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ON CHARACTERIZATION OF THE SPHERE IN E^4 BY MEANS OF THE PARALLELNESS OF CERTAIN VECTOR FIELDS

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In this paper we present a certain generalization of the results contained in [3]. Using the parallelness of a certain normal vector field associated to a given couple of tangent vector fields, we prove theorems analogous to those of [3] to get the base for other considerations.

1. Let M be a surface in the 4-dimensional Euclidean space E^4 and ∂M its boundary. Let the surface M be covered by domains U_α in such a way that in any U_α there is a field of orthonormal frames $\{M; v_1, v_2, v_3, v_4\}$ with $v_1, v_2 \in T(M)$, $v_3, v_4 \in N(M)$, $T(M)$, $N(M)$ being the tangent and the normal bundle of M, respectively. Then

$$\begin{aligned}
&\mathrm{d}M = \omega^1 v_1 + \omega^2 v_2 , \\
&\mathrm{d}v_1 = \omega_1^2 v_2 + \omega_1^3 v_3 + \omega_1^4 v_4 , \\
&\mathrm{d}v_2 = -\omega_1^2 v_1 + \omega_2^3 v_3 + \omega_2^4 v_4 , \\
&\mathrm{d}v_3 = -\omega_1^3 v_1 - \omega_2^3 v_2 + \omega_3^4 v_4 , \\
&\mathrm{d}v_4 = -\omega_1^4 v_1 - \omega_2^4 v_2 - \omega_3^4 v_3 ;
\end{aligned} \tag{1}$$

$$\begin{aligned}
&\mathrm{d}\omega^i = \omega^j \wedge \omega_j^i , \quad \mathrm{d}\omega_i^j = \omega_i^k \wedge \omega_k^j , \\
&\omega_i^j + \omega_j^i = 0 , \qquad \omega^3 = \omega^4 = 0 \qquad (i, j, k = 1, 2, 3, 4) .
\end{aligned} \tag{2}$$

Using the well-known prolongation procedure, we get the existence of real functions a_i, b_i $(i = 1, 2, 3)$, α_i, β_i $(i = 1, 2, 3, 4)$, $A_i, B_i, \ldots, E_i$ $(i = 1, 2)$ in each U_α such that

$$\begin{aligned}
&\omega_1^3 = a_1\omega^1 + a_2\omega^2 , \quad \omega_2^3 = a_2\omega^1 + a_3\omega^2 , \\
&\omega_1^4 = b_1\omega^1 + b_2\omega^2 , \quad \omega_2^4 = b_2\omega^1 + b_3\omega^2 ;
\end{aligned} \tag{3}$$

$$\begin{aligned}
&\mathrm{d}a_1 - 2a_2\omega_1^2 - b_1\omega_3^4 = \alpha_1\omega^1 + \alpha_2\omega^2 , \\
&\mathrm{d}a_2 + (a_1 - a_3)\,\omega_1^2 - b_2\omega_3^4 = \alpha_2\omega^1 + \alpha_3\omega^2 ,
\end{aligned} \tag{4}$$

$$
\begin{aligned}
&da_3 + 2a_2\omega_1^2 - b_3\omega_3^4 = \alpha_3\omega^1 + \alpha_4\omega^2 ,\\
&db_1 - 2b_2\omega_1^2 + a_1\omega_3^4 = \beta_1\omega^1 + \beta_2\omega^2 ,\\
&db_2 + (b_1 - b_3)\,\omega_1^2 + a_2\omega_3^4 = \beta_2\omega^1 + \beta_3\omega^2 ,\\
&db_3 + 2b_2\omega_1^2 + a_3\omega_3^4 = \beta_3\omega^1 + \beta_4\omega^2 ;
\end{aligned}
$$

(5)
$$
\begin{aligned}
&d\alpha_1 - 3\alpha_2\omega_1^2 - \beta_1\omega_3^4 = A_1\omega^1 + (B_1 - a_2K - \tfrac{1}{2}b_1k)\,\omega^2 ,\\
&d\alpha_2 + (\alpha_1 - 2\alpha_3)\,\omega_1^2 - \beta_2\omega_3^4 = (B_1 + a_2K + \tfrac{1}{2}b_1k)\,\omega^1 +\\
&\quad + (C_1 + a_1K - \tfrac{1}{2}b_2k)\,\omega^2 ,\\
&d\alpha_3 + (2\alpha_2 - \alpha_4)\,\omega_1^2 - \beta_3\omega_3^4 = (C_1 + a_3K + \tfrac{1}{2}b_2k)\,\omega^1 +\\
&\quad + (D_1 + a_2K - \tfrac{1}{2}b_3k)\,\omega^2 ,\\
&d\alpha_4 + 3\alpha_3\omega_1^2 - \beta_4\omega_3^4 = (D_1 - a_2K + \tfrac{1}{2}b_3k)\,\omega^1 + E_1\omega^2 ,\\
&d\beta_1 - 3\beta_2\omega_1^2 + \alpha_1\omega_3^4 = A_2\omega^1 + (B_2 - b_2K + \tfrac{1}{2}a_1k)\,\omega^2 ,\\
&d\beta_2 + (\beta_1 - 2\beta_3)\,\omega_1^2 + \alpha_2\omega_3^4 = (B_2 + b_2K - \tfrac{1}{2}a_1k)\,\omega^1 +\\
&\quad + (C_2 + b_1K + \tfrac{1}{2}a_2k)\,\omega^2 ,\\
&d\beta_3 + (2\beta_2 - \beta_4)\,\omega_1^2 + \alpha_3\omega_3^4 = (C_2 + b_3K - \tfrac{1}{2}a_2k)\,\omega^1 +\\
&\quad + (D_2 + b_2K + \tfrac{1}{2}a_3k)\,\omega^2 ,\\
&d\beta_4 + 3\beta_3\omega_1^2 + \alpha_4\omega_3^4 = (D_2 - b_2K - \tfrac{1}{2}a_3k)\,\omega^1 + E_2\omega^2 ,
\end{aligned}
$$

where

$$K = a_1a_3 - a_2^2 + b_1b_3 - b_2^2 , \quad k = (a_1 - a_3)\,b_2 - (b_1 - b_3)\,a_2 ,$$

the function K being the Gauss curvature of M. As always,

$$H = (a_1 + a_3)^2 + (b_1 + b_3)^2$$

denotes the mean curvature and

$$\xi = (a_1 + a_3)\,v_3 + (b_1 + b_3)\,v_4$$

the mean curvature vector field of M.

Let us remark that the normal vector field $n = xv_3 + yv_4$ being parallel in $N(M)$ we have $k = 0$ (see [1], p. 61), and since $v_1, v_2 \in T(M)$ generates an orthogonal conjugate net of lines on M, [2], we have

(6) $$a_2 = 0 , \quad b_2 = 0$$

and again $k = 0$ on M. In addition, in the last case, because of (4), there are real functions ϱ, σ such that

(7)
$$
\begin{aligned}
&\omega_1^2 = \varrho\omega^1 + \sigma\omega^2 ,\\
&\alpha_2 = \varrho(a_1 - a_3) , \quad \alpha_3 = \sigma(a_1 - a_3) ,\\
&\beta_2 = \varrho(b_1 - b_3) , \quad \beta_3 = \sigma(b_1 - b_3) .
\end{aligned}
$$

Like in [3], all theorems contained in this contribution are proved by means of the maximum principle.

Let $f : M \to \mathscr{R}$ be a real function. The covariant derivatives f_i, f_{ij} $(i, j = 1, 2)$ of its restriction to U_α with respect to the frames $\{M; v_1, v_2, v_3, v_4\}$ are introduced by the formulas

(8) $$df = f_1\omega^1 + f_2\omega^2 ,$$

$$df_1 - f_2\omega_1^2 = f_{11}\omega^1 + f_{12}\omega^2 , \quad df_2 + f_1\omega_1^2 = f_{12}\omega^1 + f_{22}\omega^2 .$$

We use the maximum principle in this form:

Let M be a surface in E^4 and ∂M its boundary. Let f be a real-valued function on M and f_i, f_{ij} $(i, j = 1, 2)$ its covariant derivatives. Let (i) $f \geqq 0$ *on* M; (ii) $f = 0$ *on* ∂M; (iii) f *satisfy the equation*

$$a_{11}f_{11} + 2a_{12}f_{12} + a_{22}f_{22} + a_1f_1 + a_2f_2 + a_0f = a$$

with $a_0 \leqq 0$, $a \geqq 0$ *and the quadratic form* $a_{ij}x^ix^j$ *positive definite. Then* $f = 0$ *on* M.

In the following we use the function $f : M \to \mathscr{R}$ defined by

(9) $$f = H - 4K = (a_1 - a_3)^2 + (b_1 - b_3)^2 + 4a_2^2 + 4b_2^2 ,$$

satisfying $f \geqq 0$ on M and $f = 0$ at the umbilical points of M. Using (4), (5) and (8) we get the covariant derivatives of f, in particular

(10) $$\begin{aligned} f_{11} = & -2[(a_1 - a_3)a_3 + (b_1 - b_3)b_3 - 4(a_2^2 + b_2^2)]K - \\ & - [k + 4(a_1b_2 - a_2b_1)]k + 2(\alpha_1 - \alpha_3)^2 + 2(\beta_1 - \beta_3)^2 + \\ & + 8(\alpha_2^2 + \beta_2^2) + 2(a_1 - a_3)(A_1 - C_1) + 2(b_1 - b_3)(A_2 - C_2) + \\ & + 8(a_2B_1 + b_2B_2) , \\ f_{22} = & \;\; 2[(a_1 - a_3)a_1 + (b_1 - b_3)b_1 + 4(a_2^2 + b_2^2)]K - \\ & - [k + 4(a_2b_3 - a_3b_2)]k + 2(\alpha_2 - \alpha_4)^2 + 2(\beta_2 - \beta_4)^2 + \\ & + 8(\alpha_3^2 + \beta_3^2) + 2(a_1 - a_3)(C_1 - E_1) + 2(b_1 - b_3)(C_2 - E_2) + \\ & + 8(a_2D_1 + b_2D_2) . \end{aligned}$$

2. Let M be a surface in E^4 and let $V_1, V_2 \in T(M)$ be fixed orthonormal vector fields. In all the following considerations we choose orthonormal frames $\{M; v_1, v_2, v_3, v_4\}$ of M in such a way that $V_1 = v_1$, $V_2 = v_2$ at any point $m \in M$. Define further normal vector fields $V_{ii}, V_{jii}, V_{kjii}$ $(i, j, k = 1, 2)$ by the relations

$$V_{ii} = (V_iV_i)^N , \quad V_{jii} = (V_jV_{ii})^N , \quad V_{kjii} = (V_kV_{jii})^N \quad (i, j, k = 1, 2) ,$$

where $(Y)^N$ denotes the normal component of the vector field Y.

It is easy to see that

$$(11) \qquad V_{11} = a_1 v_3 + b_1 v_4 \,, \quad V_{22} = a_3 v_3 + b_3 v_4 \,.$$

Suppose further that V_1, V_2 generate an orthogonal conjugate net of lines on M, i.e. we have (6) and (7) on M. Then we get from (11) using (1), (3) and (4)

$$\mathrm{d}V_{11} = V_{111}\omega^1 + V_{211}\omega^2 \,, \quad \mathrm{d}V_{22} = V_{122}\omega^1 + V_{222}\omega^2 \pmod{v_1, v_2}$$

with

$$(12) \qquad \begin{aligned} V_{111} &= \alpha_1 v_3 + \beta_1 v_4 \,, & V_{211} &= \alpha_2 v_3 + \beta_2 v_4 \,, \\ V_{122} &= \alpha_3 v_3 + \beta_3 v_4 \,, & V_{222} &= \alpha_4 v_3 + \beta_4 v_4 \,. \end{aligned}$$

By differentiating the relations (12) and using (1), (3), (5) we obtain

$$\begin{aligned} \mathrm{d}V_{111} &= V_{1111}\omega^1 + V_{2111}\omega^2 \,, & \mathrm{d}V_{211} &= V_{1211}\omega^1 + V_{2211}\omega^2 \,, \\ \mathrm{d}V_{122} &= V_{1122}\omega^1 + V_{2122}\omega^2 \,, & \mathrm{d}V_{222} &= V_{1222}\omega^1 + V_{2222}\omega^2 \pmod{v_1, v_2} \,, \end{aligned}$$

where

$$(13) \qquad \begin{aligned} V_{1111} &= (A_1 + 3\alpha_2\varrho)\, v_3 + (A_2 + 3\beta_2\varrho)\, v_4 \,, \\ V_{2111} &= (B_1 + 3\alpha_2\sigma)\, v_3 + (B_2 + 3\beta_2\sigma)\, v_4 \,, \\ V_{1211} &= [B_1 - (\alpha_1 - 2\alpha_3)\,\varrho]\, v_3 + [B_2 - (\beta_1 - 2\beta_3)\,\varrho]\, v_4 \,, \\ V_{2211} &= [C_1 + a_1 K - (\alpha_1 - 2\alpha_3)\,\sigma]\, v_3 + [C_2 + b_1 K - (\beta_1 - 2\beta_3)\,\sigma]\, v_4 \,, \\ V_{1122} &= [C_1 + a_3 K - (2\alpha_2 - \alpha_4)\,\varrho]\, v_3 + [C_2 + b_3 K - (2\beta_2 - \beta_4)\,\varrho]\, v_4 \,, \\ V_{2122} &= [D_1 - (2\alpha_2 - \alpha_4)\,\sigma]\, v_3 + [D_2 - (2\beta_2 - \beta_4)\,\sigma]\, v_4 \,, \\ V_{1222} &= (D_1 - 3\alpha_3\varrho)\, v_3 + (D_2 - 3\beta_3\varrho)\, v_4 \,, \\ V_{2222} &= (E_1 - 3\alpha_3\sigma)\, v_3 + (E_2 - 3\beta_3\sigma)\, v_4 \,, \end{aligned}$$

ϱ, σ being the functions defined by (7).

From (11) it follows that

$$\xi = V_{11} + V_{22} \,.$$

This vector field can be considered as a special case of the normal field

$$(14) \qquad X = PV_{11} + QV_{22} = (Pa_1 + Qa_3)\, v_3 + (Pb_1 + Qb_3)\, v_4 \,,$$

where $P, Q \in \mathscr{R}$ are constants with $P^2 + Q^2 \neq 0$.

First of all we prove that the normal vector field X is invariant on M when choosing the orthonormal frames in the above mentioned way. To this end, consider another orthonormal frame $\{M; \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$ on M such that $V_1 = \bar{v}_1$, $V_2 = \bar{v}_2$. Then we have

$$(15) \qquad \begin{aligned} \bar{v}_1 &= v_1 \,, & \bar{v}_3 &= \varepsilon \cos\sigma \,.\, v_3 - \sin\sigma \,.\, v_4 \,, \\ \bar{v}_2 &= v_2 \,, & \bar{v}_4 &= \varepsilon \sin\sigma \,.\, v_3 + \cos\sigma \,.\, v_4 \quad (\varepsilon^2 = 1) \end{aligned}$$

and the following lemma is valid:

Lemma 1. *On M, it is*

$$\bar{X} = X .$$

Proof. It is easy to see that

$$\bar{a}_1 = \varepsilon(a_1 \cos \sigma + b_1 \sin \sigma) , \quad \bar{b}_1 = -(a_1 \sin \sigma + b_1 \cos \sigma) ,$$
$$\bar{a}_3 = \varepsilon(a_3 \cos \sigma + b_3 \sin \sigma) , \quad \bar{b}_3 = -(a_3 \sin \sigma + b_3 \cos \sigma)$$

and according to (15) and the preceding equations we obtain

$$\bar{V}_{11} = V_{11} , \quad \bar{V}_{22} = V_{22} .$$

As $\bar{P} = P$, $\bar{Q} = Q$, our assertion is proved.

Now, define normal vector fields X_i, X_{ij} $(i, j = 1, 2)$ by the formulas

$$X_i = (V_i X)^N , \quad X_{ij} = (V_i X_j)^N \quad (i, j = 1, 2) ,$$

where the symbol $(Y)^N$ denotes again the normal component of the vector field Y. Then we have the following

Lemma 2. *Let $V_1, V_2 \in T(M)$ generate an orthogonal conjugate net of lines on M. Then for the normal vector field $X = PV_{11} + QV_{22}$, $P, Q \in \mathcal{R}$ we have*

(16) $$X_1 = PV_{111} + QV_{122} , \quad X_2 = PV_{211} + QV_{222} ,$$

(17) $$X_{11} = PV_{1111} + QV_{1122} , \quad X_{12} = PV_{1211} + QV_{1222} ,$$
$$X_{21} = PV_{2111} + QV_{2122} , \quad X_{22} = PV_{2211} + QV_{2222} .$$

Proof. The relation (14) yields

$$\mathrm{d}X = P\,\mathrm{d}V_{11} + Q\,\mathrm{d}V_{22}$$

and hence

$$\mathrm{d}X = (PV_{111} + QV_{122})\,\omega^1 + (PV_{211} + QV_{222})\,\omega^2 \pmod{v_1, v_2}$$

which implies (16). Further

$$\mathrm{d}X_1 = P\,\mathrm{d}V_{111} + Q\,\mathrm{d}V_{122} , \quad \mathrm{d}X_2 = P\,\mathrm{d}V_{211} + Q\,\mathrm{d}V_{222} ,$$

that is

$$\mathrm{d}X_1 = (PV_{1111} + QV_{1122})\,\omega^1 + (PV_{2111} + QV_{2122})\,\omega^2 ,$$
$$\mathrm{d}X_2 = (PV_{1211} + QV_{1222})\,\omega^1 + (PV_{2211} + QV_{2222})\,\omega^2 \pmod{v_1, v_2} .$$

This proves the validity of (17).

Thus, assuming that $V_1, V_2 \in T(M)$ generate an orthogonal conjugate net of lines on M, we have from (12) and (16)

(18) $$X_1 = (P\alpha_1 + Q\alpha_3)\,v_3 + (P\beta_1 + Q\beta_3)\,v_4 ,$$
$$X_2 = (P\alpha_2 + Q\alpha_4)\,v_3 + (P\beta_2 + Q\beta_4)\,v_4$$

and from (13) and (17)

$$(19)\qquad \begin{aligned} X_{11} &= \{PA_1 + Q(C_1 + a_3K) + \varrho[(3P - 2Q)\,\alpha_2 + Q\alpha_4]\}\, v_3 + \\ &\quad + \{PA_2 + Q(C_2 + b_3K) + \varrho[(3P - 2Q)\,\beta_2 + Q\beta_4]\}\, v_4\,, \\ X_{12} &= \{PB_1 + QD_1 - \varrho[P\alpha_1 - (2P - 3Q)\,\alpha_3]\}\, v_3 + \\ &\quad + \{PB_2 + QD_2 - \varrho[P\beta_1 - (2P - 3Q)\,\beta_3]\}\, v_4\,, \\ X_{21} &= \{PB_1 + QD_1 + \sigma[(3P - 2Q)\,\alpha_2 + Q\alpha_4]\}\, v_3 + \\ &\quad + \{PB_2 + QD_2 + \sigma[(3P - 2Q)\,\beta_2 + Q\beta_4]\}\, v_4\,, \\ X_{22} &= \{P(C_1 + a_1K) + QE_1 - \sigma[P\alpha_1 - (2P - 3Q)\,\alpha_3]\}\, v_3 + \\ &\quad + \{P(C_2 + b_1K) + QE_2 - \sigma[P\beta_1 - (2P - 3Q)\,\beta_3]\}\, v_4\,. \end{aligned}$$

By these remarks we have completed all preliminaries necessary for our considerations.

3. Now we are going to prove the basic

Theorem 1. *Let M be a surface in E^4 and ∂M its boundary. Let*

(i) $K > 0$ *on* M;

(ii) *there exist* $V_1, V_2 \in T(M)$ *generating an orthogonal conjugate net of lines on* M;

(iii) $X = PV_{11} + QV_{22}$, *where* $P, Q \in \mathscr{R}$ *satisfy the relations* $P^2 + Q^2 > 0$, $PQ \geqq 0$, *be parallel in* $N(M)$;

(iv) ∂M *consist of umbilical points.*

Then M is a part of a 2-dimensional sphere in E^4.

Proof. As remarked, we use the maximum principle for the invariant function f defined by (9). Since the assumption (ii) implies (6) in M, we have in virtue of (10)

$$(20)\quad Pf_{11} + Qf_{22} - 2[(a_1 - a_3)(Qa_1 - Pa_3) + (b_1 - b_3)(Qb_1 - Pb_3)]\,K = \\ = 2V + 2\Phi$$

where

$$(21)\qquad V = P[(\alpha_1 - \alpha_3)^2 + (\beta_1 - \beta_3)^2] + Q[(\alpha_2 - \alpha_4)^2 + (\beta_2 - \beta_4)^2] + \\ + 4P(\alpha_2^2 + \beta_2^2) + 4Q(\alpha_3^2 + \beta_3^2)$$

and

$$(22)\qquad \Phi = (a_1 - a_3)\,[P(A_1 - C_1) + Q(C_1 - E_1)] + \\ + (b_1 - b_3)\,[P(A_2 - C_2) + Q(C_2 - E_2)]\,.$$

Now, the condition (iii) for X defined by (14) yields

$$\begin{aligned} &d(Pa_1 + Qa_3) - (Pb_1 + Qb_3)\,\omega_3^4 = 0\,, \\ &d(Pb_1 + Qb_3) + (Pa_1 + Qa_3)\,\omega_3^4 = 0 \end{aligned} \tag{23}$$

and hence according to (4) and (6)

$$\begin{aligned} &P\alpha_1 + Q\alpha_3 = 0\,, \quad P\alpha_2 + Q\alpha_4 = 0\,, \\ &P\beta_1 + Q\beta_3 = 0\,, \quad P\beta_2 + Q\beta_4 = 0\,. \end{aligned} \tag{24}$$

Differentiating these equations and using (24) again, we obtain the relations

$$\begin{aligned} &[PA_1 + Q(C_1 + a_3K)]\,\omega^1 + (PB_1 + QD_1)\,\omega^2 + \\ &\quad + [(3P - 2Q)\,\alpha_2 + Q\alpha_4]\,\omega_1^2 = 0\,, \\ &[PA_2 + Q(C_2 + b_3K)]\,\omega^1 + (PB_2 + QD_2)\,\omega^2 + \\ &\quad + [(3P - 2Q)\,\beta_2 + Q\beta_4]\,\omega_1^2 = 0\,, \\ &(PB_1 + QD_1)\,\omega^1 + [P(C_1 + a_1K) + QE_1]\,\omega^2 - \\ &\quad - [P\alpha_1 - (2P - 3Q)\,\alpha_3]\,\omega_1^2 = 0\,, \\ &(PB_2 + QD_2)\,\omega^1 + [P(C_2 + b_1K) + QE_2]\,\omega^2 - \\ &\quad - [P\beta_1 - (2P - 3Q)\,\beta_3]\,\omega_1^2 = 0\,. \end{aligned} \tag{25}$$

Multiply the equations containing $A_1, \ldots, E_1$ by $a_1 - a_3$ and the relations containing $A_2, \ldots, E_2$ by $b_1 - b_3$. Then according to (7) we get in particular

$$\begin{aligned} &(a_1 - a_3)\,[PA_1 + Q(C_1 + a_3K)] + \alpha_2[(3P - 2Q)\,\alpha_2 + Q\alpha_4] = 0\,, \\ &(b_1 - b_3)\,[PA_2 + Q(C_2 + b_3K)] + \beta_2[(3P - 2Q)\,\beta_2 + Q\beta_4] = 0\,, \\ &(a_1 - a_3)\,[P(C_1 + a_1K) + QE_1] - \alpha_3[P\alpha_1 - (2P - 3Q)\,\alpha_3] = 0\,, \\ &(b_1 - b_3)\,[P(C_2 + b_1K) + QE_2] - \beta_3[P\beta_1 - (2P - 3Q)\,\beta_3] = 0 \end{aligned} \tag{26}$$

and hence

$$\begin{aligned} &(a_1 - a_3)\,[P(A_1 - C_1) + Q(C_1 - E_1)] = (a_1 - a_3)\,(Pa_1 - Qa_3)\,K - \\ &\quad - \alpha_2[(3P - 2Q)\,\alpha_2 + Q\alpha_4] - \alpha_3[P\alpha_1 - (2P - 3Q)\,\alpha_3]\,, \\ &(b_1 - b_3)\,[P(A_2 - C_2) + Q(C_2 - E_2)] = (b_1 - b_3)\,(Pb_1 - Qb_3)\,K - \\ &\quad - \beta_2[(3P - 2Q)\,\beta_2 + Q\beta_4] - \beta_3[P\beta_1 - (2P - 3Q)\,\beta_3]\,. \end{aligned}$$

Using these relations we obtain from (22)

$$\begin{aligned} \Phi = {} &[(a_1 - a_3)\,(Pa_1 - Qa_3) + (b_1 - b_3)\,(Pb_1 - Qb_3)]\,K - \\ &- \alpha_2[(3P - 2Q)\,\alpha_2 + Q\alpha_4] - \alpha_3[P\alpha_1 - (2P - 3Q)\,\alpha_3] - \\ &- \beta_2[(3P - 2Q)\,\beta_2 + Q\beta_4] - \beta_3[P\beta_1 - (2P - 3Q)\,\beta_3] \end{aligned}$$

and the equation (20) has the form

$$(27) \qquad Pf_{11} + Qf_{22} - 2(P + Q)fK = 2W$$

where

$$W = V - \alpha_2[(3P - 2Q)\alpha_2 + Q\alpha_4] - \alpha_3[P\alpha_1 - (2P - 3Q)\alpha_3] -$$
$$- \beta_2[(3P - 2Q)\beta_2 + Q\beta_4] - \beta_3[P\beta_1 - (2P - 3Q)\beta_3].$$

From this identity and from (21) we finally have

$$W = P[(\alpha_1 - \tfrac{3}{2}\alpha_3)^2 + (\beta_1 - \tfrac{3}{2}\beta_3)^2] + Q[(\alpha_4 - \tfrac{3}{2}\alpha_2)^2 + (\beta_4 - \tfrac{3}{2}\beta_2)^2] +$$
$$+ \tfrac{1}{4}(4P + 3Q)(\alpha_2^2 + \beta_2^2) + \tfrac{1}{4}(3P + 4Q)(\alpha_3^2 + \beta_3^2).$$

If $P \geqq 0$, $Q \geqq 0$ and $P^2 + Q^2 > 0$, we have $-(P + Q)K \leqq 0$, $W \geqq 0$ and the quadratic form corresponding to $Pf_{11} + Qf_{22}$ is positive definite so that, according to the maximum principle, the theorem is true. On the other hand, if $P \leqq 0$, $Q \leqq 0$ and $P^2 + Q^2 > 0$, it is $-(P + Q)K \geqq 0$, $W \leqq 0$ and the form corresponding to $Pf_{11} + Qf_{22}$ is negative definite. Then it is sufficient to multiply the equation (27) by -1 to get the condition (iii) of the maximum principle. Thus the assertion is proved.

As an immediate consequence of this theorem we introduce

Corollary 1. *Let M be a surface in E^4 possessing the properties* (i), (ii) *and* (iv) *of Theorem* 1. *Let*

(iii′) *$V_{11} \in N(M)$ or $V_{22} \in N(M)$ be parallel in $N(M)$.*
Then M is a part of a 2-dimensional sphere in E^4.

It is sufficient to put $P = 1$, $Q = 0$ or $P = 0$, $Q = 1$ in Theorem 1.

From the proof of Theorem 1, we easily see that in the case $P = Q$ we can omit the assumption (ii). But there is another interesting possibility how to do it. It is formulated in

Theorem 2. *Let M be a surface in E^4 satisfying the conditions:*

(i) *$K > 0$ on M;*

(ii) *there are orthonormal vector fields $V_1, V_2 \in T(M)$ such that linearly independent vector fileds*
$X = PV_{11} + QV_{22} \in N(M)$, $Y = RV_{11} + SV_{22} \in N(M)$, $P, Q, R, S \in \mathcal{R}$, *are parallel in $N(M)$;*

(iii) *∂M consists of umbilical points.*

Then M is a part of a 2-dimensional sphere in E^4.

Proof. The condition (ii) yields (23) and

$$d(Ra_1 + Sa_3) - (Rb_1 + Sb_3)\,\omega_3^4 = 0\,,$$
$$d(Rb_1 + Sb_3) + (Ra_1 + Sa_3)\,\omega_3^4 = 0$$

and hence according to (4)

$$(P\alpha_1 + Q\alpha_3)\,\omega^1 + (P\alpha_2 + Q\alpha_4)\,\omega^2 + 2a_2(P - Q)\,\omega_1^2 = 0\,,$$
$$(P\beta_1 + Q\beta_3)\,\omega^1 + (P\beta_2 + Q\beta_4)\,\omega^2 + 2b_2(P - Q)\,\omega_1^2 = 0\,,$$
$$(R\alpha_1 + S\alpha_3)\,\omega^1 + (R\alpha_2 + S\alpha_4)\,\omega^2 + 2a_2(R - S)\,\omega_1^2 = 0\,,$$
$$(R\beta_1 + S\beta_3)\,\omega^1 + (R\beta_2 + S\beta_4)\,\omega^2 + 2b_2(R - S)\,\omega_1^2 = 0\,,$$
$$PS - QR \neq 0\,.$$

First of all suppose $P \neq Q$, $R \neq S$. Multiply the first two equations by $R - S$ and the other two by $P - Q$. Subtracting the corresponding equations we get

$$(R - S)(P\alpha_1 + Q\alpha_3) - (P - Q)(R\alpha_1 + S\alpha_3) = 0\,,$$
$$(R - S)(P\alpha_2 + Q\alpha_4) - (P - Q)(R\alpha_2 + S\alpha_4) = 0\,,$$
$$(R - S)(P\beta_1 + Q\beta_3) - (P - Q)(R\beta_1 + S\beta_3) = 0\,,$$
$$(R - S)(P\beta_2 + Q\beta_4) - (P - Q)(R\beta_2 + S\beta_4) = 0$$

and hence

$$\alpha_1 + \alpha_3 = 0\,, \quad \alpha_2 + \alpha_4 = 0\,,$$
$$\beta_1 + \beta_3 = 0\,, \quad \beta_2 + \beta_4 = 0\,.$$

We could obtain the same relations assuming either $P = Q$ or $R = S$. The exterior differentiation of these equations and their repeated use finally implies

$$A_1 + C_1 + a_3K = 0\,, \quad C_1 + E_1 + a_1K = 0\,, \quad B_1 + D_1 = 0\,, \tag{28}$$
$$A_2 + C_2 + b_3K = 0\,, \quad C_2 + E_2 + b_1K = 0\,, \quad B_2 + D_2 = 0\,.$$

Now, consider the function f defined by (9). Since the assumption (ii) implies $k = 0$ on M, we obtain according to (10)

$$f_{11} + f_{22} - 2fK = 2V + 2\Phi + 8\varphi + 8(a_2^2 + b_2^2)\,K \tag{29}$$

where

$$V = (\alpha_1 - \alpha_3)^2 + (\alpha_2 - \alpha_4)^2 + (\beta_1 - \beta_3)^2 + (\beta_2 - \beta_4)^2 + \tag{30}$$
$$+ 4(\alpha_2^2 + \alpha_3^2) + 4(\beta_2^2 + \beta_3^2)\,,$$
$$\Phi = (a_1 - a_3)(A_1 - E_1) + (b_1 - b_3)(A_2 - E_2)\,,$$
$$\varphi = a_2(B_1 + D_1) + b_2(B_2 + D_2)\,.$$

From (28) it follows immediately that $\varphi = 0$ and

$$\Phi = [(a_1 - a_3)^2 + (b_1 - b_3)^2]\,K\,.$$

Thus the relation (29) has the form

$$f_{11} + f_{22} - 4fK = 2V$$

and the maximum principle yields our assertion.

Notice that in the case $P = Q$ we have $X = P\xi$, where ξ is the mean curvature vector field, and thus we can omit the supposition concerning the vector field Y. Analogously in the case $R = S$. (See [3].)

Corollary 2. *Let M be a surface in E^4 with the properties* (i) *and* (ii) *of Theorem* 2. *Then the condition*

(ii) *linearly independent vector fields V_{11}, $V_{22} \in N(M)$ are parallel in $N(M)$ implies that M is a part of a 2-dimensional sphere in E^4.*

We put $P = 1$, $Q = 0$, $R = 0$, $S = 1$ in Theorem 2.

Now, we introduce a certain modification of Theorem 1.

Theorem 3. *Let M be a surface in E^4, ∂M its boundary and let*

(i) *$K > 0$ on M;*

(ii) *there exist V_1, $V_2 \in T(M)$ generating an orthogonal conjugate net of lines on M;*

(iii) *$X = PV_{11} + QV_{22} \in N(M)$, $P, Q \in \mathcal{R}$, $P^2 + Q^2 > 0$, $PQ \geqq 0$, be such that $X_1, X_2 \in N(M)$ are parallel in $N(M)$;*

(iv) *∂M consist of umbilical points.*

Then M is a part of a 2-dimensional sphere in E^4.

Proof. Consider the vector field X. The parallelness of X_1, X_2 is expressed, according to (18), by the formulas

$$\begin{aligned}
&\mathrm{d}(P\alpha_1 + Q\alpha_3) - (P\beta_1 + Q\beta_3)\,\omega_3^4 = 0\,,\\
&\mathrm{d}(P\beta_1 + Q\beta_3) + (P\alpha_1 + Q\alpha_3)\,\omega_3^4 = 0\,,\\
&\mathrm{d}(P\alpha_2 + Q\alpha_4) - (P\beta_2 + Q\beta_4)\,\omega_3^4 = 0\,,\\
&\mathrm{d}(P\beta_2 + Q\beta_4) + (P\alpha_2 + Q\alpha_4)\,\omega_3^4 = 0\,.
\end{aligned}$$

Now, using (5), we obtain the equations (25) and with regard to the proof of Theorem 1 our assertion is true.

Again we have

Corollary 3. *Let M be a surface in E^4 satisfying the conditions* (i), (ii) *and* (iv) *of Theorem* 3. *Let* (iii′) $V_{111}, V_{211} \in N(M)$ *or* $V_{122}, V_{222} \in N(M)$ *be parallel in* $N(M)$. *Then* M *is a part of a 2-dimensional sphere in* E^4.

It is sufficient to put $P = 1$, $Q = 0$ or $P = 0$, $Q = 1$ in Theorem 3.

We complete the results of this corollary by

Theorem 4. *Let* M *be a surface in* E^4 *and* ∂M *its boundary. Let*

(i) $K > 0$ *on* M;

(ii) *there exist* $V_1, V_2 \in T(M)$ *generating an orthogonal conjugate net on* M;

(iii) $V_{111}, V_{222} \in N(M)$ *be parallel in* $N(M)$;

(iv) ∂M *consist of umbilical points.*

Then M *is a part of a 2-dimensional sphere in* E^4.

Proof. The assumption (ii) implies (6) and (7) on M. The condition (iii) and relations (12) yield further

$$d\alpha_1 - \beta_1\omega_3^4 = 0\,, \quad d\beta_1 + \alpha_1\omega_3^4 = 0\,,$$
$$d\alpha_4 - \beta_4\omega_3^4 = 0\,, \quad d\beta_4 + \alpha_4\omega_3^4 = 0$$

and hence using (5) and (6) we conclude

$$A_1\omega^1 + B_1\omega^2 + 3\alpha_2\omega_1^2 = 0\,, \quad A_2\omega^1 + B_2\omega^2 + 3\beta_2\omega_1^2 = 0\,,$$
$$D_1\omega^1 + E_1\omega^2 - 3\alpha_3\omega_1^2 = 0\,, \quad D_2\omega^1 + E_2\omega^2 - 3\beta_3\omega_1^2 = 0\,.$$

Thus by means of (7) we have in particular

$$(a_1 - a_3)\,A_1 + 3\alpha_2^2 = 0\,, \quad (b_1 - b_3)\,A_2 + 3\beta_2^2 = 0\,,$$
$$(a_1 - a_3)\,E_1 - 3\alpha_3^2 = 0\,, \quad (b_1 - b_3)\,E_2 - 3\beta_3^2 = 0\,. \tag{31}$$

Now, because of (6), the equation (29) has the form

$$f_{11} + f_{22} - 2fK = 2V + 2\Phi\,,$$

the functions V, Φ being defined by (30). According to (30) and (31) we get

$$\Phi = -3(\alpha_2^2 + \alpha_3^2 + \beta_2^2 + \beta_3^2)$$

so that $V + \Phi \geqq 0$. This and the maximum principle complete the proof.

4. We revert to the considerations concerning the normal vector field X and we prove the following assertion generalizing Theorem 3.

Theorem 5. *Let* M *be a surface in* E^4 *and* ∂M *its boundary. Let*

(i) $K > 0$ *on* M;

(ii) *there exist* $V_1, V_2 \in T(M)$ *generating an orthogonal conjugate net of lines on* M;

(iii) $X = PV_{11} + QV_{22} \in N(M)$, $P, Q \in \mathscr{R}$, $P^2 + Q^2 > 0$, $PQ \geqq 0$, *be such that*

(a) $\langle X_{11} + S(X_{12} - X_{21}), V_{11} - V_{22} \rangle \geqq 0$

on M, *where* $S : M \to \mathscr{R}$ *is a function with* $S^2 \leqq \frac{3}{7}$, *and*

(b) $X_2 \in N(M)$ *is parallel in* $N(M)$

or

(iii') $X = PV_{11} + QV_{22} \in N(M)$, $P, Q \in \mathscr{R}$, $P^2 + Q^2 > 0$, $PQ \geqq 0$, *be such that*

(a)' $\langle -X_{22} + S(X_{12} - X_{21}), V_{11} - V_{22} \rangle \geqq 0$

on M, $S : M \to \mathscr{R}$ *being a function satisfying* $S^2 \leqq \frac{3}{7}$, *and*

(b') $X_1 \in N(M)$ *is parallel in* $N(M)$;

(iv) ∂M *consist of umbilical points.*

Then M *is a part of a 2-dimensional sphere in* E^4.

Proof. We prove Theorem 5 under the supposition (iii), its proof with (iii') being analogous.

The condition (ii) implies (6) and (7) on M, and according to (11) and (19) the assumption (iii) (a) yields

$$(32) \qquad (a_1 - a_3)[PA_1 + Q(C_1 + a_3K)] + (b_1 - b_3)[PA_2 + Q(C_2 + b_3K)] = \\ = \langle X_{11} + S(X_{12} - X_{21}), V_{11} - V_{22} \rangle - \\ - \alpha_2[(3P - 2Q)\alpha_2 + Q\alpha_4] - \beta_2[(3P - 2Q)\beta_2 + Q\beta_4] + \\ + S[P(\alpha_1\alpha_2 + \beta_1\beta_2) + (P + Q)(\alpha_2\alpha_3 + \beta_2\beta_3) + Q(\alpha_3\alpha_4 + \beta_3\beta_4)] \,.$$

The condition (iii) (b) is expressed by the two last equations of (25) from the proof of Theorem 3. Following the proof of Theorem 1, we have the last two equations of (26) and adding them we obtain

$$(a_1 - a_3)[P(C_1 + a_1K) + QE_1] + (b_1 - b_3)[P(C_2 + b_1K) + QE_2] = \\ = \alpha_3[P\alpha_1 - (2P - 3Q)\alpha_3] + \beta_3[P\beta_1 - (2P - 3Q)\beta_3] \,.$$

Using this relation and (32), we get from (22)

$$(33) \qquad \Phi = \langle X_{11} + S(X_{12} - X_{21}), V_{11} - V_{22} \rangle + \\ + [(a_1 - a_3)(Pa_1 - Qa_3) + (b_1 - b_3)(Pb_1 - Qb_3)]K - \\ - \alpha_2[(3P - 2Q)\alpha_2 + Q\alpha_4] - \beta_2[(3P - 2Q)\beta_2 + Q\beta_4] - \\ - \alpha_3[P\alpha_1 - (2P - 3Q)\alpha_3] - \beta_3[P\beta_1 - (2P - 3Q)\beta_3] + \\ + S[P(\alpha_1\alpha_2 + \beta_1\beta_2) + (P + Q)(\alpha_2\alpha_3 + \beta_2\beta_3) + Q(\alpha_3\alpha_4 + \beta_3\beta_4)]$$

and thus the equation (20) has the form (27) with

$$W = \langle X_{11} + S(X_{12} - X_{21}),\ V_{11} - V_{22}\rangle + V -$$
$$- (3P - 2Q)(\alpha_2^2 + \beta_2^2) + (2P - 3Q)(\alpha_3^2 + \beta_3^2) -$$
$$- P(\alpha_1\alpha_3 + \beta_1\beta_3) - Q(\alpha_2\alpha_4 + \beta_2\beta_4) +$$
$$+ S[P(\alpha_1\alpha_2 + \beta_1\beta_2) + (P + Q)(\alpha_2\alpha_3 + \beta_2\beta_3) + Q(\alpha_3\alpha_4 + \beta_3\beta_4)],$$

V being the function defined by (21). Using (21) we obtain

$$W = \langle X_{11} + S(X_{12} - X_{21}),\ V_{11} - V_{22}\rangle +$$
$$+ P[(\alpha_1 - \alpha_3)^2 + (\beta_1 - \beta_3)^2] + Q[(\alpha_2 - \alpha_4)^2 + (\beta_2 - \beta_4)^2] +$$
$$+ (P + 2Q)(\alpha_2^2 + \beta_2^2) + (2P + Q)(\alpha_3^2 + \beta_3^2) -$$
$$- P(\alpha_1\alpha_3 + \beta_1\beta_3) - Q(\alpha_2\alpha_4 + \beta_2\beta_4) +$$
$$+ S[P(\alpha_1\alpha_2 + \beta_1\beta_2) + (P + Q)(\alpha_2\alpha_3 + \beta_2\beta_3) + Q(\alpha_3\alpha_4 + \beta_3\beta_4)]$$

and hence

(34)
$$W = \langle X_{11} + S(X_{12} - X_{21}),\ V_{11} - V_{22}\rangle +$$
$$+ P[(\alpha_1 - \tfrac{3}{2}\alpha_3 + \tfrac{1}{2}S\alpha_2)^2 + (\beta_1 - \tfrac{3}{2}\beta_3 + \tfrac{1}{2}S\beta_2)^2] +$$
$$+ Q[(\alpha_4 - \tfrac{3}{2}\alpha_2 + \tfrac{1}{2}S\alpha_3)^2 + (\beta_4 - \tfrac{3}{2}\beta_2 + \tfrac{1}{2}S\beta_3)^2] +$$
$$+ \tfrac{1}{4}P[\varphi(\alpha_2, \alpha_3) + \varphi(\beta_2, \beta_3)] + \tfrac{1}{4}Q[\varphi(\alpha_3, \alpha_2) + \varphi(\beta_3, \beta_2)]$$

where

$$\varphi(x, y) = (4 - S^2)\,x^2 + 10xy + 3y^2\,.$$

The quadratic form $\varphi(x, y)$ being non-negative for all x and y in virtue of $|S| \leqq \sqrt{\frac{3}{7}}$, we have $W \geqq 0$. Considerations analogous to those from the proof of Theorem 1 imply the validity of our assertion.

Remark. The special case $(P = 1,\ Q = 1)$ of the preceding theorem was proved in [3] under the supposition that S satisfies the inequality $|S| \leqq 4\sqrt{(2)} - 5$. As $\frac{3}{7} < 4\sqrt{(2)} - 5$, the result obtained in [3] is a little better than that of Theorem 5. However, in the case $PQ > 0$, we can replace the inequality $|S| \leqq \sqrt{\frac{3}{7}}$ independent of P, Q by a more suitable one. In fact, the last two terms on the right-hand side of (34) are equal to the sum of two quadratic forms of the type

$$(4P + 3Q - PS^2)\,x^2 + 10(P + Q)\,Sxy + (3P + 4Q - QS^2)\,y^2$$

which are non-negative for all S satisfying

$$S^2 \leqq (PQ)^{-1}\,[14(P + Q)^2 + PQ] - 2|P + Q|\sqrt{[49(P + Q)^2 + 4PQ]}\,.$$

A special case of Theorem 5 is this

Corollary 4. *Let M be a surface in E^4 possessing the properties* (i), (ii) *and* (iv) *of Theorem* 5. *Let*

(iii) $X = PV_{11} + QV_{22} \in N(M)$, $P, Q \in \mathscr{R}$, $P^2 + Q^2 > 0$, $PQ \geqq 0$, *be such that*

(a) $X_{11} + S(X_{12} - X_{21}) = 0$ *on* M, $S : M \to \mathscr{R}$ *being a function with* $S^2 \leqq \frac{3}{7}$, *and*

(b) $X_2 \in N(M)$ *is parallel in* $N(M)$

or

(iii') $X = PV_{11} + QV_{22} \in N(M)$, $P, Q \in \mathscr{R}$, $P^2 + Q^2 > 0$, $PQ \geqq 0$, *be such that*

(a') $-X_{22} + S(X_{12} - X_{21}) = 0$ *on* M, *where* $S : M \to \mathscr{R}$ *is a function satisfying* $S^2 \leqq \frac{3}{7}$, *and*

(b') $X_1 \in N(M)$ *is parallel in* $N(M)$.

Then M is a part of a 2-dimensional sphere in E^4.

From the other special cases of Theorem 5 concerning the vector fields V_{11}, V_{22} we introduce only those restricted by $S = 0$.

Corollary 5. *Let M be a surface in E^4 satisfying the conditions* (i), (ii) *and* (iv) *of Theorem* 5. *Let*

(iii) (a) $\langle V_{1111}, V_{11} - V_{22} \rangle \geqq 0$ *on* M *and*

(b) $V_{211} \in N(M)$ *be parallel in* $N(M)$

or

(a) $\langle V_{1122}, V_{11} - V_{22} \rangle \geqq 0$ *on* M *and*

(b) $V_{222} \in N(M)$ *be parallel in* $N(M)$

or

(iii') (a') $\langle -V_{2211}, V_{11} - V_{22} \rangle \geqq 0$ *on* M *and*

(b') $V_{111} \in N(M)$ *be parallel in* $N(M)$

or

(a') $\langle -V_{2222}, V_{11} - V_{22} \rangle \geqq 0$ *on* M *and*

(b') $V_{122} \in N(M)$ *be parallel in* $N(M)$.

Then M is a part of a 2-dimensional sphere in E^4.

The result follows from Theorem 5 by Lemma 2 for $P = 1$, $Q = 0$ or $P = 0$, $Q = 1$ and $S = 0$.

We complete the assertions of Corollary 5 by

Theorem 6. *Let M be a surface in E^4 and ∂M its boundary. Let*

(i) $K > 0$ *on* M;

(ii) *there exist* $V_1, V_2 \in T(M)$ *generating an orthogonal conjugate net on* M;

(iii) (a) $\langle V_{1111}, V_{11} - V_{22}\rangle \geqq 0$ *on* M *and*

(b) $V_{222} \in N(M)$ *be parallel in* $N(M)$

or

(iii′) (a′) $\langle -V_{2222}, V_{11} - V_{22}\rangle \geqq 0$ *on* M *and*

(b′) $V_{111} \in N(M)$ *be parallel in* $N(M)$;

(iv) ∂M *consist of umbilical points.*

Then M *is a part of a 2-dimensional sphere in* E^4.

Proof. We have (6) and (7) on M. From (11), (13) we get

$$\langle V_{1111}, V_{11} - V_{22}\rangle = (a_1 - a_3) A_1 + (b_1 - b_3) A_2 + 3(\alpha_2^2 + \beta_2^2)\,.$$

The condition (iii) (b) is expressed by the two last equations of (31) from the proof of Theorem 4. Thus (30) implies

$$_{\nu}\Phi = \langle V_{1111}, V_{11} - V_{22}\rangle - 3(\alpha_2^2 + \alpha_3^2 + \beta_2^2 + \beta_3^2)$$

and hence

(35) $$f_{11} + f_{22} - 2fK = 2W$$

where

$$W = V + \Phi = \langle V_{1111}, V_{11} - V_{22}\rangle +$$
$$+ (\alpha_1 - \alpha_3)^2 + (\alpha_2 - \alpha_4)^2 + (\beta_1 - \beta_3)^2 + (\beta_2 - \beta_4)^2 + \alpha_2^2 + \alpha_3^2 + \beta_2^2 + \beta_3^2\,,$$

V being defined by (30). The maximum principle completes our proof.

A generalization of Theorem 5 is given by the following

Theorem 7. *Let* M *be a surface in* E^4 *and* ∂M *its boundary. Let*

(i) $K > 0$ *on* M;

(ii) *there exist* $V_1, V_2 \in T(M)$ *generating an orthogonal conjugate net of lines on* M;

(iii) $X = PV_{11} + QV_{22} \in N(M)$, $P, Q \in \mathcal{R}$, $P^2 + Q^2 > 0$, $PQ \geqq 0$, *be such that*

$$\langle X_{11} - X_{22} + S(X_{12} - X_{21}), V_{11} - V_{22}\rangle \geqq 0$$

on M, $S : M \to \mathcal{R}$ *being a function with* $S^2 \leqq \frac{3}{7}$;

(iv) ∂M *consist of umbilical points.*

Then M *is a part of a 2-dimensional sphere in* E^4.

Proof. We choose orthonormal frames in the usual way and we have the relations (6) and (7), and the equations (20), (21) and (22) on M.

Using (11) and (19) we see that the expression (22) has the form

$$\begin{aligned}\Phi = \langle X_{11} - X_{22} + S(X_{12} - X_{21}), V_{11} - V_{22}\rangle + \\ + [(a_1 - a_3)(Pa_1 - Qa_3) + (b_1 - b_3)(Pb_1 - Qb_3)]\,K - \\ - \alpha_2[(3P - 2Q)\,\alpha_2 + Q\alpha_4] - \beta_2[(3P - 2Q)\,\beta_2 + Q\beta_4] - \\ - \alpha_3[P\alpha_1 - (2P - 3Q)\,\alpha_3] - \beta_3[P\beta_1 - (2P - 3Q)\,\beta_3] + \\ + S[P(\alpha_1\alpha_2 + \beta_1\beta_2) + (P + Q)(\alpha_2\alpha_3 + \beta_2\beta_3) + Q(\alpha_3\alpha_4 + \beta_3\beta_4)]\,.\end{aligned}$$

This relation is, however, formally the same as (33), so that we have (27) where W is given by (34) when writing $X_{11} - X_{22} + S(X_{12} - X_{21})$ instead of $X_{11} + S(X_{12} - X_{21})$. Thus the assertion is proved.

First of all let us introduce this trivial

Corollary 6. *Let M be a surface in E^4 possessing the properties* (i), (ii) *and* (iv) *of Theorem 7. Let*

(iii) $X = PV_{11} + QV_{22} \in N(M)$, $P, Q \in \mathcal{R}$, $P^2 + Q^2 > 0$, $PQ \geqq 0$, *be such that* $X_{11} - X_{22} + S(X_{12} - X_{21}) = 0$ *on M, S being a real-valued function on M such that* $S^2 \leqq \frac{3}{7}$.

Then M is a part of a 2-dimensional sphere in E^4.

Theorem 7, as a very special case, contains these two results:

Corollary 7. *Let M be a surface in E^4 satisfying the conditions* (i), (ii) *and* (iv) *of Theorem 7. Let*

(iii) $\langle V_{1111} - V_{2211}, V_{11} - V_{22}\rangle \geqq 0$ *on M*

or

(iii′) $\langle V_{1122} - V_{2222}, V_{11} - V_{22}\rangle \geqq 0$ *on M.*

Then M is a part of a 2-dimensional sphere in E^4.

Both the assertions follow from Theorem 7 and Lemma 2 for $P = 1$, $Q = 0$ or $P = 0$, $Q = 1$ and $S = 0$.

We complete again these two results by

Theorem 8. *Let M be a surface in E^4 and let*

(i) $K > 0$ *on M;*

(ii) *there exist $V_1, V_2 \in T(M)$ generating an orthogonal conjugate net on M;*

(iii) $\langle V_{1111} - V_{2222}, V_{11} - V_{22}\rangle \geqq 0$ *on M;*

(iv) *∂M consist of umbilical points.*