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ON EXCEPTIONAL VALUES OF HOLOMORPHIC MAPPINGS OF RIEMANN SURFACES

ALOIS KLÍČ, Praha

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INTRODUCTION

In the framework of the classical Nevanlinna theory, see [4]–[9], sufficient conditions for $\delta(a) = 0$ have been examined. In this paper we shall solve the same problem for the case of holomorphic mappings $f : \mathbf{V} \to \mathbf{M}$, where $\mathbf{V}$ is an open Riemann surface and $\mathbf{M}$ is a closed Riemann surface. Our basic reference is [2].

The layout of this paper is as follows. Section 1 contains basic concepts and denotations and First and Second Main Theorems. Basic concepts from the Ahlfors theory of covering surfaces are briefly introduced as necessary for the formulation of the Ahlfors covering theorem. The Ahlfors covering theorem is used in Section 2 to derive a generalization of Cartan's formula. Sufficient conditions for $\delta(a) = 0$ are treated in Sections 3 and 4; they are followed by several closing notes in Section 5.

1. DENOTATIONS, BASIC CONCEPTS AND THEOREMS

From now on, it is assumed that $\mathbf{V}$ is an open Riemann surface, $\mathbf{M}$ is a compact Riemann surface and $f : \mathbf{V} \to \mathbf{M}$ is a holomorphic mapping. This standard notation will be adhered to throughout the paper. We shall assume that a *harmonic exhaustion* exists on $\mathbf{V}$.

Definition 1.1. A function $\tau : \mathbf{V} \to [0, s)$, $(s \leqq \infty)$ will be called a *harmonic exhaustion* on the open Riemann surface $\mathbf{V}$ iff

(i) $\tau : \mathbf{V} \to [0, s)$ is onto.
(ii) $\tau \in \mathbf{C}^\infty(\mathbf{V})$.
(iii) τ is proper, i.e. the inverse image of a compact set is compact.
(iv) τ is eventually harmonic, i.e. there exists a number $r(\tau)$, $0 \leqq r(\tau) < s$, such that τ is harmonic on $\{p : \tau(p) \geqq r(\tau)\}$.

If $s < \infty$, we say that $\mathbf{V}$ admits a *finite harmonic exhaustion.* If $s = \infty$, we say that $\mathbf{V}$ admits an *infinite harmonic exhaustion.*

Theorem 1.0. *A Riemann surface is parabolic if and only if it carries a harmonic exhaustion with $s = \infty$.*

The first part of Theorem 1.0 was proved by NAKAI [12]. If the surface carries a harmonic exhaustion with $s = \infty$, then such surface must be parabolic, because the harmonic measure of the ideal boundary is zero, see [13], p. 204, 6E.

Example. In the classical case of meromorphic functions on $\mathbf{V} = \mathbf{C}$, $\log |z|$ outside of a certain disc can be used as an exhaustion function.

Denotation 1.1. The following denotation will be used:

(i) $V[r] = \{p \in \mathbf{V},\ \tau(p) \leqq r\}$,

(ii) $\partial V[r] = \{p \in \mathbf{V},\ \tau(p) = r\}$,

(iii) $n(r, a)$ denotes the number of pre-images of a in $V[r]$, counting multiplicity,

(iv) $N(r, a) = \int_{r_0}^{r} n(t, a)\, dt$, where $r_0 \geqq r(\tau)$ is a given number.

Definition 1.2. A real number r is called a *critical value* of a function τ, if $\tau^{-1}(r)$ contains a critical point of τ.

Remark 1.1. $\int_{\partial V[r]} * \mathrm{d}\tau$ is a constant for all $r \geqq r(\tau)$.

For the proof see Corollary 4.3 in [2]; an easy proof can be obtained with help of Green's formula, see [13], p. 133. If we apply Greens formula to the function τ that is harmonic on the compact bordered surface $W = \overline{V[r] \setminus V[r_0]}$, we obtain

$$0 = \int_{\partial W} * \mathrm{d}\tau = \int_{\partial V[r]} * \mathrm{d}\tau - \int_{\partial V[r_0]} * \mathrm{d}\tau .$$

Thus $\int_{\partial V[r]} * \mathrm{d}\tau = \text{const.}$ for every $r \geqq r_0 \geqq r(\tau)$, QED.

With the help of the function τ a holomorphic function ζ is constructed. This function will be used as a coordinate function (or local parameter). Let us suppose that (r_1, r_2) does not contain any critical value of τ, and suppose W is one component of Int $(V[r_2] \setminus V[r_1])$. Let γ be one of the level curves of τ in W, say, $\gamma = W \cap \partial V[r]$ for some $r \in (r_1, r_2)$. We call $\int_\gamma * \mathrm{d}\tau = \Gamma$ a period of $* \mathrm{d}\tau$ and define the conjugate harmonic function ϱ of τ by $\varrho(p) = \int_{p_0}^{p} * \mathrm{d}\tau$, where p_0 is a fixed point of γ. Since $\mathrm{d}(*\mathrm{d}\tau) = 0$, ϱ is defined up to periods, i.e. up to integral multiples of Γ and $\mathrm{d}\varrho = *\mathrm{d}\tau$. Consequently, $\zeta = \tau + \mathrm{i}\varrho$ is a holomorphic function on W, but it is multi-valued. Let $\alpha = \{p \in W,\ \varrho(p) = 0 \pmod{\Gamma}\}$.

Lemma 1.1. *For $\zeta = \tau + \mathrm{i}\varrho$ as above, $\zeta : W \setminus \alpha \to (r_1, r_2) \times (0, \Gamma) \subset \mathbf{C}$ is biholomorphic.*

Proof see [2].

Definition 1.3. The holomorphic coordinate function ζ from Lemma 1.1 in a component of $\tau^{-1}((r_1, r_2))$ is called *a special coordinate function*.

Remark 1.2. It is well known (see Lemma 1.2 in [2]) that on every compact Riemann surface $\boldsymbol{M}$ a Hermitian metric of constant Gaussian curvature can be introduced. Let us denote this metric by G and its volume element by Ω. In local coordinates we have:

$$G = g(\mathrm{d}x^2 + \mathrm{d}y^2), \quad \Omega = g\,\mathrm{d}x \wedge \mathrm{d}y .$$

By a suitable choice of a multiplicative constant the standardization $\int_{\boldsymbol{M}} \Omega = 1$ is easily achieved.

Theorem 1.1. *Let $a \in \boldsymbol{M}$ be a fixed point of $\boldsymbol{M}$. Then there exists a real-valued function u_a with the following properties*:

1. u_a *is* $\mathbf{C}^\infty$ *in* $\boldsymbol{M} \setminus \{a\}$.
2. $\dfrac{1}{2\pi}\,\mathrm{dd^c}u_a = \Omega$ *in* $\boldsymbol{M} \setminus \{a\}$.
3. *If z is any a-centered holomorphic coordinate function in a neighborhood U of a, then $u_a(z) + \log|z|$ is $\mathbf{C}^\infty$ on U.*
4. $u_a \geqq 0$.

Proof see [2].

Denotation 1.2. Let us set

(i) $$v(r) = \int_{V[r]} f^*\Omega ,$$

(ii) $$m(r, a) = \frac{1}{2\pi}\int_{\partial V[r]} f^*u_a * \mathrm{d}\tau ,$$

(iii) $$T(r) \int_{r_0}^{r} v(t)\,\mathrm{d}t ,$$

$T(r)$ is the *Nevanlinna characteristic function* of the mapping f,

(iv) $$E(r) = \int_{r_0}^{r} \chi(t)\,\mathrm{d}t ,$$

where $\chi(t)$ denotes the *Euler characteristic* of $V[t]$ ($\chi(\boldsymbol{S}) = +2$, where $\boldsymbol{S}$ denotes the Riemann sphere),

(v) $$\chi = \limsup_{r \to s} \frac{-E(r)}{T(r)} .$$

Remark 1.3. Let us recall that for a nonconstant holomorphic mapping $f: \boldsymbol{V} \to \boldsymbol{M}$, where $\boldsymbol{V}$ is a parabolic, we have $T(r) \to \infty$ as $r \to \infty$, for

$$T(r) = \int_{r_0}^{r} v(t)\,\mathrm{d}t > v(r_0)(r - r_0) \to \infty \quad \text{as} \quad r \to \infty .$$

Remark 1.4 .A nonnegative function h is defined by the relation

(1) $$f^*\Omega = h\,\mathrm{d}\tau \wedge *\mathrm{d}\tau .$$

The function h is not defined at the critical points of τ; see [2], p. 501.

Theorem 1.2. (First Main Theorem). *For every* $r \geqq r(\tau)$,

(2) $$m(r, a) + N(r, a) = T(r) + m(r_0, a) .$$

Definition 1.4. Let us put (for $a \in \boldsymbol{M}$)

$$\delta(a) = \limsup_{r \to s} \left(1 - \frac{N(r, a)}{T(r)}\right).$$

The quantity $\delta(a)$ is called *the defect of the value a*. If $\delta(a) > 0$ then the value a is said to be the *deficient* or *Nevanlinna exceptional value.*

Remark 1.5. For a mapping with an unbounded characteristic function $T(r)$ (i.e. $\lim_{r \to s} T(r) = \infty$), the quantity $\delta(a)$ can be defined by the relation

(3) $$\delta(a) = \liminf_{r \to s} \frac{m(r, a)}{T(r)}$$

as can be seen from relation (2) in Theorem 1.2.

Theorem 1.3 (Defect relations). *If* $f: \boldsymbol{V} \to \boldsymbol{M}$ *is holomorphic and* $\boldsymbol{V}$ *admits an infinite harmonic exhaustion, then*

(4) $$\sum_{a \in \boldsymbol{M}} \delta(a) \leqq \chi(\boldsymbol{M}) + \chi .$$

If $\boldsymbol{V}$ *admits a finite harmonic exhaustion, then*

(5) $$\sum_{a \in \boldsymbol{M}} \delta(a) \leqq \chi(\boldsymbol{M}) + \chi + \varepsilon ,$$

where

$$\varepsilon = \limsup_{r \to s} \frac{1}{T(r)} \left\{\frac{L}{4\pi} \log\left(T(r) + \text{const.}\right) + 2 \log \frac{1}{s - r} + \text{const.}\right\}.$$

Basic concepts from the Ahlfors theory of covering surfaces will now be introduced. For a more detailed study we refer to [11], Vol. 2, Chap. VII and X.

Let W and W_0 be two topological spaces and $F : W \to W_0$ a continuous mapping from W into W_0. Let $\mathfrak{N}$ be the set of ordered pairs $[p, F(p)] = [p, p_0]$, $p \in W$ $p_0 \in W_0$, $p_0 = F(p)$. The set $\mathfrak{N}$ will be endowed with a topology in this manner: By a neighborhood of a point $[q, q_0] \in \mathfrak{N}$ we understand the set of pairs $[p, p_0] \in \mathfrak{N}$, where p moves through a neighborhood of the point q. The set $\mathfrak{N}$, endowed with this topology, will be denoted by $(W_0)_F^W$ and $(W_0)_F^W$ will be called a *covering space of the base space* W_0. The point $p_0 \in W_0$ associated with the point $[p, p_0] \in (W_0)_F^W$ is called the *trace point* of $[p, p_0]$; one also says that $[p, p_0]$ *lies over* p_0.

Let W_0 be a compact surface with a *normal metric*, let W be an arbitrary topological surface and F an *inner* mapping. (For definitions of the *normal* metric and the *inner* mapping we refer to [11].) The metric on the base surface W_0 is carried over to the surface W with help of the mapping F in the following manner:

(i) **Length of a curve.** The properties of the inner mapping imply: Each curve β on W can be decomposed into parts, on which the mapping F is *topological* and with each such part the length of its image on the base surface W_0 is associated. The total length of the curve β is equal to the sum of the lengths of these parts.

(ii) **Area.** Let $D \subset W$ be a compact region. We can decompose the region D into parts, on which the mapping F is *topological* and we associate with each such part the area of its image on the base surface W_0. The total area of the region D is equal to the sum of the area of these parts.

Let $W_r \subset W$ be a compact polyhedral region with a boundary ∂W_r. The covering surface $(W_0)_F^{W_r}$ is contained in the covering surface $(W_0)_F^W$.

Definition 1.5. (i) The quantity

$$S_r = \frac{A_r}{A_0}$$

(where A_r is the total area of W_r and A_0 is the area of the base surface W_0) is called the *mean sheat number of* W_r *over* W_0.

(ii) Let γ be a *curve* on W_0. The quantity

$$s_r(\gamma) = \frac{L_r(\gamma)}{L_0(\gamma)}$$

(where $L_0(\gamma)$ stands for the length of γ and $L_r(\gamma)$ for the total length of the arcs *lying over* γ on W_r), is called the *mean sheet number over the curve* γ.

Theorem 1.4 (Ahlfors Covering Theorem). *Let* γ *be a regular curve on the base* *surface W_0. Then there exists a finite number k dependent only on W_0 and γ, such that*

$$|S_r - s_r(\gamma)| \leqq kL_r$$

where L_r is the length of ∂W_r.

2. GENERALIZATION OF CARTAN'S FORMULA

Let us interpret the mapping $f : \mathbf{V} \to \mathbf{M}$ as a covering mapping. Then $(\mathbf{M})_f^{\mathbf{V}}$ is a covering surface. It is possible to assume the metric G on the surface $\mathbf{M}$ (from Remark 1.2) to be normal. By the uniformization theorem, every compact Riemann surface is covered by either the complex plane, or the unit disc, or the Riemann sphere. These three spaces can be equipped with a Hermitian metric that is normal. Consequently, the surface $\mathbf{M}$ can also be equipped with such a metric.

The quantity

$$v(r) = \int_{V[r]} f^*\Omega$$

is the *mean sheet number* of $V[r]$ over $\mathbf{M}$ (for $\int_{\mathbf{M}} \Omega = 1$). The quantity $s_r(\gamma)$ will be denoted from now on by $s(r, \gamma)$. If ds denotes the *element of arc-length* of the curve γ, then the following identity is evident:

$$s(r, \gamma) = \frac{1}{L_0(\gamma)} \int_\gamma n(r, a)\, ds(a)\,. \tag{7}$$

With this notation, the relation (6) from Theorem 1.4 yields

$$|v(r) - s(r, \gamma)| \leqq k\, L(r)\,, \tag{8}$$

where $L(r)$ is the length of the curve $\partial V[r]$ in the metric $f^*\Omega$. Thus

$$L(r) = \int_{\partial V[r]} h^{1/2} * d\tau\,, \tag{9}$$

where the function h is defined by the relation (1) from Remark 1.3.

The following well-known lemma will be used:

Lemma 2.1. *Suppose ψ is a once continuously differentiable, positive, increasing function on $[r_0, \infty)$. Then for any real number $\varepsilon > 0$, $\psi'(t) \leqq \{\psi(t)\}^{1+\varepsilon}$ on $[r_0, \infty)$ except on an open set $I \subset [r_0, \infty)$ for which $\int_I dt < \infty$.*

Definition 2.1. Let $S(r, f)$ denote the quantity defined by

$$S(r, f) = o\{T(r)\}$$

for $r \to \infty$, $r \in [r_0, \infty) \setminus I$, $\int_I dx < \infty$.

Lemma 2.2. *Let $f : \mathbf{V} \to \mathbf{M}$ be a nonconstant holomorphic mapping from a parabolic Riemann surface $\mathbf{V}$ into $\mathbf{M}$. Then*

$$\int_{r_0}^{r} L(t)\,\mathrm{d}t = S(r, f)\,.$$

Proof. Let K_0 denote the set of critical values of the function τ. Then

$$\int_{r_0}^{r} L(t)\,\mathrm{d}t = \int_{(r_0,r)\setminus K_0} L(t)\,\mathrm{d}t = \int_{(r_0,r)\setminus K_0} \left(\int_{\partial V[t]} h^{1/2} *\mathrm{d}\tau \right) \mathrm{d}t\,. \tag{10}$$

If the Schwarz inequality is applied to the inner integral, we obtain

$$\int_{\partial V[t]} h^{1/2} *\mathrm{d}\tau \leqq \left\{ \int_{\partial V[t]} h *\mathrm{d}\tau \right\}^{1/2} \left\{ \int_{\partial V[t]} *\mathrm{d}\tau \right\}^{1/2} = \sqrt{L} \left\{ \int_{\partial V[t]} h *\mathrm{d}\tau \right\}^{1/2}. \tag{11}$$

The following result is obtained from (10), (11) by repeated application of the Schwarz inequality:

$$\begin{aligned}
\int_{r_0}^{r} L(t)\,\mathrm{d}t &\leqq \sqrt{L} \int_{(r_0,r)\setminus K_0} \left(\int_{\partial V[t]} h *\mathrm{d}\tau \right)^{1/2} \mathrm{d}t \leqq \\
&\leqq \sqrt{L} \left\{ \int_{(r_0,r)\setminus K_0} \left(\int_{\partial V[t]} h *\mathrm{d}\tau \right) \mathrm{d}t \right\}^{1/2} \left(\int_{r_0}^{r} \mathrm{d}t \right)^{1/2} = \\
&= \sqrt{L} \sqrt{(r - r_0)} \left\{ \int_{(r_0,r)\setminus K_0} \left(\int_{\partial V[t]} h *\mathrm{d}\tau \right) \mathrm{d}t \right\}^{1/2} \leqq \\
&\leqq \sqrt{L} \sqrt{(r - r_0)} \sqrt{\left(\int_{V[r]\setminus V[r_0]} h\,\mathrm{d}\tau \wedge *\mathrm{d}\tau \right)} = \\
&= \sqrt{L} \sqrt{(r - r_0)} \sqrt{(v(r) - v(r_0))} \leqq \sqrt{L} \sqrt{(r\,.\,v(r))}\,,
\end{aligned}$$

i.e. we proved

$$\int_{r_0}^{r} L(t)\,\mathrm{d}t \leqq \sqrt{L} \sqrt{(r\,.\,v(r))}\,. \tag{12}$$

As

$$r\,.\,v(r) = r \frac{\mathrm{d}T(r)}{\mathrm{d}r} = \frac{\mathrm{d}T(r)}{\mathrm{d}\ln r}\,,$$

Lemma 2.1 applied to the function $\psi(t) = T(r)$, where $t = \ln r$ and $\varepsilon = \frac{1}{2}$ yields

$$r\,v(r) \leqq \{T(r)\}^{3/2} \tag{13}$$

outside of the set I for which $\int_I \ln x\,\mathrm{d}x < \infty$. Finally, as a consequence of (12), (13), we obtain

$$\int_{r_0}^{r} L(t)\,\mathrm{d}t \leqq \sqrt{L}\,\{T(r)\}^{3/4} = S(r, f)\,, \qquad \text{QED.}$$

Lemma 2.3. *$n(t, a)$ is a measurable function on $[0, s) \times \boldsymbol{M}$.*

Corollary. *$n(t, a)$ is an integrable function on $[r_1, r_2] \times \gamma$, where $[r_1, r_2]$ is any finite closed subinterval of $(r(\tau), s)$ and γ is a regular curve on $\boldsymbol{M}$.*

Proof of Lemma 2.3. Let $(t, a) \in [0, s) \times \boldsymbol{M}$, $(t_i, a_i) \to (t, a)$ as $i \to \infty$. We will prove

$$\limsup_{i \to \infty} n(t_i, a_i) \leqq n(t, a) ,$$

which is equivalent to the semi-continuity. Let $\{t_i\} = \{r_i\} \cup \{s_i\}$ where $r_i \geqq t$, $s_i \leqq t$. It suffices to prove

(i) $\limsup_{i \to \infty} n(r_i, a_i) \leqq n(t, a)$,

(ii) $\limsup_{i \to \infty} n(s_i, a_i) \leqq n(t, a)$.

First we prove the following statement: *There exists such a neighborhood U of the point a that*

(iii) *$n(t, a') \leqq n(t, a)$ for all $a' \in U$.*

Let us denote $f^{-1}(a) \cap V[t] = \{p_1, \ldots, p_k, q_1, \ldots, q_m\}$ where $\{p_1, \ldots, p_k\} \in V[t] \setminus \partial V[t]$ and $\{q_1, \ldots, q_m\} \in \partial V[t]$. We can choose neighborhoods $U_1, \ldots, U_k$ of the points $p_1, \ldots, p_k$ with the following properties:

a) $U_i \subset V[t] \setminus \partial V[t]$ for $i = 1, 2, \ldots, k$;

b) If f assumes the value a at the point p_i with a multiplicity m_i $(i = 1, 2, \ldots, k)$, then for an arbitrary $a' \in U$ exactly m_i distinct simple roots of the equation $f(z) = a'$ lie in U_i. Hence for an arbitrary $a' \in U$, the contributions of the points from U_i to $n(t, a')$ and $n(t, a)$ are equal.

We will now investigate the points $q_1, \ldots, q_m$. Let the neighborhoods $V_1, \ldots, V_m$ of the points $q_1, \ldots, q_m$ have the property *b)* above. Then for an arbitrary $a' \in U$ the contributions of the points from V_j to $n(t, a')$ are less or equal than to $n(t, a)$, as some preimages of the point a' can lie outside of $V[t]$.

We will first prove (ii). It is self-evident that $n(s_i, a_i) \leqq n(t, a_i)$ and according to (iii) $n(t, a_i) \leqq n(t, a)$ for every i sufficiently large. Hence (ii) is valid. As for (i) we will prove it by contradiction. Assume (i) is false, then

$$\limsup_{i \to \infty} n(r_i, a_i) > n(t, a) .$$

Passing to a subsequence if necessary, we may assume that we have $\lim_{i \to \infty} n(r_i, a_i) > n(t, a)$, where $r_1 \geqq r_2 \ldots \geqq t$, and furthermore that $V[r_1]$ encloses the same number of preimages of the point a as $V[t]$, i.e., $n(r_1, a) = n(t, a)$. For every i sufficiently large we have $n(r_i, a_i) \leqq n(r_1, a_i) \leqq n(r_1, a) = n(t, a)$, which contradicts our assumption, therefore (i) is valid.

Proof of Corollary. The function $n(t, a)$ is semi-continuous on the set $[0, s) \times \boldsymbol{M}$, hence $n(t, a)$ is also semi-continuous on the set $[0, s) \times \gamma$. The set $[r_1, r_2] \times \gamma$ is a compact set, thus the semi-continuous function $n(t, a)$ is bounded on $[r_1, r_2] \times \gamma$ and hence integrable, QED.

Remark 2.1. Let γ be a curve on $\boldsymbol{M}$ and $L_0(\gamma)$ its length. Let $\mathrm{d}s$ be the element of arc-length of the curve γ and

$$\mathrm{d}s^0 = \frac{\mathrm{d}s}{L_0(\gamma)}.$$

The relation (7) can now be written as

$$s(r, \gamma) = \int_\gamma n(r, a)\, \mathrm{d}s^0(a).$$

Theorem 2.1 (Generalized Cartan's formula). *Let $f : \boldsymbol{V} \to \boldsymbol{M}$ be a holomorphic mapping from a parabolic Riemann surface $\boldsymbol{V}$ into $\boldsymbol{M}$, γ a regular curve on $\boldsymbol{M}$. Then*

$$T(r) = \int_\gamma N(r, a)\, \mathrm{d}s^0(a) + S(r, f). \tag{14}$$

Proof. From the relation (8) we obtain by integration

$$\left| \int_{r_0}^{r} v(t)\, \mathrm{d}t - \int_{r_0}^{r} s(t, \gamma)\, \mathrm{d}t \right| \leqq k . \int_{r_0}^{r} L(t)\, \mathrm{d}t. \tag{15}$$

Furthermore, (Fubini Theorem and Lemma 2.3)

$$\int_{r_0}^{r} s(t, \gamma)\, \mathrm{d}t = \int_{r_0}^{r} \left(\int_\gamma n(t, a)\, \mathrm{d}s^0(a) \right) \mathrm{d}t =$$

$$= \int_\gamma \left(\int_{r_0}^{r} n(t, a)\, \mathrm{d}t \right) \mathrm{d}s^0(a) = \int_\gamma N(r, a)\, \mathrm{d}s^0(a).$$

Substitution in the relation (15) yields

$$\left| T(r) - \int_\gamma N(r\ a)\, \mathrm{d}s^0(a) \right| \leqq S(r, f). \tag{16}$$

QED.

3. THE CASE OF A PARABOLIC RIEMANN SURFACE

Theorem 3.1. *Let $f : \boldsymbol{V} \to \boldsymbol{M}$ be a transcendental[1]) holomorphic mapping from a parabolic Riemann surface $\boldsymbol{V}$ into $\boldsymbol{M}$ $a_0 \in \boldsymbol{M}$, and $D(a_0)$ its neighborhood. Let the portion of surface $(\boldsymbol{M})_f^{\boldsymbol{V}}$ over the neighborhood $D(a_0)$ consist of a system of*

[1]) Let us recall that the mapping f: $\boldsymbol{V} \to \boldsymbol{M}$ is *transcendental* iff $\lim_{r \to \infty} (r/T(r)) = 0$, see [2].

regions $G_\nu \subset (\mathbf{M})_f^{\mathbf{V}}$ *with the following properties: If* $G_\nu \subset (\mathbf{M})_f^{\mathbf{V}}$ *is an arbitrary domain over* $D(a_0)$, *then over every point* $a \in D(a_0)$ *there lie just* λ_ν *points and* $1 \leqq \lambda_\nu \leqq \Lambda < \infty$; *every ramification point of an order* m *is counted* $(m + 1)$-*times. Then the value* a_0 *is not a deficient value of the mapping* f, *i.e.*

$$\delta(a_0) = 0\,.$$

Proof. Let $a_0 \in \mathbf{M}$ and let $D(a_0)$ be its neighborhood with the property required by Theorem 3.1. Then every neighborhood $U(a_0) \subset D(a_0)$ has this property. We can choose a chart $\{U, \varphi\}$ in the neighborhood of the point a_0 so that $\varphi(a_0) = 0$ and

$$\varphi(\bar{D}(a_0)) = \{z \in \mathbf{C}\ |z| \leqq 1\}\,.$$

Then

$$\gamma = \varphi^{-1}(\{z \in \mathbf{C},\ |z| = R,\ R < 1\})$$

is a regular curve in $D(a_0)$ $a_0 \in \operatorname{Int}\gamma$, where

$$\operatorname{Int}\gamma = \varphi^{-1}(\{z \in \mathbf{C},\ |z| < R\})\,.$$

Let d be the distance between γ and $\partial D(a_0)$ on the surface $\mathbf{M}$. Let $\bar{F}_\nu$ denote the closed set of the points from G_ν, lying over $\overline{\operatorname{Int}\gamma}$. The set $\bar{F}_\nu$ consists of not more than λ_ν closed regions. It is known that $\mathbf{V}$ and $(\mathbf{M})_f^{\mathbf{V}}$ are homeomorphic (even conformally equivalent), i.e. a homeomorphic mapping ψ exists,

$$\psi : \mathbf{V} \to (\mathbf{M})_f^{\mathbf{V}}\,.$$

Let us denote $D_\nu = \psi^{-1}(G_\nu)$, $\bar{C}_\nu = \psi^{-1}(\bar{F}_\nu)$. Further, let a_γ be an arbitrary point on γ. Every set D_ν contains the same number of a_0-points and a_γ-points, namely λ_ν. All a_0-points and a_γ-points are contained in $\bigcup_\nu \bar{C}_\nu \subset \bigcup_\nu D_\nu$. If $\bar{C}_\nu \subset V[r]$, then the functions $n(r, a_0)$ and $n(r, a_\gamma)$ have the same increment on the set $V[r]$. The increment equals λ_ν.

Let $k(r)$ denote the number of sets $\bar{C}_\nu$ that have a nonempty intersection with both $V[r]$ and $\mathbf{V} \setminus V[r]$. Then

$$n(r, a_\gamma) - n(r, a_0) \leqq \Lambda\, k(r)\,, \tag{17}$$

as the number of a_γ-points in $\bar{C}_\nu$ is less than or equal to Λ. If $k(r) \geqq 2$, then the whole $\partial V[r]$ cannot lie in a single set D_ν. If $\partial V[r]$ intersects $k(r) \geqq 2$ distinct sets $\bar{C}_\nu$, then $f(\partial V[r])$ intersects "*the ring*" $D(a_0) \setminus \overline{\operatorname{Int}\gamma}$ at least $2\,k(r)$-times, connecting the points on $\partial D(a_0)$ with the points on γ. Thus

$$L(r) \geqq 2\,k(r)\,.\,d\,, \tag{18}$$

$$k(r) \leqq \frac{1}{2d}\,L(r)\,. \tag{19}$$

From the relations (17) and (18) we obtain

$$n(r, a_\gamma) - n(r, a_0) \leqq A \max(1, k(r)) \leqq A\left(1 + \frac{1}{2d} L(r)\right). \tag{20}$$

Integrating the relation (20) from r_0 to r we have

$$N(r, a_\gamma) - N(r, a_0) \leqq A\left(r + \frac{1}{2d}\int_{r_0}^{r} L(t)\, \mathrm{d}t\right). \tag{21}$$

Further integration along γ gives

$$\int_\gamma N(r, a_\gamma)\, \mathrm{d}s^0 - N(r, a_0) \leqq A\left(r + \frac{1}{2d}\int_{r_0}^{r} L(t)\, \mathrm{d}t\right). \tag{22}$$

Using Theorem 2.1 and Lemma 2.2 we have

$$T(r) - N(r, a_0) \leqq A\left(r + \frac{1}{2d} S(r, f)\right). \tag{23}$$

Using Theorem 1.2 on the left hand side of (23) we obtain

$$m(r, a_0) \leqq S(r, f)\,, \quad \text{i.e.} \quad \delta(a_0) = 0\,, \qquad \text{QED.}$$

4. THE CASE OF A HYPERBOLIC RIEMANN SURFACE

We shall prove a theorem analogous to Theorem 3.1 for an open Riemann surface with *finite harmonic exhaustion*. The following well-known lemma will be used (see [2]):

Lemma 4.1. *Suppose ψ is a once differentiable positive increasing function on $[0, s)$, $s < \infty$. Then for every real number $\varepsilon > 0$,*

$$\psi'(r) \leqq \frac{1}{s - r}\{\psi(r)\}^{1+\varepsilon}$$

for all $r \in [0, s) \setminus I_1$, where $I_1 \subset [0, s)$ is an open set such that

$$\int_{I_1} \mathrm{d}\log(s - r) > -\infty\,.$$

We recall that the relation (12) is also valid for a Riemann surface admitting finite harmonic exhaustion. Since

$$r < s < \infty$$

in this case, the relation (12) can be rewritten as

$$\int_{r_0}^{r} L(t)\,\mathrm{d}t \leqq \sqrt{L}\sqrt{s}\sqrt{v(r)}\,. \tag{24}$$

If Lemma 4.1 is applied to the function $T(r)$, the result is

$$v(r) \leqq \frac{1}{s-r}\{T(r)\}^{1+\varepsilon}.$$

The inequality

$$\sqrt{v(r)} \leqq \frac{1}{\sqrt{(s-r)}}\{T(r)\}^{1/2+\varepsilon} \quad (\varepsilon > 0 \textit{ arbitrary}) \tag{25}$$

is valid on $[0, s) \setminus I_1$.

Let $T(r)$ grow so rapidly that

$$\frac{1}{\sqrt{(s-r)}}\{T(r)\}^{1/2+\varepsilon} = o\{T(r)\} \tag{26}$$

is valid for $\varepsilon \in (0, 1/2)$. Then Lemma 2.2 is also true for a surface with finite harmonic exhaustion.

Definition 4.1. Let $Q(r, f)$ denote the quantity defined by

$$Q(r, f) = o\{T(r)\}$$

for $r \to s$, $r \in [r_0, s) \setminus I_1$, where I_1 is the set from Lemma 4.1.

Lemma 4.2. *Let $f: \mathbf{V} \to \mathbf{M}$ be a holomorphic mapping from an open Riemann surface $\mathbf{V}$, admitting finite harmonic exhaustion, into $\mathbf{M}$. If the relation (26) is valid, then*

$$\int_{r_0}^{r} L(t)\,\mathrm{d}t = Q(r, f)\,.$$

Proof follows at once from the relations (24), (25), (26). Furthermore, the following theorems are true.

Theorem 4.1. *Let $f: \mathbf{V} \to \mathbf{M}$ be a holomorphic mapping from an open Riemann surface $\mathbf{V}$, admitting finite harmonic exhaustion, into $\mathbf{M}$ and let γ be a regular curve on $\mathbf{M}$. If the relation (26) is valid, then*

$$T(r) = \int_{\gamma} N(r, a)\,\mathrm{d}s^0(a) + Q(r, f)\,.$$

(This is *generalized Cartan's formula* in the case $s < \infty$.)

Theorem 4.2. *Let $f : \mathbf{V} \to \mathbf{M}$ be a holomorphic mapping from an open Riemann surface $\mathbf{V}$, admitting finite harmonic exhaustion, into $\mathbf{M}$. Let the relation (26) be valid. Then under the assumptions of Theorem* 3.1 *its assertion also holds, i.e.* $\delta(a_0) = 0$.

The proofs of Theorems 4.1 and 4.2 are analogous to the proofs of Theorems 2.1 and 3.1. For this reason they are omitted.

Remark 4.1. It is possible to give a weaker condition on the rapidity of the growth $T(r)$ than (26). If Lemma 4.1 is applied to the function $\log T(r)$, the result is

$$\frac{T'(r)}{T(r)} \leqq \frac{1}{s - r} [\log T(r)]^{1+\varepsilon},$$

i.e.

$$v(r) \leqq \frac{1}{s - r} T(r) [\log T(r)]^{1+\varepsilon}.$$

Then instead of (26) we can introduce the condition

$$(26)' \qquad \frac{1}{\sqrt{(s - r)}} \sqrt{T(r)} [\log T(r)]^{1/2+\varepsilon} = o\{T(r)\}.$$

Remark 4.2. In this remark an example of a mapping satisfying (26) is given. Let $D = \{z \in \mathbf{C}, |z| < 1\}$ and consider the function

$$f(z) = -1 + \exp \frac{2\pi i}{(1 - z)^3}$$

defined on D. The point $z = 1$ is an essential singularity of f. The zeros of f in D are located at $z_k = 1 - (1/\sqrt[3]{k})$, $k = 1, 2, \ldots$ and so a circle of radius $r < 1$ encloses at most $1/(1 - r)^3$ zeros of f, up to a constant which is independent of r. Further, we define a harmonic exhaustion of D with help of the function $\tau(z) = \log e|z|$; then $\tau : D \to [0, 1)$. Let us put $r_0 = 2/e$. Then

$$\int_{r_0}^{r} n(t, 0)\, dt \geqq \int_{r_0}^{r} \left[\frac{1}{(1 - e^{t-1})^3} + \text{const.}\right] dt \geqq \frac{1}{(1 - e^{r-1})^2} = O\left\{\frac{1}{(1 - r)^2}\right\}.$$

From First Main Theorem we obtain

$$T(r) \geqq O\left\{\frac{1}{(1 - r)^2}\right\},$$

hence $T(r)$ satisfies the condition (26).

For the function $f(z) = -1 + \exp [2\pi i/(1 - z)^2]$ we similarly obtain $T(r) = O\{1/(1 - r)\}$ so that the condition (26) is not fulfilled.

In this section we mention several consequences of the preceding theorems. Most of these results, known from [1], Chap. VI, are obtained here in a quite different way. The following lemma will be needed:

Lemma 5.1 (Generalized l'Hospital rule). *Let g and h be differentiable functions on an interval $[a, b)$, $b \leqq \infty$, such that h' exists and is nowhere zero on $[a, b)$. If*

$$\lim_{x \to b-} g(x) = \lim_{x \to b-} h(x) = 0$$

or if

$$\lim_{x \to b-} g(x) = \lim_{x \to b-} h(x) = \infty ,$$

then

$$\liminf_{x \to b-} \frac{g'}{h'} \leqq \liminf_{x \to b-} \frac{g}{h} \leqq \limsup_{x \to b-} \frac{g}{h} \leqq \limsup_{x \to b-} \frac{g'}{h'} . \tag{27}$$

Proof see [12].

(i) As a consequence of Lemma 2.2 we have

$$\liminf_{r \to \infty} \frac{\int_{r_0}^{r} L(t)\, dt}{T(r)} = 0 .$$

Because $(\int_{r_0}^{r} L(t)\, dt)' = L(r)$ and $(T(r))' = v(r)$, we obtain using Lemma 5.1

$$\liminf_{r \to \infty} \frac{L(r)}{v(r)} \leqq \liminf_{r \to \infty} \frac{\int_{r_0}^{r} L(t)\, dt}{T(r)} = 0 . \tag{28}$$

The relation (28) implies the following assertion:

If $f : \mathbf{V} \to \mathbf{M}$ is a holomorphic mapping of a parabolic Riemann surface $\mathbf{V}$ into $\mathbf{M}$, then the covering surface $(\mathbf{M})_f^{\mathbf{V}}$ is regularly exhaustible.

(ii) Similarly, we have as a consequence of Lemma 4.2:

If $f : \mathbf{V} \to \mathbf{M}$ is a holomorphic mapping of an open Riemann surface $\mathbf{V}$, admitting finite harmonic exhaustion, into $\mathbf{M}$, for which the relation (26) is valid, then the covering surface $(\mathbf{M})_f^{\mathbf{V}}$ is regularly exhaustible.

(iii) In the case of a parabolic Riemann surface H. Wu gave in [2] a simple proof that $f(\mathbf{V})$ is *open dense* in $\mathbf{M}$. A much stronger result is known (see [1]): If $\mathbf{V}$ is parabolic, then $\mathbf{M} \setminus f(\mathbf{V})$ has the *capacity zero.*

The following assertion follows from Theorem 2.1:

If $\mathbf{V}$ is a parabolic Riemann surface and $f : \mathbf{V} \to \mathbf{M}$ a holomorphic mapping, then for an arbitrary regular curve γ on $\mathbf{M}$ the intersection

$$f(\mathbf{V}) \cap \gamma$$

is nonempty.