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EMBEDDING TREES INTO CLIQUE-BRIDGE-CLIQUE GRAPHS

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The paper [2] concerns embedding trees into graphs which have exactly two blocks, each of them being a clique. Now we shall study a similar problem – embedding trees into graphs which consist of two vertex-disjoint cliques and of a bridge between them. Such a graph will be called a clique-bridge-clique graph (shortly *CBC*-graph).

Let n and k be two positive integers, $2 \leqq k \leqq [\frac{1}{2}n]$. By $H_n(k)$ we denote the *CBC*-graph in which one of the mentioned cliques has k and the other $n - k$ vertices. We shall investigate the conditions for a tree with n vertices to be embeddable into $H_n(k)$.

We shall use some concepts from [2]. A median of a tree T with n vertices is a vertex a of T at which the vertex deviation $m_1(a)$ attains its minimum. The vertex deviation is defined by

$$m_1(a) = \frac{1}{n} \sum_{x \in V} d(a, x),$$

where V is the vertex set of T and $d(a, x)$ denotes the distance between a and x in T. A tree has either exactly one median, or exactly two medians which are joined by an edge.

We recall also the definition of a branch. Let a be a vertex of a tree T. We define a binary relation E on the set of vertices of T which are distinct from a such that $(x, y) \in E$ if and only if the vertex a does not separate x from y in T (this means that the path connecting x and y in T does not contain a). The relation E is evidently an equivalence. The subtree of T induced by the union of one class of E with the one-element set $\{a\}$ is called a branch of T with the knag a.

Theorem 1. *Let n be an even positive integer, $n \geqq 4$. A tree T with n vertices can be embedded into $H_n(\frac{1}{2}n)$ if and only if it has two medians.*

Prpof. The weight of a vertex v of a tree T is defined in [1] as the maximal number of edges of a branch with the knag v. In [3] it is proved that a vertex of a tree has the minimal weight if and only if it is a median of this tree. Let T be a tree with n vertices and two medians. By Theorem 3 from [2] it can be embedded into the graph $G_n(\frac{1}{2}n)$ consisting of two blocks which are both cliques, one of them has $\frac{1}{2}n$, the other $\frac{1}{2}n + 1$ vertices. Let the former be B_1, the latter B_2. Let a be the vertex of T which is mapped onto the cut vertex of $G_n(\frac{1}{2}n)$ in this embedding. The weight of a is evidently at most $\frac{1}{2}n$, the weight of any vertex mapped onto a vertex of B_1 which is not a cut vertex is greater than $\frac{1}{2}n$, because there exists a branch with this vertex as a knag which contains all vertices which are embedded into B_2. Thus a is a median of T and the other median a' of T is mapped onto a vertex of B_2. The vertex a cannot be joined in T with other vertex embedded into B_2 than a'. If we delete from $G_n(\frac{1}{2}n)$ all edges joining a with vertices of B_2 except for the edge aa', we obtain the graph $H_n(\frac{1}{2}n)$ and T is embedded into $H_n(\frac{1}{2}n)$. On the other hand, let a tree T be embedded into $H_n(\frac{1}{2}n)$, let a and a' be the vertices of T which are mapped onto the end vertices of the bridge of $H_n(\frac{1}{2}n)$ in this embedding. Then evidently the weights of a and a' are both equal to $\frac{1}{2}n$ and the weights of all other vertices are greater. Therefore a and a' are medians of T.

Theorem 2. *Let T be a tree with $n \geqq 4$ vertices. The tree T can be embedded into $H_n([\frac{1}{2}n])$ if and only if the weight of its median is $[\frac{1}{2}(n + 1)]$.*

Proof. First we shall prove necessity of the condition. If n is even, then $[\frac{1}{2}(n + 1)] = [\frac{1}{2}n] = \frac{1}{2}n$. By Theorem 1 the tree T can be embedded into $H_n(\frac{1}{2}n)$ if and only if it has two medians. Thus let T have two medians a and a'. Let B (or B') be the branch of T with the knag a (or a') which contains a' (or a, respectively). If B has less than $w(a)$ edges (where $w(a)$ denotes the weight of a), then there exists a branch with the knag a other than B which has $w(a)$ edges. It is a proper subtree of B', therefore B' has more than $w(a)$ edges and a' is not a median, which is a contradiction. Therefore B has $w(a)$ edges and analogously B' has $w(a') = w(a)$ edges. The branches B and B' have exactly one common edge aa' and their union is the whole tree T, therefore $n - 1 = 2\,w(a) - 1$ and $w(a) = \frac{1}{2}n$. We have proved necessity of the condition for n even. Now let n be odd. Then $[\frac{1}{2}(n + 1)] = \frac{1}{2}(n + 1)$, $[\frac{1}{2}n] = \frac{1}{2}(n - 1)$. Suppose that the weight $w(a)$ of a median a of T is greater than $\frac{1}{2}(n + 1)$. Let b be the vertex adjacent to a and belonging to the branch with the knag a which has $w(a)$ edges. The branch with the knag b which contains a has $n - w(a)$ edges, the sum of numbers of edges of other branches with the knag b is $w(a) - 1$. Thus $w(b) \leqq \min(w(a) - 1, n - w(a))$. We have $w(a) - 1 > \frac{1}{2}(n + 1) - 1 = \frac{1}{2}(n - 1)$, $n - w(a) < n - \frac{1}{2}(n + 1) = \frac{1}{2}(n - 1)$, therefore $w(b) \leqq \frac{1}{2}(n - 1) < w(a)$ and this is a contradiction with the assumption that a is a median of T. Therefore $w(a) \leqq \frac{1}{2}(n + 1)$. Now let v be a vertex of T which is not a median of T; let again a be a median of T. Let B (or B') be the branch of T with the knag v (or a) which contains a (or v, respectively). If there is a branch with the knag v with $w(v)$ edges other than B, then B' contains all this branch and, moreover, the path connecting a and v, thus it has more than $w(v)$ edges and $w(a) > w(v)$, which is a contradiction. Thus B has $w(v)$ edges. Suppose that $w(v) < \frac{1}{2}(n + 1)$. The sum of numbers of edges of branches with the knag a other than B' is less than $w(v)$, therefore B' has at least $n - w(v)$ edges and $w(a) \geqq n - w(v)$, which implies $w(v) \geqq n - w(a)$. As $w(a) \leqq \frac{1}{2}(n + 1)$, we have $w(v) \geqq n - \frac{1}{2}(n + 1) = \frac{1}{2}(n - 1)$. We have proved that $w(v)$ can be less than $\frac{1}{2}(n - 1)$ only if v is a median of T. Let T be embedded into $H_n(\frac{1}{2}(n - 1))$. Let B_1 be the clique of $H_n(\frac{1}{2}(n - 1))$ with $\frac{1}{2}(n - 1)$ vertices, let u be the vertex of T which is mapped onto the end vertex of the bridge of $H_n(\frac{1}{2}(n - 1))$ belonging to B_1. The vertices of T which are mapped onto vertices of $H_n(\frac{1}{2}(n - 1))$ not belonging to B_1 together with u form a branch of T with the knag u. This branch has $\frac{1}{2}(n + 1)$ edges, thus $w(u) = \frac{1}{2}(n + 1)$ and u is a median of T.

Now we shall prove sufficiency of the condition. Let $w(a) = [\frac{1}{2}(n + 1)]$ for a median a of T. Then evidently T can be embedded into $H_n([\frac{1}{2}n])$ so that a is mapped onto the end vertex of the bridge belonging to the clique with $[\frac{1}{2}n]$ vertices and all vertices of the branch with the knag a having $w(a)$ edges except for a are mapped onto the vertices of the other clique.

Theorem 3. *Let T be a tree with $n \geqq 4$ vertices, let T contain a subtree T' which is a snake and one terminal vertex of which is a median of T. Let T' have $[\frac{1}{2}n]$ vertices. Then T can be embedded into $H_n(k)$ for all $k = 2, \ldots, [\frac{1}{2}n]$.*

Remark. A snake is a tree which consists of vertices and edges of one simple path.

Proof. Let the vertices of the snake T' be $u_0, \ldots, u_m$, where $m = [\frac{1}{2}n]$. Let u_m be the median of T. Then for each $k = 2, \ldots, [\frac{1}{2}n]$ we can embed T into $H_n(k)$ so that the end vertices of the bridge coincide with the vertices u_k and u_{k+1}.

Theorem 4. *Let $n \geqq 4$ be a positive integer, let K be a subset of the number set $\{2, \ldots, [\frac{1}{2}n]\}$. Then there exists a tree with n vertices which can be embedded into $H_n(k)$ for each $k \in K$ and cannot be embedded into $H_n(k)$ for $k \notin K$.*

Proof. We shall use the concept of caterpillar (introduced by F. HARARY). A caterpillar is a tree with the property that after deleting all terminal vertices from it a snake is obtained. This snake is called the body of the caterpillar [4]. If the vertices of the body are $u_0, \ldots, u_m$ and the edges u_iu_{i+1} for $i = 0, 1, \ldots, m - 1$, then we denote by α_i the number of terminal vertices of the caterpillar which are adjacent to u_i for $i = 0, 1, \ldots, m$. Thus we assign a vector $[\alpha_0, \ldots, \alpha_m]$ to the caterpillar. For $K \neq \emptyset$ the required tree is a caterpillar with the vector $[\alpha_0, \ldots, \alpha_m]$ which is described as follows. Let $K = \{k_1, \ldots, k_q\}$ and let $k_i < k_j$ for $i < j$. Then $m = 2q - 1$, $\alpha_0 = k_1$, $\alpha_i = k_{i+1} - k_i$ for $i = 1, \ldots, q - 1$. Further $\alpha_{m-i} = \alpha_i$ for $i = 0, 1, \ldots, q - 1$. The caterpillar C can be embedded into $H_n(k_i)$ so that onto the end vertices of the bridge of $H_n(k_i)$ the vertices u_{i-1}, u_i or the vertices u_{m-i+1}, u_{m-i} are mapped. On the other hand, the unique edges of C which can be mapped onto