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LAPLACE TRANSFORM OF EXPONENTIALLY LIPSCHITZIAN VECTOR-VALUED FUNCTIONS

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The purpose of this note is to give the theory – in the form as definitive as possible – of the Laplace transform of exponentially Lipschitzian vector-valued functions whose most important part was proved and applied in [1] (see especially Section 4).

We shall use the following notation: (1) R – the real number field, (2) R^+ – the set of all positive real numbers, (3) (ω, ∞) – the set of all real numbers greater than ω if $\omega \in R$, (4) E – an arbitrary Banach space over R, (5) $M_1 \to M_2$ – the set of all mappings of the whole set M_1 into the set M_2.

1. Lemma. *For every $\alpha \geqq 0$, $\chi > 1$ and $r \in \{0, 1, \ldots\}$ such that $r > \chi\alpha$, the following inequality holds:*

$$\left(\frac{r}{r-\alpha}\right)^r \leqq e^{(\chi/(\chi-1))\alpha}.$$

Proof. We have under our assumptions

$$\frac{r}{r-\alpha} = \frac{1}{1-\dfrac{\alpha}{r}} \leqq \frac{1}{1-\dfrac{1}{\chi}} = \frac{\chi}{\chi-1}$$

which implies

$$\left(\frac{r}{r-\alpha}\right)^r = \left(1+\frac{\alpha}{r-\alpha}\right)^r = \left(e^{\alpha/(r-\alpha)}\right)^r = e^{(r/(r-\alpha))\alpha} \leqq e^{(\chi/(\chi-1))\alpha}.$$

2. Lemma. *For every $\alpha \geqq 0$ and $r \in \{2, 3, \ldots\}$ such that $r > \alpha^2$, we have*

$$\left(\frac{r}{r-\alpha}\right)^r \leqq e^{(\sqrt{r}/(\sqrt{r}-1))\alpha}.$$

Proof. We have $\sqrt{r} > \alpha$, i.e. $r > \alpha \sqrt{(r)}$. Hence we can choose $\chi = \sqrt{r}$ in Lemma 1 and the desired inequality follows.

3. Lemma. *For every $\omega \geqq 0$, $0 < t_1 < t_2$ and $p \in \{0, 1, \ldots\}$ such that $p > (\omega t_2 + 1)^2$ we have*

$$\int_{t_1}^{t_2} \frac{1}{\left(1 - \dfrac{\omega\tau}{p+1}\right)^{p+2}} \, \mathrm{d}\tau \leqq \frac{1}{1 - \dfrac{\omega t_2}{p+1}} \int_{t_1}^{t_2} e^{(\sqrt{(p+1)}/(\sqrt{(p+1)}-1))\omega\tau} \, \mathrm{d}\tau \, .$$

Proof. It follows by means of Lemma 2 with $\alpha = \omega\tau$ and $r = p + 1$ that

$$\int_{t_1}^{t_2} \frac{1}{\left(1 - \dfrac{\omega\tau}{p+1}\right)^{p+2}} \, \mathrm{d}\tau \leqq \frac{1}{1 - \dfrac{\omega t_2}{p+1}} \int_{t_1}^{t_2} \frac{1}{\left(1 - \dfrac{\omega\tau}{p+1}\right)^{p+1}} \, \mathrm{d}\tau \leqq$$

$$\leqq \frac{1}{1 - \dfrac{\omega t_2}{p+1}} \int_{t_1}^{t_2} e^{(\sqrt{(p+1)}/(\sqrt{(p+1)}-1))\omega\tau} \, \mathrm{d}\tau \, .$$

4. Theorem. *Let ω be a nonnegative constant, $F \in (\omega, \infty) \to E$ and let M_0, M_1 be two nonnegative constants. Then*

(A_1) *the function F is infinitely differentiable on (ω, ∞),*

(A_2) $\left\| \dfrac{\mathrm{d}^p}{\mathrm{d}\lambda^p} F(\lambda) \right\| \leqq \dfrac{M_0 p!}{(\lambda - \omega)^{p+1}}$ *for every $\lambda > \omega$ and $p \in \{0, 1, \ldots\}$,*

(A_3) $\left\| \dfrac{\mathrm{d}^p}{\mathrm{d}\lambda^p} \lambda F(\lambda) \right\| \leqq \dfrac{M_1 p!}{(\lambda - \omega)^{p+1}}$ *for every $\lambda > \omega$ and $p \in \{1, 2, \ldots,\}$,*

if and only if there exists a function $f \in R^+ \to E$ such that

(B_1) $\|f(t)\| \leqq M_0 e^{\omega t}$ *for any $t \in R^+$,*

(B_2) $\|f(t_1) - f(t_2)\| \leqq M_1 \displaystyle\int_{t_1}^{t_2} e^{\omega\tau} \, \mathrm{d}\tau$ *for any $t_1, t_2 \in R^+$, $t_1 < t_2$,*

(B_3) $F(\lambda) = \displaystyle\int_0^\infty e^{-\lambda\tau} f(\tau) \, \mathrm{d}\tau$ *for any $\lambda > \omega$.*

Proof. "Only if" part. Let us first denote

(1) $G(\mu) = F(\mu + \omega)$ for any $\mu > 0$.

It follows from $(A_1)-(A_3)$ that

(2) the function G is infinitely differentiable on R^+,

(3) $$\left\| \frac{d^p}{d\mu^p} G(\mu) \right\| \leqq \frac{M_0 p!}{\mu^{p+1}} \quad \text{for any} \quad \mu > 0 \quad \text{and} \quad p \in \{0, 1, \ldots\},$$

(4) $$\left\| \frac{d^p}{d\mu^p} (\mu\, G(\mu)) \right\| = \left\| \frac{d^p}{d\mu^p} (\mu\, F(\mu + \omega)) \right\| =$$

$$= \left\| \frac{d^p}{d\mu^p} [(\mu + \omega) F(\mu + \omega) - \omega F(\mu + \omega)] \right\| \leqq \frac{(M_1 + \omega M_0)\, p!}{\mu^{p+1}}$$

for any $\mu > 0$ and $p \in \{1, 2, \ldots\}$.

Let us now denote

(5) $$g_q(t) = \frac{(-1)^q}{q!} \left(\frac{q+1}{t}\right)^{q+1} G^{(q)}\left(\frac{q+1}{t}\right) \quad \text{for} \quad t \in R^+ \quad \text{and} \quad q \in \{0,1,\ldots\}.$$

By (2) and (3) we obtain

(6) the function g_q is differentiable on R^+ for every $q \in \{0, 1, \ldots\}$,

(7) $\|g_q(t)\| \leqq M_0$ for every $t \in R^+$ and $q \in \{0, 1, \ldots\}$,

(8) $$g_q'(t) = \frac{(-1)^{q+1}}{q!} (q+1) \frac{q+1}{t^2} \left(\frac{q+1}{t}\right)^q G^{(q)}\left(\frac{q+1}{t}\right) +$$

$$+ \frac{(-1)^{q+1}}{q!} \left(\frac{q+1}{t}\right)^{q+1} \frac{q+1}{t^2} G^{(q+1)}\left(\frac{q+1}{t}\right) =$$

$$= \frac{(-1)^{q+1}}{(q+1)!} \left(\frac{q+1}{t}\right)^{q+2} \left[(q+1)\, G^{(q)}\left(\frac{q+1}{t}\right) + \frac{q+1}{t} G^{(q+1)}\left(\frac{q+1}{t}\right)\right]$$

for every $t \in R^+$ and $q \in \{0, 1, \ldots\}$.

Now we need to estimate the growth of g_q'. To this aim, let us denote

(9) $H(\mu) = \mu\, G(\mu)$ for $\mu > 0$.

It is clear that

(10) $H^{(q+1)}(\mu) = (q + 1)\, G^{(q)}(\mu) + \mu\, G^{(q+1)}(\mu)$ for any $\mu > 0$ and $q \in \{0, 1, \ldots\}$.

Now (9) and (10) permit us to rewrite (8) in the form

(11) $$g_q'(t) = \frac{(-1)^{q+1}}{(q+1)!} \left(\frac{q+1}{t}\right)^{q+2} H^{(q+1)}\left(\frac{q+1}{t}\right).$$

On the other hand, we have by (4) and (9) that

(12) $\|H^{(q+1)}(\mu)\| \leqq \dfrac{(M_1 + \omega M_0)(q+1)!}{\mu^{q+2}}$ for any $\mu > 0$ and $q \in \{0, 1, \ldots\}$.

We see from (11) and (12) that $\|g_q'(t)\| \leqq M_1 + \omega M_0$ for every $t \in R^+$ and $q \in \in \{0, 1, \ldots\}$ which implies

(13) $\|g_q(t_1) - g_q(t_2)\| \leqq (M_1 + \omega M_0)|t_1 - t_2|$ for every $t_1, t_2 \in R^+$ and $q \in \{0, 1, \ldots\}$.

In view of (6) and (7) we can define

(14) $G_q(\mu) = \displaystyle\int_0^\infty e^{-\mu\tau} g_q(\tau)\,d\tau$ for every $\mu > 0$ and $q \in \{0, 1, \ldots\}$.

It follows easily that

(15) the functions G_q are infinitely differentiable on R^+ for all $q \in \{0, 1, \ldots\}$.

Now we proceed to the decisive step of the proof.

According to (A_1) and (A_2), the hypotheses of Lemma [1] 4.15 are fulfilled for the function G and consequently, (5), (14) and (15) imply

(16) $G_q^{(p)}(\mu) \xrightarrow[q \to \infty]{} G^{(p)}(\mu)$ for any $\mu > 0$ and $p \in \{0, 1, \ldots\}$.

This result enables us to construct a function g whose Laplace transform is G. Indeed, by A.3 from the Appendix we obtain from (6), (7), (13) and (14) that

$$\left\| \frac{(-1)^p}{p!} \left(\frac{p+1}{t}\right)^{p+1} G_q^{(p)}\left(\frac{p+1}{t}\right) - g_q(t) \right\| \leqq \frac{(M_1 + \omega M_0)t}{\sqrt[4]{(p+1)}} + \frac{2M_0}{\sqrt{(p+1)}}$$

for every $t \in R^+$ and $p, q \in \{0, 1, \ldots\}$ which implies

(17)
$$\left\| \frac{(-1)^{p_1}}{p_1!} \left(\frac{p_1+1}{t}\right)^{p_1+1} G_q^{(p_1)}\left(\frac{p_1+1}{t}\right) - \right.$$
$$\left. - \frac{(-1)^{p_2}}{p_2!} \left(\frac{p_2+1}{t}\right)^{p_2+1} G_q^{(p_2)}\left(\frac{p_2+1}{t}\right) \right\| \leqq$$
$$\leqq (M_1 + \omega M_0)\, t \left(\frac{1}{\sqrt[4]{(p_1+1)}} + \frac{1}{\sqrt[4]{(p_2+1)}}\right) + 2M_0 \left(\frac{1}{\sqrt{(p_1+1)}} + \right.$$
$$\left. + \frac{1}{\sqrt{(p_2+1)}}\right) \text{ for every } t \in R^+ \text{ and } p_1, p_2, q \in 0, 1, \ldots .$$

It follows from (16) and (17) ($q \to \infty$) that

$$(18)\quad \left\| \frac{(-1)^{p_1}}{p_1!}\left(\frac{p_1+1}{t}\right)^{p_1+1} G^{(p_1)}\left(\frac{p_1+1}{t}\right) - \right.$$

$$\left. - \frac{(-1)^{p_2}}{p_2!}\left(\frac{p_2+1}{t}\right)^{p_2+1} G^{(p_2)}\left(\frac{p_2+1}{t}\right)\right\| \leqq$$

$$\leqq (M_1 + \omega M_0)\, t\left(\frac{1}{\sqrt{(p_1+1)}} + \frac{1}{\sqrt[4]{(p_2+1)}}\right) + 2M_0\left(\frac{1}{\sqrt{(p_1+1)}} + \right.$$

$$\left. + \frac{1}{\sqrt{(p_2+1)}}\right) \quad \text{for every} \quad t \in R^+ \quad \text{and} \quad p_1, p_2 \in \{0, 1, \ldots\}.$$

In view of (5), we can write (18) in the form

$$(19)\quad \|g_{p_1}(t) - g_{p_2}(t)\| \leqq (M_1 + \omega M_0)\, t\left(\frac{1}{\sqrt[4]{(p_1+1)}} + \frac{1}{\sqrt[4]{(p_2+1)}}\right) +$$

$$+ 2M_0\left(\frac{1}{\sqrt{(p_1+1)}} + \frac{1}{\sqrt{(p_2+1)}}\right) \quad \text{for every} \quad t \in R^+ \quad \text{and}$$

$$p_1, p_2 \in \{0, 1, \ldots\}.$$

By (19) we can write

$$(20)\quad g(t) = \lim_{p\to\infty} g_p(t) \quad \text{for} \quad t \in R^+.$$

It follows from (7) and (20) that

$$(21)\quad \|g(t)\| \leqq M_0 \quad \text{for every} \quad t \in R^+.$$

Further, (19) and (20) give

$$(22)\quad \|g_p(t) - g(t)\| \leqq \frac{(M_1 + \omega M_0)\, t}{\sqrt[4]{(p+1)}} + \frac{2M_0}{\sqrt{(p+1)}} \quad \text{for every}$$

$$t \in R^+ \quad \text{and} \quad p \in \{0, 1, \ldots\}.$$

It follows from (6) and (22) that

(23) the function g is continuous on R^+.

Finally by (6), (7), (22) and (23) we conclude that

$$(24)\quad \int_0^\infty e^{-\mu\tau} g_p(\tau)\, d\tau \to_{p\to\infty} \int_0^\infty e^{-\mu\tau} g(\tau)\, d\tau \quad \text{for every} \quad \mu > 0.$$

On the other hand, (14) and (16) give

$$(25)\quad \int_0^\infty e^{\mu\tau} g_p(\tau)\, d\tau \to_{p\to\infty} G(\mu) \quad \text{for every} \quad \mu > 0.$$

Thus, it follows from (24) and (25) that

$$(26)\quad G(\mu) = \int_0^\infty e^{-\mu\tau} g(\tau)\,\mathrm{d}\tau \quad \text{for every} \quad \mu > 0\,.$$

The desired function f will be now defined by

$$(27)\quad f(t) = e^{\omega t} g(t) \quad \text{for any} \quad t \in R^+\,.$$

Our final task in this part of the proof is to verify the properties (B_1), (B_2), (B_3) for the function f defined by (27).

First by (21) and (23)

$$(28)\quad \|f(t)\| \leqq M_0 e^{\omega t} \quad \text{for every} \quad t \in R^+\,,$$

(29) the function f is continuous on R^+.

Further by (1) and (26)

$$(30)\quad \int_0^\infty e^{-\lambda\tau} f(\tau)\,\mathrm{d}\tau \rightleftharpoons F(\lambda) \quad \text{for every} \quad \lambda > \omega\,.$$

Now we shall prove

$$(31)\quad \|f(t_1) - f(t_2)\| \leqq M_1 \int_{t_1}^{t_2} e^{\omega\tau}\,\mathrm{d}\tau \quad \text{for every} \quad t_1, t_2 \in R^+\,, \quad t_1 < t_2\,.$$

To this aim, let us define

$$(32)\quad f_p(t) = \frac{(-1)^p}{p!}\left(\frac{p+1}{t}\right)^{p+1} F^{(p)}\left(\frac{p+1}{t}\right) \quad \text{for every} \quad p \in \{0, 1, \ldots\} \quad \text{and}$$
$$0 < t < (p+1)/(\omega+1)\,.$$

Using 4.4 and 4.10 from [1] we obtain by (28)–(30) and (32) that

$$(33)\quad f_p(t) \to_{p\to\infty} f(t) \quad \text{for every} \quad t \in R^+\,.$$

On the other hand, by (A_1),

(34) the functions f_p are differentiable on $(0, (p+1)/(\omega+1))$ for every $p \in \{0, 1, \ldots\}$,

$$(35)\quad f_p'(t) = \frac{(-1)^{p+1}}{p!}(p+1)\frac{p+1}{t^2}\left(\frac{p+1}{t}\right)^p F^{(p)}\left(\frac{p+1}{t}\right) +$$
$$+ \frac{(-1)^{p+1}}{p!}\left(\frac{p+1}{t}\right)^{p+1}\frac{p+1}{t^2}F^{(p+1)}\left(\frac{p+1}{t}\right) =$$
$$= \frac{(-1)^{p+1}}{(p+1)!}\left(\frac{p+1}{t}\right)^{p+2}\left[(p+1)F^{(p)}\left(\frac{p+1}{t}\right) + \frac{p+1}{t}F^{(p+1)}\left(\frac{p+1}{t}\right)\right]$$

for every $p \in \{0, 1, \ldots\}$ and $0 < t < (p+1)/(\omega+1)$.

For the sake of brevity, we denote

(36) $J(\lambda) = \lambda F(\lambda)$ for $\lambda > \omega$.

It is clear that

(37) $J^{(p+1)}(\lambda) = (p + 1) F^{(p)}(\lambda) + \lambda F^{(p+1)}(\lambda)$ for every $\lambda > \omega$ and $p \in \{0, 1, \ldots\}$.

Now by (35)–(37) we have

(38) $$f_p'(t) = \frac{(-1)^{p+1}}{(p+1)!}\left(\frac{p+1}{t}\right)^{p+2} J^{(p+1)}\left(\frac{p+1}{t}\right) \quad \text{for every} \quad p \in \{0, 1, \ldots\}$$

and $0 < t < (p + 1)/(\omega + 1)$.

On the other hand, by (A_3) and (36)

(39) $$\|J^{(p+1)}(\lambda)\| \leqq \frac{M_1(p+1)!}{(\lambda-\omega)^{p+2}} \quad \text{for every} \quad \lambda > \omega \quad \text{and} \quad p \in \{0, 1, \ldots\}.$$

It follows from (38) and (39) that

$$\|f_p'(t)\| \leqq M_1 \left(\frac{1}{1 - \dfrac{\omega t}{p+1}}\right)^{p+2}$$

for every $p \in \{0, 1, \ldots\}$ and $0 < t < (p + 1)/(\omega + 1)$ which implies

(40) $$\|f_p(t_1) - f_p(t_2)\| \leqq M_1 \int_{t_1}^{t_2} \frac{1}{\left(1 - \dfrac{\omega\tau}{p+1}\right)^{p+2}}\, d\tau \quad \text{for every} \quad p \in \{0, 1, \ldots\}$$

and $0 < t_1 < t_2 < (p + 1)/(\omega + 1)$.

Using Lemma 3 we get from (40) that

(41) $$\|f_p(t_1) - f_p(t_2)\| \leqq M_1 \frac{1}{1 - \dfrac{\omega t_2}{p+1}} \int_{t_1}^{2t} e^{(\sqrt{(p+1)}/(\sqrt{(p+1)}-1))\omega\tau}\, d\tau \quad \text{for every}$$

$p \in \{0, 1, \ldots\}$ and $0 < t_1 < t_2 < (p + 1)/(\omega + 1)$.

Letting $p \to \infty$ in (41) and using (33), we obtain at once (31).

Since the properties (B_1), (B_2), (B_3) of the function f are contained in (28), (30) and (31), the proof of the "only if" part is complete.

"If" part. Let f be a fixed function with the properties (B_1), (B_2), (B_3).

It follows from (B_2) that

(1) the function f is continuous on R^+.

Now we obtain easily from (1) and from (B_1) and (B_3) that

(2) the properties (A_1), (A_2) hold.

To prove (A_3) let us first denote $f_h(t) = \frac{1}{h}\int_t^{t+h} f(\tau)\,d\tau$ for any $h > 0$ and $t > 0$.

It follows from (B_1) that

$$(3) \quad \|f_h(t)\| \leqq M_0 \frac{1}{h}\int_t^{t+h} e^{\omega\tau}\,d\tau = M_0 e^{\omega t}\frac{1}{h}\int_0^h e^{\omega\tau}\,d\tau \quad \text{for any} \quad h > 0 \quad \text{and} \quad t > 0\,.$$

Further we see easily from (1) that

(4) the function f_h is continuous for any $h > 0$,

(5) $f_h(t) \to f(t)$ $(h \to 0_+)$ for any $t \in R^+$.

On the other hand, by (B_2) we have

$$(6) \quad \left\|\frac{1}{h}(f(t+h) - f(t))\right\| \leqq M_1 \frac{1}{h}\int_t^{t+h} e^{\omega\tau}\,d\tau = M_1 e^{\omega t}\frac{1}{h}\int_0^h e^{\omega\tau}\,d\tau$$

for any $h > 0$ and $t > 0$.

Moreover, a simple calculation shows

$$(7) \quad f_h(t) = \frac{1}{h}\int_0^t (f(\tau + h) - f(\tau))\,d\tau + \frac{1}{h}\int_0^h f(\tau)\,d\tau \quad \text{for any} \quad h > 0 \quad \text{and} \quad t > 0\,.$$

Let us now write $F_h(\lambda) = \int_0^\infty e^{-\lambda\tau} f_h(\tau)\,d\tau$ for $h > 0$ and $\lambda > \omega$, which is admissible thanks to (3) and (4).

We get easily from (3)–(5) that

(8) $F_h^{(p)}(\lambda) \to F^{(p)}(\lambda)$ $(h \to 0_+)$ for any $\lambda > \omega$ and $p \in \{0, 1, \ldots\}$.

On the other hand, if follows from (7) that

$$F_h(\lambda) = \frac{1}{\lambda}\int_0^\infty e^{-\lambda\tau}\left[\frac{1}{h}(f(\tau + h) - f(\tau))\right]d\tau + \frac{1}{\lambda}\frac{1}{h}\int_0^h f(\tau)\,d\tau\,,$$

i.e.

$$(9) \quad \lambda F_h(\lambda) = \int_0^\infty e^{-\lambda\tau}\left[\frac{1}{h}(f(\tau + h) - f(\tau))\right]d\tau + \frac{1}{h}\int_0^h f(\tau)\,d\tau$$

for every $h > 0$ and $\lambda > \omega$.

It follows from (6) and (9) that

(10) $$\left\|\frac{d^p}{d\lambda^p}\lambda F_h(\lambda)\right\| \leqq M_1 \frac{p!}{(\lambda-\omega)^{p+1}}\frac{1}{h}\int_0^h e^{\omega\tau}\,d\tau \quad \text{for every} \quad h > 0\,, \quad \lambda > \omega$$

and $p \in \{1, 2, \ldots\}$.

Using (8) and (10) we see immediately that

(11) the property (A_3) holds.

By (2) and (11), the proof of the "if" part is complete.

5. Remark. We have here the opportunity to correct a mistake in Proposition 4.9 of [1] which is true for $\omega = 0$, but generally ω must be replaced by 2ω. The same is true in Proposition 1.4 in [1] which was used in the proof of 4.9. In the Appendix to this note we shall give a modified and improved version of the above mentioned Proposition 4.9 from [1].

APPENDIX

The aim of this Appendix is to examine the so called inversion problem for the Laplace transform of exponentially bounded functions.

A.1. Lemma. *For every $t \in R^+$ and $p \in \{0, 1, \ldots\}$, we have*

$$\frac{1}{p!}\left(\frac{p+1}{t}\right)^{p+1}\int_0^\infty e^{-((p+1)/t)\tau}\tau^p\,d\tau = 1\,.$$

Proof. Cf. [1], Proposition 4.6.

A.2. Lemma. *For every $t \in R^+$, $\chi \in R$ and $p \in \{0, 1, \ldots\}$ such that $p + 1 > 2\chi t$, the following inequality holds:*

$$\frac{1}{p!}\left(\frac{p+1}{t}\right)^{p+1}\left[\int_0^{t-t/\sqrt[4]{(p+1)}} e^{-((p+1)/t-\chi)\tau}\tau^p\,d\tau + \right.$$

$$\left. + \int_{t+t/\sqrt[4]{(p+1)}}^\infty e^{-((p+1)/t-\chi)\tau}\tau^p\,d\tau\right] \leqq \frac{1}{\sqrt{(p+1)}}\,e^{7\chi t}\,.$$

Proof. Let $t \in R^+$, $\chi \in R$ and $p \in \{0, 1, \ldots\}$ be fixed so that $p + 1 > 2\chi t$. Let us recall that clearly

(1) $$\frac{\chi t}{p+1} < \frac{1}{2}\,,$$

(2) $$\frac{\sqrt{(p+1)}}{t^2}(\tau - t)^2 \geqq 1 \quad \text{for every} \quad \tau \in R \quad \text{such that} \quad |\tau - t| > t/\sqrt[4]{(p+1)}\,.$$

Using (1) and (2) we get

$$\frac{1}{p!}\left(\frac{p+1}{t}\right)^{p+1}\left[\int_0^{t-t/\sqrt[4]{(p+1)}} e^{-((p+1)/t-\chi)\tau}\tau^p\, d\tau + \right.$$

$$\left. + \int_{t+t/\sqrt[4]{(p+1)}}^{\infty} e^{-((p+1)/t-\chi)\tau}\tau^p\, d\tau\right] \leqq$$

$$\leqq \frac{1}{p!}\left(\frac{p+1}{t}\right)^{p+1}\frac{\sqrt{(p+1)}}{t^2}\left[\int_0^{t-t/\sqrt[4]{(p+1)}} e^{-((p+1)/t-\chi)\tau}\tau^p(\tau-t)^2\, d\tau + \right.$$

$$\left. + \int_{t+t/\sqrt[4]{(p+1)}}^{\infty} e^{-((p+1))/t-\chi)\tau}\tau^p(\tau-t)^2\, d\tau\right] \leqq$$

$$\leqq \frac{1}{p!}\left(\frac{p+1}{t}\right)^{p+1}\frac{\sqrt{(p+1)}}{t^2}\int_0^{\infty} e^{-((p+1)/t-\chi)\tau}\tau^p(\tau-t)^2\, d\tau =$$

$$= \frac{1}{p!}\left(\frac{p+1}{t}\right)^{p+1}\frac{\sqrt{(p+1)}}{t^2}\left[\int_0^{\infty} e^{-((p+1)t-\chi)\tau}\tau^{p+2}\, d\tau - \right.$$

$$\left. - 2t\int_0^{\infty} e^{-((p+1)/t-\chi)\tau}\tau^{p+1}\, d\tau + t^2\int_0^{\infty} e^{-((p+1)/t-\chi)\tau}\tau^p\, d\tau\right] =$$

$$= \frac{1}{p!}\left(\frac{p+1}{t}\right)^{p+1}\frac{\sqrt{(p+1)}}{t^2}\left[(p+2)!\left(\frac{t}{p+1-\chi t}\right)^{p+3} - \right.$$

$$\left. - 2t(p+1)!\left(\frac{t}{p+1-\chi t}\right)^{p+2} + t^2 p!\left(\frac{t}{p+1-\chi t}\right)^{p+1}\right] =$$

$$= \frac{(p+1)^{p+1}\sqrt{(p+1)}}{p!}\left[\frac{(p+2)!}{(p+1-\chi t)^{p+3}} - \frac{2(p+1)!}{(p+1-\chi t)^{p+2}} + \frac{p!}{(p+1-\chi t)^{p+1}}\right]$$

$$\sqrt{(p+1)}\left(\frac{p+1}{p+1-\chi t}\right)^{p+1}\left[\frac{(p+1)(p+2)}{(p+1-\chi t)^2} - \frac{2(p+1)}{p+1-\chi t} + 1\right] =$$

$$= \sqrt{(p+1)}\left(\frac{p+1}{p+1-\chi t}\right)^{p+1} \times$$

$$\times \frac{(p+1)(p+2) - 2(p+1)(p+1-\chi t) + (p+1-\chi t)^2}{(p+1-\chi t)^2} \times$$

$$\times \sqrt{(p+1)}\left(\frac{p+1}{p+1-\chi t}\right)^{p+1}\frac{p+1+(\chi t)^2}{(p+1-\chi t)^2} =$$

$$= \sqrt{(p+1)}\frac{1}{(p+1)^2}\left(\frac{p+1}{p+1-\chi t}\right)^{p+3}(p+1+(\chi t)^2) =$$

$$= \frac{1}{\sqrt{(p+1)}}\left(\frac{p+1}{p+1-\chi t}\right)^{p+3}\left(1+\frac{(\chi t)^2}{p+1}\right) =$$

$$= \frac{1}{\sqrt{(p+1)}}\left(\frac{1}{1-\dfrac{\chi t}{p+1}}\right)^{p+3}(1+\chi t) =$$

$$= \frac{1}{\sqrt{(p+1)}}\left(1+\frac{\dfrac{\chi t}{p+1}}{1-\dfrac{\chi t}{p+1}}\right)^{p+3} e^{\chi t} = \frac{1}{\sqrt{(p+1)}}\left(1+2\,\frac{\chi t}{p+1}\right)^{p+3} e^{\chi t} =$$

$$= \frac{1}{\sqrt{(p+1)}}\left(e^{(2\chi t/(p+1))}\right)^{p+3} e^{\chi t} \leqq \frac{1}{\sqrt{(p+1)}}\, e^{7\chi t}\,.$$

A.3. Proposition. *Let $f \in R^+ \to E$ and let ω be a nonnegative constant. If*

(α) *the function f is continuous on R^+,*

(β) *the function $e^{-\omega t} f(t)$ is bounded on R^+,*
then for every $t \in R^+$ and $p \in \{0, 1, \ldots\}$ such that $p + 1 > 2\omega t$, the following inequality holds:

$$\left\| \frac{1}{p!}\left(\frac{p+1}{t}\right)^{p+1} \int_0^\infty e^{-((p+1)/t)\tau} \tau^p f(\tau)\, d\tau - f(t) \right\| \leqq$$

$$\leqq \sup_{|\tau - t| < t/\sqrt[4]{(p+1)}} (\|f(\tau) - f(t)\|) + \frac{1}{\sqrt{(p+1)}} \left[\|f(t)\| + e^{7\omega t} \sup_{t\in R^+} (e^{-\omega t}\|f(t)\|)\right].$$

Proof. Let us denote for the sake of simplicity

(1) $M = \sup_{t\in R^+} (e^{-\omega t}\|f(t)\|)$,

(2) $Z_{t,p} = \left(t - \dfrac{t}{\sqrt[4]{(p+1)}},\ t + \dfrac{t}{\sqrt[4]{(p+1)}}\right)$ for every $t \in R^+$ and $p \in \{0, 1, \ldots\}$.

By use of the preceding Lemma A.1 and Lemma A.2 with $\chi = 0$ and $\chi = \omega$ we get, with regard to (1) and (2),

(3) $$\left\| \frac{1}{p!}\left(\frac{p+1}{t}\right)^{p+1} \int_0^\infty e^{-((p+1)/t)\tau} \tau^p f(\tau)\, d\tau - f(t) \right\| =$$

$$= \left\| \frac{1}{p!}\left(\frac{p+1}{t}\right)^{p+1} \int_0^\infty e^{-((p+1)/t)\tau} \tau^p (f(\tau) - f(t))\, d\tau \right\| \leqq$$