Γ. Then $d_\Phi\omega = 0$ if and only if $(d_\Phi\omega)_{112} = 0$.*

Proof. Locally, $(d_\Phi\omega)_{112} = -a_{\alpha\beta}A^\alpha_{jk}a^\beta_i dx^j \wedge dx^k \wedge dx^i + a_{\alpha\beta}A^\alpha_{jk}dx^j \wedge dx^k \wedge \wedge dy^\beta$. This yields our assertion.

Proposition 19. *Let ω be a 2-form on E. Then*

$$ (d(v^*\omega))_{112} = -(d_\Phi\omega)_{112} $$

for any connection Γ on E.

Proof. $(dv^*\omega)_{112}(X, Y, Z) = dv^*\omega(hX, hY, vZ) = hX(v^*\omega(hY, vZ)) - hY(v^*\omega(hX, vZ)) + vZ(v^*\omega(hX, hY)) - v^*\omega([hX, hY], vZ) + v^*\omega([X, vZ], hY) - v^*\omega([hY, vZ], hX) = -\omega(v[hX, hY], vZ) = -(d_\Phi\omega)_{112}(X, Y, Z)$.

Corollary of Proposition 18, 19 and Lemma 6. *Let ω be a connecting 2-form on E. Then a connection Γ is integrable if and only if $(dv^*\omega)_{112} = 0$.*

Proposition 20. *Let ω be a connecting 2-form on E. Then the connection ${}^{\omega}\Gamma$ is integrable if and only if*

$$ (dd_v\omega)_{112} = 0 \, . $$