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GROUPOIDS WITH A CLOSURE CONDITION

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INTRODUCTION

J. ACZEL [1] while classifying regular algebraic nets corresponding to the classes of isotopic quasigroups and satisfying various closure conditions, $\boldsymbol{T}$, $\boldsymbol{R}$, $\boldsymbol{B}$, $\boldsymbol{B}_1$, $\boldsymbol{B}_2$ and $\boldsymbol{H}$, also proved that in a quasigroup, associativity implies Reidemeister condition $\boldsymbol{R}$, i.e., for all x_i, y_i $(i = 1, 2, 3, 4)$ in the quasigroup,

$$x_1 \cdot y_2 = x_2 \cdot y_1 \;(= q)\,, \quad x_1 \cdot y_4 = x_2 \cdot y_3 \;(= r)\,, \quad x_3 \cdot y_2 = x_4 \cdot y_1 \;(= p)$$

imply that

$$x_3 \cdot y_4 = x_4 \cdot y_3 \;(= s)\,.$$

Further, Aczel [2] proved that Reidemeister condition is necessary and sufficient for associativity of loops.

By putting $x_1 = y_1 = e$ the identity of the loop in condition $\boldsymbol{R}$ we get condition $\boldsymbol{R}'$, i.e.,

$$\text{if} \quad y_2 = x_2 \quad \text{and} \quad y_4 = x_2 \cdot y_3\,, \quad x_4 = x_3 \cdot y_2 \quad \text{then} \quad x_3 \cdot y_4 = x_4 \cdot y_3\,.$$

$\boldsymbol{R}'$ is equivalent to Reidemeister condition $\boldsymbol{R}$ in a loop; further, $\boldsymbol{R}'$ is equivalent to assiciativity in any groupoid.

Here we define another closure condition $\boldsymbol{N}$ for a groupoid $(G, \cdot)$, namely, for all x_i, y_i in G $(i = 1, 2, 3, 4)$,

$$x_1 \cdot y_2 = x_3 \cdot y_1 \;(= q)\,, \quad x_1 \cdot y_4 = x_2 \cdot y_3 \;(= r) \quad x_2 \cdot y_2 = x_4 \cdot y_1 \;(= p)$$

imply that

$$x_3 \cdot y_4 = x_4 \cdot y_3 \;(= s)\,.$$

This condition is certainly different from Reidemeister, Bol and Hexagonal closure conditions.

We observe that in a loop, closure condition $\boldsymbol{N}$ is equivalent to associativity together with commutativity. In a groupoid with an identity, if we put $x_1 = y_1 = e$

the identity of the groupoid in condition N then we get condition N', i.e.,

$$\text{if} \quad y_2 = x_3 \quad \text{and} \quad y_4 = x_2 \,.\, y_3 \,, \quad x_4 = x_2 \,.\, y_2 \quad \text{then} \quad x_3 \,.\, y_4 = x_4 \,.\, y_3 \,.$$

Here N' is equivalent to N_r-associativity (discussed in [3]) in any groupoid.

A groupoid $(G, \cdot)$ satisfies N_r-associativity [3] if $(a \,.\, b) \,.\, c = b \,.\, (a \,.\, c)$ for all a, b, c in G.

In this paper we wish to investigate some of the properties of groupoids satisfying closure condition N. We shall see under what conditions these groupoids are semigroups or groups.

FOTEDAR in [6] has investigated necessary and sufficient conditions for an isotope of a given groupoid to be a semigroup or group. He appears to be successful in giving a partial solution of the problem, assuming the presence of an identity element in the given groupoid. In this connection he has stated that generalized associative law holds in a groupoid $(G, \cdot)$, if there exists a pair of elements (a, b) in G such that

$$[\{x(by)\}\, a]\, z = x[b\{(ya)\, z\}]$$

for all x, y, z in G, and then he has proved the following.

Theorem. *If $(G, \cdot)$ is a groupoid with unit element 1 then the groupoid $(G, 0)$ isotopic to $(G, \cdot)$ under the isotopy $x \, 0 \, y = x^\alpha \,.\, y^\beta$ is a semigroup iff there exists a pair of elements a, b in G such that*

$$x^\alpha = xa\,, \quad x^\beta = bx$$

for all x in G and $(G, \cdot)$ satisfies the g.a.l.

$$[\{x(by)\}\, a]\, z = x[b\{(ya)\, z\}]$$

for all x, y, z in G. Moreover, $(G, 0)$ is a group iff in addition to the above conditions, $(G, \cdot)$ is a quasigroup.

Finally, he remarks that the presence of an identity element in $(G, \cdot)$ introduces an element of incompleteness in the solution of the problem. In this context, we may invoke the following analogue of a famous theorem due to A. A. ALBERT, proved by N. J. S. HUGHES in 1957.

If a groupoid with a unit element is isotopic to a semigroup, then they are isomorphic.

This result rules out the possibility of finding necessary and sufficient conditions for an isotope of a given groupoid with a unit element to be a semigroup since in that case the groupoid itself becomes a semigroup. Therefore, the theorem proved in this regard states only the following.

If a groupoid with a unit element contains a pair of right non-singular and left non-singular elements such that the g.a.l. is satisfied then it is a semigroup.

Theorem 6 of this paper gives us necessary and sufficient conditions for an isotope of a given groupoid to be an abelian group.

Definition 1. A groupoid $(G, \cdot)$ is called *left equally cancellative* if and only if for all x_1, x_2, y_1 in G,

$$y_1 . x_1 = y_1 . x_2 \quad \text{implies} \quad y . x_1 = y . x_2 \quad \text{for all } y \text{ in } G .$$

Similarly, *right equally cancellative* groupoids are defined. Further, a groupoid is said to be *equally cancellative* if it is both left and right equally cancellative.

Definition 2. A groupoid $(G, \cdot)$ is called *left cancellative* if and only if for all x_1, x_2, y in G,

$$y . x_1 = y . x_2 \quad \text{implies} \quad x_1 = x_2 .$$

Similarly, *right cancellativity* and *(two sided) cancellativity* are defined.

Definition 3. A groupoid $(G, \cdot)$ is an *N-groupoid* if and only if for all x_i, y_i in G $(i = 1, 2, 3, 4)$,

$$x_1 . y_2 = x_3 . y_1 , \quad x_1 . y_4 = x_2 . y_3 ,$$

$$x_2 . y_2 = x_4 . y_1 \quad \text{imply} \quad x_3 . y_4 = x_4 . y_3 .$$

We shall denote this groupoid by $(G_N, \cdot)$.

When used in connection with a groupoid $(G, \cdot)$ the product xy will be equivalent to $x . y$.

The first theorem gives us alternative axioms for an abelian group.

Theorem 1. *A groupoid $(G, \cdot)$ satisfies the conditions*

(i) *for all x_2, x_3, y_3 in G,*

$$y_4 = x_2 y_3 \quad \textit{and} \quad x_4 = x_2 x_3 \quad \textit{imply} \quad x_3 y_4 = x_4 y_3 ;$$

(ii) *$x\,a = b$ is uniquely solvable in x for all a, b in G, if and only if $(G, \cdot)$ is an abelian group.*

Proof. If condition (i) holds in G then by putting the values of $y_4 = x_2 y_3$ and $x_4 = x_2 x_3$ in $x_3 y_4 = x_4 y_3$, we have, for all x_2, x_3, y_3 in G,

$$x_3(x_2 y_3) = (x_2 x_3)\, y_3 \quad (N_r\text{-associativity}) .$$

Next we prove that $(G, \cdot)$ is commutative.

Let $a, b \in G$, by condition (ii) there exists a unique x in G such that $x\,a = b$.

Now,

$$a\,b = a(x\,a) = (x\,a)\,a = b\,a .$$

Then commutativity of $(G, \cdot)$ and condition (ii) imply that $a\,x = b$ is also uniquely solvable in x for every a, b in G, hence $(G, \cdot)$ is a quasigroup. Again commutativity and N_r-associativity imply that $(G, \cdot)$ is associative as well.

Therefore $(G, \cdot)$ is an abelian group.

The converse of Theorem 1 is trivially true.

Theorem 2. *Every N-groupoid is equally cancellative.*

Proof. Suppose $(G_N, \cdot)$ is an N-groupoid and

$$r_1 s_1 = r_2 s_1 , \quad r_1, r_2, s_1 \text{ in } G_N .$$

In closure condition N

$$x_1 y_2 = x_3 y_1 , \quad x_2 y_2 = x_4 y_1 , \quad x_1 y_4 = x_2 y_3$$

imply

$$x_3 y_4 = x_4 y_3 ;$$

put

$$x_1 = x_2 = x_3 = r_1 , \quad x_4 = r_2 ,$$

$$y_1 = y_2 = s_1 , \qquad y_3 = y_4 = s , \quad s \text{ in } G_N .$$

This gives

$$r_1 s_1 = r_2 s_1 \quad \text{implies} \quad r_1 s = r_2 s \quad \text{for all s in } G .$$

Similarly, right equal cancellativity can be proved.

Lemma 1. *Every equally cancellative groupoid with an identity element is cancellative.*

Theorem 3. *A groupoid with an identity element is an N-groupoid if and only if it is a cancellative abelian semigroup.*

Proof. Suppose $(G_N, \cdot)$ is an N-groupoid with an identity element e.

Let x_1, x_2 in G_N, then by condition N equations

$$x_1 x_2 = (x_1 x_2)\, e , \quad x_1 e = e x_1 , \quad e x_2 = x_2 e$$

imply

$$(x_1 x_2)\, e = x_2 x_1 ,$$

i.e., $x_1 x_2 = x_2 x_1$. Hence $(G_N, \cdot)$ is commutative.

Further, if x_1, x_2, x_3 are in G_N then

$$x_2 x_1 = (x_1 x_2)\, e , \quad x_2 x_3 = (x_3 x_2)\, e , \quad (x_3 x_2)\, x_1 = ((x_3 x_2)\, x_1)\, e$$

imply

$$(x_1 x_2)\, x_3 = ((x_3 x_2)\, x_1)\, e .$$

This gives us $(x_1 x_2)\, x_3 = (x_3 x_2)\, x_1$, which reduces to associativity in commutative groupoids.

Cancellativity follows from Theorem 2 and Lemma 1.

Conversely, suppose $(G, \cdot)$ is a cancellative abelian semigroup.

Let $x_i, y_i \in G$ $(i = 1, 2, 3, 4)$ be such that

$$x_1y_2 = x_3y_1\,, \quad x_1y_4 = x_2y_3\,, \quad x_2y_2 = x_4y_1\,.$$

Then

$$(x_1y_2)(x_4y_1)(x_2y_3) = (x_3y_1)(x_2y_2)(x_1y_4)\,.$$

Now applying associativity, commutativity and cancellativity we get

$$x_4y_3 = x_3y_4\,.$$

Hence $(G, \cdot)$ satisfies condition N.

Corollary 3.1. *A loop satisfies closure condition N if and only if it is an abelian group.*

Corollary 3.2. *A groupoid $(G, \cdot)$ is an N-groupoid with an identity element e such that, for all b in G, $a\,.\,b = e$ has a solution in a, if and only if $(G, \cdot)$ is an abelian group.*

Theorem 4. *If $(G_N, \cdot)$ is an N-groupoid with elements r, s such that, for all y in G_N, there exists x in G_N satisfying $xy = s$ and $rG = G$ then for each pair of elements c, d in G_N the equations*

$$xc = d\,, \quad cy = d$$

are solvable for x and y in G_N.

Proof. We shall first prove that $G_Nt = G_N$, where $t \in G_N$ is such that $r\,t = s$.

For an arbitrary y in G_N there exist x_1, y_1, x_2 in G_N such that $x_1y = s$, $x_1t = ry_1$, $x_2y_1 = s$.

By closure condition N, equations

$$x_1t = ry_1\,, \quad x_1y = rt\,, \quad rt = x_2y_1 \quad \text{imply} \quad ry = x_2t\,.$$

As y is abitrary, we have $G_Nt = G_N$.

The next step is to show that for all x in G_N, $xy = s$ has a solution in y.

Let $x \in G_N$, then there exist y_1, x_1, y_2, x_2 in G_N such that $xt = ry_1$, $x_1y_1 = rt$, $x_1t = ry_2$, $x_2y_2 = rt$.

Due to closure condition N, equations

$$rt = x_2y_2\,, \quad rt = x_1y_1\,, \quad \text{and} \quad x_1t = ry_2 \quad \text{imply} \quad x_2t = ry_1\,;$$

from $xt = ry_1$ we get $x_2t = x\,t$; but G_N is equally cancellative by Theorem 2, therefore $x_2y_2 = xy_2$ and then $x_2y_2 = rt = s$, which gives $xy_2 = s$.

Now in order to complete the proof of the theorem, let $x \in G_N$; then there exist y_1, x_1 in G_N such that

$$xt = ry_1 \quad \text{and} \quad x_1y_1 = rt\,;$$

and for arbitrary y_2 in G_N, there exist y_3 in G_N such that

$$x_1 y_2 = r y_3 .$$

From these three equations, applying closure condition N, we get

$$x y_3 = r y_2 .$$

As y_2 is arbitrary and $rG_N = G_N$ we have

$$x \,.\, G_N = G_N \quad \text{for all } x \text{ in } G_N .$$

By symmetric considerations

$$G_N \,.\, y = G_N \quad \text{for all } y \text{ in } G_N .$$

Thus the theorem is proved.

Theorem 5. *Let $(G_N, \cdot)$ be an N-groupoid with elements r, s in G_N such that for all y in G_N there exists x in G_N satisfying $xy = s$ and $rG_N = G_N$. Then a new operation $+$ can be defined on the elements of G_N as follows: For arbitrary but fixed m and n in G_N, define*

$$xn + my = xy \quad \textit{for all } x, y \textit{ in } G_N .$$

Then $+$ is a well defined binary operation on G_N, under which G_N forms an abelian group.

Proof. Consider the mappings from G_N to G_N, given by

$$x \to xn \;(= X) \quad \text{and} \quad y \to my \;(= Y) .$$

Then it follows from Theorem 4 that these mappings are onto G_N.

Next, we shall show that $+$ is a well defined operation. From Theorem 2 we see that

$$x_1 n = x_2 n \quad \text{implies} \quad x_1 y = x_2 y \quad \text{for all } y \text{ in } G_N ,$$

so that

$$x_1 y_1 = x_2 y_1 . \tag{1}$$

Similarly, $my_1 = my_2$ gives

$$x_2 y_1 = x_2 y_2 \tag{2}$$

so that if $x_1 n = x_2 n$ and $my_1 = my_2$ then (1) and (2) imply that

$$x_1 y_1 = x_2 y_2$$

and hence $+$ is a well defined operation.

Here mn acts as identity element for the groupoid $(G_N, +)$. Indeed,

$$mn + my = my , \quad xn + mn = xn .$$

Groupoid $(G_N, +)$ also retains the N-groupoid property of $(G_N, \cdot)$. Let

$$X_1 + Y_2 = X_3 + Y_1 , \quad X_1 + Y_4 = X_2 + Y_3 \quad \text{and} \quad X_2 + Y_2 = X_4 + Y_1 ,$$

where X_i, Y_i in G_N $(i = 1, 2, 3, 4)$.

Then there exist x_i, y_i in G_N $(i = 1, 2, 3, 4)$ such that

$$x_i n = X_i \quad \text{and} \quad m y_i = Y_i$$

and thus

$$x_1 y_2 = x_3 y_1 , \quad x_1 y_4 = x_2 y_3 \quad \text{and} \quad x_2 y_2 = x_4 y_1 ,$$

which implies $x_3 y_4 = x_4 y_3$, which in turn gives

$$X_3 + Y_4 = X_4 + Y_3 .$$

Then Theorem 3 implies that $(G_N, +)$ is an abelian semigroup. For completing the proof of the theorem we have to establish the quasigroup property also in $(G_N, +)$.

Let $C, D \in G_N$, then we shall show that $C + Y = D$ is uniquely solvable in Y. By Theorem 4 there exist c, y in G_N such that $cn = C$ and $cy = D$. Assume $my = Y$, then $C + Y = D$. Further, Theorem 3 gives the uniqueness of the solution. The proof for the solution on the left is similar.

Thus the theorem is proved.

Now we shall take up the problem attempted by Fotedar [6].

Since we know that every isotope of a given groupoid is isomorphic to a principal isotope, there is no loss of generality in the theory of isotopy in restricting our attention to principal isotopes, a fact pointed out in various papers of Albert and Bruck.

Here we start with

Theorem 6. *A groupoid $(G, +)$ isotopic to a given groupoid $(G, \cdot)$ under the isotopy $x + y = x^\alpha y^\beta$ is an abelian group if and only if the following conditions are satisfied*:

$$x^{\alpha^{-1}} = xn , \quad x^{\beta^{-1}} = mx \quad \textit{for some } n, m \textit{ in } G ,$$

for all x in G; and $(G, \cdot)$ is an N-groupoid such that there exists a pair of elements r, s in G satisfying $rG = G$ and for all y in G, $xy = s$ is solvable in x.

Proof. Suppose $(G, +)$ is a groupoid isotopic to a given groupoid $(G, \cdot)$ under the isotopy

(1) $$x + y = x^\alpha y^\beta .$$

As α and β are permutations of G therefore, without loss of generality, we can consider the above equation in the form

(2) $$x^{\alpha^{-1}} + x^{\beta^{-1}} = xy .$$

From the condition of the theorem

$$x^{\alpha^{-1}} = xn \quad \text{and} \quad x^{\beta^{-1}} = mx$$

we obtain

$$xn + my = xy \quad \text{for some } n, m \text{ in } G .$$

Now, as $(G, \cdot)$ satisfies the conditions of Theorem 5, $(G, +)$ is an abelian group.

Conversely, suppose $(G, +)$ is an abelian group isotopic to a groupoid $(G, \cdot)$ under the isotopy (1). We can consider (1) in the form of (2), i.e.,

$$x^{\alpha^{-1}} + y^{\beta^{-1}} = xy .$$

Let e be the identity element of $(G, +)$ and denote $e^{\alpha} = m$ and $e^{\beta} = n$, then putting $n^{\beta^{-1}} = e$ and $m^{\alpha^{-1}} = e$ separately in the relation (2) we get

$$x^{\alpha^{-1}} = xn \quad \text{and} \quad x^{\beta^{-1}} = my ,$$

hence

$$xn + my = xy . \tag{3}$$

Thus, every element of $(G, +)$ can be obtained as right and left translations of n and m, respectively, by some elements of $(G, \cdot)$. Further, the composition $+$ of G is defined by the relation (3).

In view of Theorem 3, $(G, +)$ satisfies the N-groupoid property. We have to show the N-groupoid property in $(G, \cdot)$. Let

$$x_1 y_2 = x_3 y_1 , \quad x_1 y_4 = x_2 y_3 \quad \text{and} \quad x_2 y_2 = x_4 y_1 ,$$

where x_i, y_i in G $(i = 1, 2, 3, 4)$; then there exist X_i, Y_i in G $(i = 1, 2, 3, 4)$ such that

$$x_i n = X_i \quad \text{and} \quad m y_i = Y_i$$

and thus,

$$X_1 + Y_2 = X_3 + Y_1 , \quad X_1 + Y_4 = X_2 + Y_3 \quad \text{and} \quad X_2 + Y_2 = X_4 + Y_1 ,$$

which means $X_3 + Y_4 = X_4 + Y_3$ which in turn gives $x_3 y_4 = x_4 y_3$.

Next, as the quasigroup property is invariant under isotopy, $(G, \cdot)$ is a quasigroup, and further, this implies the conditions of the theorem.

This proves the theorem completely.

In addition, if we assume that $(G, \cdot)$ is a finite groupoid then we can restate our Theorem 6 with a slight modification. We start with

Theorem 6′. *If $(G, \cdot)$ is a finite groupoid then $(G, \cdot)$ is isotopic to an abelian group $(G, +)$ under the isotopy $x + y = x^{\alpha} y^{\beta}$ if and only if $(G, \cdot)$ is an N-groupoid and there exists a pair of elements r, s in G such that $rG = G$ and $xy = s$ has a solution in x for all y in G.*

Proof. In a groupoid $(G, \cdot)$ satisfying the conditions of Theorem 5, we define a composition $+$ by

$$xn + my = xy \quad \text{for some } n, m \text{ in } G .$$