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QUASI-ORDERS OF ALGEBRAS

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In this paper the set $\mathcal{Q}(\mathfrak{A})$ of all quasi-orders of an arbitrary partial algebra $\mathfrak{A} = (A, F)$ is studied, in particular, properties of this set provided $\mathfrak{A}$ is a group are shown.

In the first section it is proved that $\mathcal{Q}(\mathfrak{A})$ ordered by inclusion is an algebraic lattice and its compact elements are described. The methods and the results of Schmidt's book [2] are essentially used here. In the second section the lattice $\mathcal{Q}(\mathfrak{G})$ for an arbitrary group $\mathfrak{G} = (G, +)$ is characterized by means of the set $\mathcal{P}(\mathfrak{G})$ of all invariant subsemigroups with 0 of G. $\mathcal{P}(\mathfrak{G})$ ordered by inclusion is a lattice isomorphic to $\mathcal{Q}(\mathfrak{G})$. Constructions of the lattice operations in both of these lattices are shown and it is proved that, in general, this lattices are not modular.

BASIC CONCEPTS AND NOTATIONS

Let $A \neq \emptyset$ be a set, n a positive integer, R an n-ary relation on A. A mapping $f : R \to A$ is called an *n-ary partial operation on A*. In this case let us write also $R = D(f, A)$. The arity of f is denoted by n_f. If $D(f, A) = A^n$, then we call f an *n-ary operation on A*.

A partial algebra $\mathfrak{A}$ is an ordered pair (A, F), where $A \neq \emptyset$ is a set and F is a family of finitary partial operations on A. If each $f \in F$ is an operation on A, then $\mathfrak{A}$ is called an *algebra*.

If $\mathfrak{A} = (A, F)$ is a partial algebra, then the elements of F are called *fundamental operations on* $\mathfrak{A}$. Let i, n be positive integers, $i \leqq n$. Then $e^{i,n}$ denotes the i-th n-ary projection on A, i.e. the operation on A such that for each $a_1, \ldots, a_n \in A$ it is $a_1 \ldots \ldots a_n e^{i,n} = a_i$. Let $F^* = F \cup \{e^{i,n};\ i, n \in N,\ i \leqq n\}$. Let $X \neq \emptyset$ be a set and let $w = w(x_1, \ldots, x_m)$ be a word generated by F^* on X. Let $a_1, \ldots, a_k$ $(k \leqq m)$ be elements of A, $1 \leqq i_1, \ldots, i_k \leqq m$, and let us substitute the elements $a_1, \ldots, a_k$ for $x_{i_1}, \ldots, x_{i_k}$. Then we obtain an $(n - k)$-ary partial operation on A that we denote by $w(\ldots, a_1, \ldots, a_k, \ldots)$. This partial operation is called an *algebraic function on* $\mathfrak{A}$ *induced by* w. If $w \in F^*$, then each unary algebraic function induced by w will be called an *elementary translation on* $\mathfrak{A}$. Each product of elementary translations on $\mathfrak{A}$ is called a *translation on* $\mathfrak{A}$.

1. THE LATTICE OF ALL QUASI-ORDERS OF A PARTIAL ALGEBRA

Let $A \neq \emptyset$ be a set and let Q be a binary relation on A. Q is a quasi-order of A if it is reflexive and transitive. An antisymmetric quasi-order of A is called *an order of A*. A quasi-ordered set (qo-set) is a pair (A, Q), where $A \neq \emptyset$ is a set and Q is a quasi-order of A. Similarly an ordered set (po-set).

For any binary relation R, aRb will denote $(a, b) \in R$. Let $\mathfrak{A} = (A, F)$ be a partial algebra and let Q be a quasi-order of the set A. Then Q is called a *quasi-order of the partial algebra* $\mathfrak{A}$ if it satisfies the property (C):

(C) If $f \in F$, both $a_1 \dots a_{n_f} f$ and $b_1 \dots b_{n_f} f$ are defined and $a_i Q b_i$ $(a_i, b_i \in A, i = 1, \dots, n_f)$, then $a_1 \dots a_{n_f} f Q b_1 \dots b_{n_f} f$. A quasi-order Q of $\mathfrak{A}$ is called *strong* if, whenever $a_i Q b_i$ $(a_i, b_i \in A, i = 1, \dots, n_f)$ and $a_1 \dots a_{n_f} f (b_1 \dots b_{n_f} f)$ exists, then also $b_1 \dots b_{n_f} f (a_1 \dots a_{n_f} f)$ exists and $a_1 \dots a_{n_f} f Q b_1 \dots b_{n_f} f$.

For a partial algebra $\mathfrak{A} = (A, F)$, let us introduce the following notation:

$\mathcal{Q}_0(A)$ denotes the set of all quasi-orders of the set A,

$\mathcal{Q}(\mathfrak{A})$ denotes the set of all quasi-orders of $\mathfrak{A}$,

$\mathcal{Q}_s(\mathfrak{A})$ denotes the set of all strong quasi-orders of $\mathfrak{A}$.

We consider the sets $\mathcal{Q}_0(A)$, $\mathcal{Q}(\mathfrak{A})$ and $\mathcal{Q}_s(\mathfrak{A})$ ordered by inclusion. It is clear that $\mathcal{Q}_0(A)$ is a complete lattice in which the infimum of each system of elements is formed by its intersection and the supremum by its transitive hull. $A \times A$ is the greatest element, $\Delta_A = \{(a, a); a \in A\}$ is the smallest element in $\mathcal{Q}_0(A)$. In the paper $\cup$ and $\cap$ denote the set-theoretical intersection and union, respectively, $\vee$ and $\wedge$ denote the lattice operations sup and inf, respectively.

Lemma 1.1. *Let* $\mathfrak{A} = (A, F)$ *be a partial algebra,* $Q_\alpha \in \mathcal{Q}(\mathfrak{A})$ $(\alpha \in I)$. *Then* $\bigcap_{\alpha \in I} Q_\alpha \in \mathcal{Q}(\mathfrak{A})$.

Proof. It is $\bigcap_{\alpha \in I} Q_\alpha \in \mathcal{Q}_0(A)$. Let $f \in F$ and let $a_i (\bigcap_{\alpha \in I} Q_\alpha) b_i$ $(i = 1, \dots, n_f)$. Then $a_i Q_\alpha b_i$ for all $\alpha \in I$ and thus if $a_1 \dots a_{n_f} f$, $b_1 \dots b_{n_f} f$ are defined it follows that $a_1 \dots a_{n_f} f Q_\alpha b_1 \dots b_{n_f} f$ for all $\alpha \in I$. This means $a_1 \dots a_{n_f} f (\bigcap_{\alpha \in I} Q_\alpha) b_1 \dots b_{n_f} f$.

Corollary 1.1.1. *For a partial algebra* $\mathfrak{A} = (A, F)$, $\mathcal{Q}(\mathfrak{A})$ *is a complete lattice that is a closed* $\wedge$*-subsemilattice of the lattice* $\mathcal{Q}_0(A)$. *The lattices* $\mathcal{Q}(\mathfrak{A})$ *and* $\mathcal{Q}_0(A)$ *have the same greatest and smallest elements.*

Lemma 1.2. *If* Q_α $(\alpha \in I)$ *are strong quasi-orders of a partial algebra* $\mathfrak{A} = (A, F)$, *then the transitive hull of the system* $\{Q_\alpha; \alpha \in I\}$ *is also a strong quasi-order of* $\mathfrak{A}$.

Proof. Let us denote the transitive hull of $\{Q_\alpha; \alpha \in I\}$ by Q. It is $Q \in \mathcal{Q}_0(A)$. Let $f \in F$, $a_i Q b_i$ $(a_i, b_i \in A, i = 1, \dots, n_f)$ and let $a_1 \dots a_{n_f} f$ be defined. Then there exists a sequence

$$a_i = z_1^i, z_2^i, \dots, z_{k_i}^i = b_i$$

of elements of A such that

$$z^i_{j-1} Q^i_{\alpha_j} z^i_j\,, \quad j = 2, \ldots, k_i\,, \quad Q^i_{\alpha_j} \in \{Q_\alpha;\, \alpha \in I\}\,.$$

From the reflexivity of quasi-orders it follows that we can suppose

$$k_1 = k_2 = \ldots = k_{n_f} \quad \text{and} \quad Q^1_{\alpha_j} = Q^2_{\alpha_j} = \ldots = Q^{n_f}_{\alpha_j} = Q_{\alpha_j}\,.$$

Then

$$a_1 Q_{\alpha_1} z^1_2, \ldots, a_{n_f} Q_{\alpha_1} z^{n_f}_2\,.$$

If $a_1 \ldots a_{n_f} f$ exists, then there also exists $z^1_2 \ldots z^{n_f}_2 f$ and it is $a_1 \ldots a_{n_f} f Q_{\alpha_1} z^1_2 \ldots z^{n_f}_2 f$. Similarly we obtain $z^1_2 \ldots z^{n_f}_2 f Q_{\alpha_2} z^1_3 \ldots z^{n_f}_3 f$, etc. Therefore $a_1 \ldots a_{n_f} f Q b_1 \ldots b_{n_f} f$. Analogously for the case that $b_1 \ldots b_{n_f}$ exists.

Corollary 1.2.1. *If $\mathfrak{A} = (A, F)$ is a partial algebra, then $\mathcal{Q}_s(\mathfrak{A})$ is a principal ideal in $\mathcal{Q}(\mathfrak{A})$ that is a closed complete sublattice of $\mathcal{Q}_0(A)$.*

Corollary 1.2.2. *If $\mathfrak{A} = (A, F)$ is an algebra, then $\mathcal{Q}(\mathfrak{A})$ is a closed complete sublattice of $\mathcal{Q}_0(A)$.*

Lemma 1.3. *Let ϱ be a reflexive binary relation on a set $A \neq \emptyset$. Then $R = \bigcup_{n=1}^{\infty} \varrho^n$ is the smallest quasi-order of A that contains ϱ.*

Let $(A, \leqq)$ be a po-set. A family S of elements of A is called *directed* if each finite subset $\subseteq S$ has an upper bound in S.

Lemma 1.4. *Let $\{Q_\alpha;\, \alpha \in I\}$ be a directed family of quasi-orders of a partial algebra $\mathfrak{A} = (A, F)$. Then $\bigcup_{\alpha \in I} Q_\alpha = \bigvee_{\alpha \in I} Q_\alpha$ in $\mathcal{Q}_0(A)$ and $\bigcup_{\alpha \in I} Q_\alpha \in \mathcal{Q}(\mathfrak{A})$.*

Proof. It is $\bigcup_{\alpha \in I} Q_\alpha \subseteq \bigvee_{\alpha \in I} {}_{\mathcal{Q}_0(A)} Q_\alpha$.

Let $a(\bigvee_{\alpha \in I} {}_{\mathcal{Q}_0(A)} Q_\alpha)\, b$. Then there exists a sequence

$$a = z_0\,, \quad z_1, \ldots, z_n = b$$

of elements of A such that

$$z_{i-1} Q_{\alpha_i} z_i \; (i = 1, \ldots, n)\,, \quad Q_{\alpha_i} \in \{Q_\alpha;\, \alpha \in I\}\,.$$

Since $\{Q_\alpha;\, \alpha \in I\}$ is a directed family, there exists an element Q of this family such that $Q_{\alpha_i} \subseteq Q$ $(i = 1, \ldots, n)$. Therefore $z_{i-1} Q z_i$ $(i = 1, \ldots, n)$, and so aQb. This means that $a(\bigcup_{\alpha \in I} Q_\alpha)\, b$ and $\bigvee_{\alpha \in I} {}_{\mathcal{Q}_0(A)} Q_\alpha \subseteq \bigcup_{\alpha \in I} Q_\alpha$.

Let us show that $\bigvee_{\alpha \in I} {}_{\mathcal{Q}_0(A)} Q_\alpha \in \mathcal{Q}(\mathfrak{A})$. Let $f \in F$, $a_i(\bigvee_{\alpha \in I} {}_{\mathcal{Q}_0(A)} Q_\alpha)\, b_i$ $(a_i, b_i \in A,\ i = 1, \ldots, n_f)$, and let $a_1 \ldots a_{n_f} f$ and $b_1 \ldots b_{n_f} f$ exist. Then for each $i = 1, \ldots, n_f$ there exists a sequence

$$a_i = z^i_0,\ z^i_1, \ldots, z^i_{k_i} = b_i$$

of elements of A such that $z_j^i Q_{i_j} z_{j+1}^i$, $Q_{i_j} \in \{Q_\alpha; \alpha \in I\}$. Since the family $\{Q_\alpha; \alpha \in I\}$ is directed, there exists $Q \in \{Q_\alpha; \alpha \in I\}$ for which $Q_{i_j} \subseteq Q$ $(i = 1, \ldots, n_f,\ j = 1, \ldots, k_i)$. Then $z_j^i Q z_{j+1}^i$, and so $a_i Q b_i$. By condition (C) we obtain $a_1 \ldots a_{n_f} f Q b_1 \ldots b_{n_f} f$, therefore also $a_1 \ldots a_{n_f} f (\bigvee_{\alpha \in I} {}_{\mathcal{Q}_0(A)} Q_\alpha)\, b_1 \ldots b_{n_f} f$.

A complete lattice L is called *algebraic* if each element of L is the supremum of a set of compact elements.

Lemma 1.5. *Let $A \neq \emptyset$ be a set. Then the lattice $\mathcal{Q}_0(A)$ is algebraic.*

Proof. It is known that the lattice $\mathcal{R}_0(A)$ of all reflexive relations on the set $A \neq \emptyset$ is algebraic. The infimum (the supremum) in $\mathcal{R}_0(A)$ is formed by the intersection (by the union). The smallest element in $\mathcal{R}_0(A)$ is Δ_A, the greatest element is $A \times A$. It is clear that $\mathcal{Q}_0(A)$ is a closed $\wedge$-subsemilattice of $\mathcal{R}_0(A)$. By the proof of Lemma 1.4, every directed family $\{R_\alpha; \alpha \in I\}$ of elements of $\mathcal{Q}_0(A)$ fulfils $\bigvee_{\alpha \in I} {}_{\mathcal{Q}_0(A)} R_\alpha = \bigcup_{\alpha \in I} R_\alpha$, thus $\bigvee_{\alpha \in I} {}_{\mathcal{R}_0(A)} R_\alpha \in \mathcal{Q}_0(A)$. Δ_A, $A \times A \in \mathcal{Q}_0(A)$, therefore by [2, Folgerung 4.7] $\mathcal{Q}_0(A)$ is an algebraic lattice.

Let $(A, \leqq)$ be a po-set. A closure operator in A is a function $\lambda : A \to A$ such that for each $a, b \in A$

(i) $a \leqq a\lambda$;

(ii) $a \leqq b$ implies $a\lambda \leqq b\lambda$;

(iii) $(a\lambda)\,\lambda = a\lambda$;

(iv) if A contains the smallest element 0, then $0\lambda = 0$.

Let L be an algebraic lattice. A closure operator in L is called *algebraic* if it holds for each compact element $a \in L$: If $a \leqq x\lambda$, then there exists a compact element $x' \leqq x$ such that $a \leqq x'\lambda$.

Let $\mathfrak{A} = (A, F)$ be a partial algebra and let $R \subseteq A \times A$. Since $A \times A \in \mathcal{Q}(\mathfrak{A})$, then by Lemma 1.1 there exists a smallest quasi-order Q_R of $\mathfrak{A}$ that contains R. It is clear that a function $\lambda : \mathcal{Q}_0(A) \to \mathcal{Q}_0(A)$ such that $R\lambda = Q_R$ for each $R \in \mathcal{Q}_0(A)$ is a closure operator in $\mathcal{Q}_0(A)$.

Theorem 1.6. *λ is an algebraic operator.*

Proof. By Lemma 1.5, $\mathcal{Q}_0(A)$ is an algebraic lattice. Then from Lemma 1.4 and [2, Lemma 4.7] it follows that λ is algebraic.

Corollary 1.6.1. *$\mathcal{Q}(\mathfrak{A})$ is an algebraic lattice.*

Proof. The lattice $\mathcal{Q}_0(A)$ and the operator λ are algebraic, thus the assertion follows from [2, Lemma 4.2].

Corollary 1.6.2. *The lattice* $\mathcal{Q}_s(\mathfrak{A})$ *is algebraic.*

Proof follows from the fact that $\mathcal{Q}_s(\mathfrak{A})$ is a principal ideal in $\mathcal{Q}(\mathfrak{A})$.

Lemma 1.7. *Let* $\mathfrak{A} = (A, F)$ *be a partial algebra and let* R, R_α $(\alpha \in I)$ *be binary relations on* A *such that* $R = \bigcup_{\alpha \in I} R_\alpha$. *Then* $Q_R = \bigvee_{\alpha \in I}{}_{\mathcal{Q}(\mathfrak{A})} Q_{R_\alpha}$.

Proof. It is $R_\alpha \subseteq R$, thus $\bigvee_{\alpha \in I} Q_{R_\alpha} \subseteq Q_R$. If $Q \in \mathcal{Q}(\mathfrak{A})$, $Q \supseteq \bigvee_{\alpha \in I} Q_{R_\alpha}$, then $Q \supseteq R_\alpha$ for each $\alpha \in I$ and then also $Q \supseteq \bigcup_{\alpha \in I} R_\alpha$. This implies $Q = Q_Q \supseteq Q_R$. Therefore $\bigvee_{\alpha \in I} Q_{R_\alpha} \supseteq Q_R$, i.e. $Q_R = \bigvee_{\alpha \in I} Q_{R_\alpha}$.

For $a, b \in A$ we denote $Q_{\{(a,b)\}}$ by $Q_{a,b}$.

Corollary 1.7.1. *If* $R \subseteq A \times A$, *then* $Q_R = \bigvee_{(a,b) \in R} Q_{a,b}$.

Let now $\mathfrak{A} = (A, F)$ be a partial algebra and let R be a binary relation on A. Then

R^T denotes the transitive hull of R, i.e. $R^T = \bigcup_{n=1}^{\infty} R^n$;

R^F denotes the set of all $(u, v) \in A \times A$ such that for an appropriate algebraic function $x_1 \dots x_n p$ there exist $(a_i, b_i) \in R$ $(i = 1, \dots, n)$ such that $u = a_1 \dots a_n p$, $v = b_1 \dots b_n p$;

R^U denotes the set of all $(u, v) \in A \times A$ such that for an appropriate unary algebraic function p there exists $(a, b) \in R$ such that $u = ap$, $v = bp$;

$R^{U'}$ denotes the set of all $(u, v) \in A \times A$ such that for an appropriate translation p there exists $(a, b) \in R$ such that $u = ap$, $v = bp$.

It is clear that T, F, U, U' are closure operators in the complete lattice $\exp(A \times A)$.

Let us denote

$$R_0 = R,\ R_1 = R_0^F,\ R_2 = R_1^T,\ R_3 = R_2^F, \dots, R_{2i} = R_{2i-1}^T,\ R_{2i+1} = R_{2i}^F, \dots .$$

It holds $R_0 \subseteq R_1 \subseteq \dots$. Let us denote $\bar{R} = \bigcup_{i=1}^{\infty} R_i$ for $R \neq \emptyset$ and $\bar{\emptyset} = \Delta_A$. It is clear that $\bar{R}^T = \bar{R}^F = \bar{R}$.

Theorem 1.8. *Let* $\mathfrak{A} = (A, F)$ *be a partial algebra and let* $R \subseteq A \times A$. *Then* $Q_R = \bar{R}$.

Proof. It holds $R \subseteq \bar{R} \subseteq Q_R$. Let us show that $\bar{R} \in \mathcal{Q}(\mathfrak{A})$. Let $c \in A$, $(x_1, x_2) \in R$ and let us consider the algebraic function $xp = cxe^{1,2}$. Then $(c, c) \in R^F$ and therefore $(c, c) \in \bar{R}$. This means $\bar{R}$ is reflexive. Further $R_{2i-1}R_{2i-1} \subseteq R_{2i}$, thus $\bar{R}\bar{R} \subseteq \bar{R}$. Hence $\bar{R}$ is transitive.

Let now $f \in F$, $a_1 \bar{R} b_1, \dots, a_{n_f} \bar{R} b_{n_f}$ and let us assume that $a_1 \dots a_{n_f} f$, $b_1 \dots b_{n_f} f$ exist. Then there exists i such that $(a_j, b_j) \in R_{2i}$ $(j = 1, \dots, n_f)$ and so $a_1 \dots$ $\dots a_{n_f} f R_{2i+1} b_1 \dots b_{n_f} f$. Therefore $\bar{R}$ satisfies the condition (C).

Theorem 1.9. *Let $\mathfrak{A} = (A, F)$ be an algebra, $R \subseteq A \times A$. Then $(R^U)^T = (R^F)^T$, $(R^U)^T = ((R^U)^T)^U$.*

Proof. Since $R^U \subseteq R^F$, then $(R^U)^T \subseteq (R^F)^T$. Let $(c, d) \in (R^F)^T$. Then there exists a sequence

$$c = z_0 , \quad z_1, \ldots, z_n = d$$

of elements of A such that $(z_{i-1}, z_i) \in R^F$ $(i = 1, \ldots, n)$. This means that for an appropriate algebraic function $x_1 \ldots x_k p$ it holds $z_{i-1} = a_1 \ldots a_k p$, $z_i = b_1 \ldots b_k p$, where $(a_j, b_j) \in R$ $(j = 1, \ldots, k)$.

Let us introduce the following unary functions:

$$xP_1 = xa_2a_3 \ldots a_k p, \; xP_2 = b_1 x a_3 \ldots a_k p, \ldots, xP_k = b_1 b_2 \ldots b_{k-1} x p .$$

It is $a_1P_1 = z_{i-1}$, $b_jP_j = a_{j+1}P_{j+1}$, $b_kP_k = z_i$ $(j = 1, \ldots, k - 1)$, i.e. $(z_{i-1}, z_i) \in \in (R^U)^T$. Thus $(R^U)^T = (R^F)^T$.

Let $(c, d) \in ((R^U)^T)^U$. Thus there exist $(a_1, b_1), \ldots, (a_n, b_n) \in R$ such that for appropriate unary algebraic functions $p_1, p_2, \ldots, p_n, q$ it holds

$$c' = a_1 p_1, \; b_1 p_1 = a_2 p_2, \; b_2 p_2 = a_3 p_3, \ldots, b_n p_n = d'$$

and

$$c = c'q , \quad d = d'q .$$

Let $P_i = p_i q$. Then

$$a_1 P_1 = c, \; b_j P_j = a_{j+1} P_{j+1}, \; b_n P_n = d \quad (j = 1, \ldots, n - 1) .$$

Therefore $(c, d) \in (R^U)^T$, and so $(R^U)^T = ((R^U)^T)^U$.

Theorem 1.10. *Let $\mathfrak{A} = (A, F)$ be an algebra and let R be a binary relation on A. Then $Q_R = (R^U)^T$ (i.e. for $c, d \in A$ it holds $cQ_R d$ if and only if there exist $c = = z_0, \ldots, z_n = d \in A$, $(a_i, b_i) \in R$ $(i = 1, \ldots, n)$, and unary algebraic functions $p_1, \ldots, p_n$ such that $a_i p_i = z_{i-1}$, $b_i p_i = z_i$ for $i = 1, \ldots, n$).*

Proof. The assertion follows immediately from Theorems 1.8 and 1.9.

Corollary 1.10.1. *Let $\mathfrak{A} = (A, F)$ be an algebra, $a, b, x, y \in A$. Then $xQ_{a,b}y$ if and only if there exist a sequence $x = z_0, z_1, \ldots, z_n = y$ of elements of A and a sequence of unary algebraic functions $p_0, p_1, \ldots, p_{n-1}$ on F such that $z_i = ap_i$, $z_{i+1} = bp_i$ $(i = 1, \ldots, n - 1)$.*

Theorem 1.11. *Let $\mathfrak{A} = (A, F)$ be an algebra, $a, b, x, y \in A$. Then $xQ_{a,b}y$ if and only if there exist elements $x = z_0, z_1, \ldots, z_n = y$ of A and translations $p_0, \ldots, p_{n-1}$ such that $z_i = ap_i$, $z_{i+1} = bp_i$ $(i = 1, \ldots, n - 1)$.*

Proof. Let us show that $(R^{U'})^T = (R^U)^T$. If $(u, v) \in R^U$, then there exist $(a, b) \in R$ and an appropriate unary algebraic function p such that $u = ap$, $v = bp$. Therefore, translations $t_1, \ldots, t_n$ and a word w of A such that $w(t_1, \ldots, t_n) = p$ must exist. Thus

$$xF_i = w(bt_1, \ldots, bt_{i-1}, xt_i, at_{i+1}, \ldots, at_n)$$

is a translation such that

$$bF_i = aF_{i+1} \ (i = 1, \ldots, n-1), \quad aF_1 = ap = u, \quad bF_n = bp = v,$$

i.e. $(u, v) \in (R^{U'})^T$. Therefore $R^U \subseteq (R^{U'})^T$ and so $(R^U)^T \subseteq (R^{U'})^T$. Finally, since $R^{U'} \subseteq R^U$, it holds $(R^U)^T = (R^{U'})^T$.

Now we shall describe the set $\mathcal{Q}(\mathfrak{A})^*$ of all compact elements in the lattice $\mathcal{Q}(\mathfrak{A})$ of a partial algebra $\mathfrak{A} = (A, F)$.

Theorem 1.12. *Let Q be a quasi-order of a partial algebra $\mathfrak{A} = (A, F)$. Then $Q \in \mathcal{Q}(\mathfrak{A})^*$ if and only if there exists a finite binary relation R on A such that $Q = Q_R$.*

Proof. Let $Q \in \mathcal{Q}(\mathfrak{A})$. Then $\Delta_A \subseteq Q$. For $R \subseteq A \times A$ it is $R \subseteq Q_R$ and thus $R \cup \Delta_A \subseteq Q_R$. Therefore $Q_{R \cup \Delta_A} \subseteq Q_R$, and so $Q_{R \cup \Delta_A} = Q_R$.

By Lemma 1.6, the closure operator $R\lambda = Q_R$ on the lattice $\mathcal{R}_0(A)$ of all reflexive relations on A is algebraic. Thus, by [2, Lemma 4.3], $R' \in \mathcal{Q}(\mathfrak{A})$ is compact in $\mathcal{Q}(\mathfrak{A})$ if and only if $R'' = R' \cup \Delta_A$ is a compact element in $\mathcal{R}_0(A)$. But this is satisfied (by [2, p. 33]) if and only if there exists a finite relation $R \subseteq A \times A$ such that $R' \cup \Delta_A = R \cup \Delta_A$.

Theorem 1.13. *Let $\mathfrak{A} = (A, F)$ be a partial algebra. Then the lattice of all ideals in $\mathcal{Q}(\mathfrak{A})^*$ is isomorphic to $\mathcal{Q}(\mathfrak{A})$.*

Proof follows from [2, proof of Lemma 3.9].

2. THE LATTICE OF ALL QUASI-ORDERS OF A GROUP

Let $\mathfrak{G} = (G, +)$ be a group, $R \in \mathcal{Q}(\mathfrak{G})$. Then the pair $\mathfrak{G}, R$ is called a *quasi-ordered group* (qo-group). This qo-group will be denoted by $\mathfrak{G} = (G, +, R) = (G, R)$. Let us denote $P_R = \{x \in G; 0Rx\}$, where 0 is the zero-element of the group $(G, +)$. P_R is called the *positive cone of the qo-group* (G, R).

For a system $R_\alpha \in \mathcal{Q}(\mathfrak{G})$ $(\alpha \in A)$, we shall often denote the corresponding positive cones by P_α instead of P_{R_α} $(\alpha \in A)$.

Lemma 2.1. *Let $\mathfrak{G} = (G, R)$ be a qo-group. Then P_R is an invariant subsemigroup with 0 of $\mathfrak{G}$.*

Lemma 2.2. *Let S be an invariant subsemigroup with 0 of a group $\mathfrak{G} = (G, +)$. The the binary relation R defined by*

$$aRb \quad \textit{iff} \quad -a + b \in S \quad (\textit{iff } b - a \in S) \quad \textit{for all} \quad a, b \in G$$

is a quasi-order of the group $\mathfrak{G}$.

Supplement. $S = P_R$.

Proof. If aRb, $x \in G$, then $-x - a + b + x \in S$, $-a - x + x + b \in S$, therefore $-(a + x) + (b + x) \in S$, $-(x + a) + (x + b) \in S$, and so $(a + x) R(b + x)$, $(x + a) R(x + b)$.

Proof of Supplement. 1. If $x \in S$, then $-0 + x \in S$. Thus $0Rx$, i.e. $x \in P_R$. 2. Let $y \in P_R$, i.e. $0Ry$. Therefore $-0 + y = y \in S$.

Let us denote by $\mathscr{P}(\mathfrak{G})$ the set of all invariant subsemigroups with 0 of G. It is clear that the correspondence $R \mapsto P_R$ (for each $R \in \mathscr{Q}(\mathfrak{G})$) is a one-to-one mapping between $\mathscr{Q}(\mathfrak{G})$ and $\mathscr{P}(\mathfrak{G})$.

Further, for $R_1, R_2 \in \mathscr{Q}(\mathfrak{G})$ it is $R_1 \subseteq R_2$ iff $P_{R_1} \subseteq P_{R_2}$. Therefore the ordered sets $(\mathscr{Q}(\mathfrak{G}), \subseteq)$ and $(\mathscr{P}(\mathfrak{G}), \subseteq)$ are isomorphic.

Theorem 2.3. *$\mathscr{P}(\mathfrak{G})$ ordered by inclusion is an algebraic lattice.*

Supplement. *Let $P_\alpha \in \mathscr{P}(\mathfrak{G})$, $\alpha \in A$. Then*

a) $\bigwedge_{\alpha \in A} P_\alpha = \bigcap_{\alpha \in A} P_\alpha$;

b) $\bigvee_{\alpha \in A} P_\alpha = \sum_{\alpha \in A} P_\alpha$;

in particular,

c) $P_{\alpha_1} \vee P_{\alpha_2} = P_{\alpha_1} + P_{\alpha_2} = P_{\alpha_2} + P_{\alpha_1}$.

Proof. Since $\mathscr{P}(\mathfrak{G})$ is isomorphic to $\mathscr{Q}(\mathfrak{G})$, $\mathscr{P}(\mathfrak{G})$ is (by Corollary 1.6.1) an algebraic lattice.

a) Let $P_\alpha \in \mathscr{P}(\mathfrak{G})$ $(\alpha \in A)$, $P = \bigcap_{\alpha \in A} P_\alpha$. It is evident that $P \in \mathscr{P}(\mathfrak{G})$.

b) It is clear that $\bar{P} = \sum_{\alpha \in A} P_\alpha$ is the smallest subsemigroup with 0 containing P_α $(\alpha \in A)$. Let us show that $\bar{P}$ is invariant. If $x = a_{\alpha_1} + a_{\alpha_2} + \ldots + a_{\alpha_n} \in \bar{P}$ $(a_{\alpha_i} \in P_{\alpha_i}$, $i = 1, 2, \ldots, n)$, $z \in G$, then

$$-z + x + z = (-z + a_{\alpha_1} + z) + (-z + a_{\alpha_2} + z) + \ldots + (-z + a_{\alpha_n} + z) \in \bar{P}.$$

c) If A is an invariant subsemigroup of $\mathfrak{G}$, then for each $z \in G$ it holds $-z + A + z \subseteq A$, thus $A + z \subseteq z + A$. Therefore also $A + (-z) \subseteq (-z) + A$, i.e. $z + A + (-z) \subseteq A$, then $z + A \subseteq A + z$, and so $A + z = z + A$. If now

$$x = a_1 + b_1 + a_2 + b_2 + \ldots + a_n + b_n$$
$$(a_i \in P_1,\ b_i \in P_2,\ i = 1, 2, \ldots, n),$$

then

$$x = (a_1 + a_2) + (b_1' + b_2) + a_3 + b_3 + \ldots + a_n + b_n =$$
$$= a_1' + b_2' + a_3 + b_3 + \ldots + a_n + b_n = \ldots = a + b\,,$$

where $a \in P_1$, $b \in P_2$.

Corollary 2.3.1. *For the infimum and the supremum in the algebraic lattice $\mathcal{Q}(\mathfrak{G})$ it holds: Let $R_\alpha \in \mathcal{Q}(\mathfrak{G})$ $(\alpha \in A)$. Then*

a) $\bigwedge_{\alpha \in A} R_\alpha = \bigcap_{\alpha \in A} R_\alpha$;

b) *if* $a(\bigvee_{\alpha \in A} R_\alpha)\, b$, *then for each* $i \in A$ *there exist* $x, x' \in \bigvee_{\alpha \in A} P_\alpha$ *such that* $(a + x)\,.$ $.\, R_i(b - x')$;

c) *if there exist* $x, x' \in \bigvee_{\alpha \in A} P_\alpha$ *and* $i \in A$ *such that* $(a + x)\, R_i(b - x')$, *then* $a(\bigvee_{\alpha \in A} R_\alpha)\, b$.

Proof. a) The assertion a) follows from Lemma 1.1.

b) Let us denote $R = \bigvee_{\alpha \in A} {}_{\mathcal{Q}(\mathfrak{G})} R_\alpha$, $P = \bigvee_{\alpha \in A} {}_{\mathcal{P}(\mathfrak{G})} P_\alpha$. Further, let aRb. Then $-a + b \in \in P$, thus $-a + b = x_{i_1} + \ldots + x_{i_r} + x_i + x_{j_s} + \ldots + x_{j_1}$, where $x_{i_m} \in P_{i_m}$, $x_{j_n} \in P_{j_n}$, $x_i \in P_i$, $i_1, \ldots, i_r$, $j_1, \ldots, j_s$, $i \in A$. (If in the partition there is no element of P_i, we can add $x_i = 0$.) Let us denote $x_{i_1} + \ldots + x_{i_r} = x$, $(-x_{j_1}) + \ldots + (-x_{j_s}) = -x'$. Then $-(a + x) + (b - x') \in P_i$, therefore $(a + x)\, R_i(b - x')$.

c) Let now $x, x' \in P$, $i \in A$, $(a + x)\, R_i(b - x')$. Then $-(a + x) + (b - x') = x_i$, $x_i \in P_i$, and so $-a + b = x + x_i + x'$. If $x = x_{i_1} + \ldots + x_{i_k}$, $x' = x_{j_1} + \ldots + x_{j_l}$, then $-a + b = x_{i_1} + \ldots + x_{i_k} + x_i + x_{j_1} + \ldots + x_{j_l}$. This means $-a + b \in P$, and thus aRb.

Theorem 2.4. *The set $\mathcal{P}_1(\mathfrak{G})$ of all invariant subsemigroups P with 0 of a group G such that $P \cap -P = \{0\}$ is a closed $\wedge$-subsemilattice of the lattice $\mathcal{P}(\mathfrak{G})$.*

Proof. In $\mathcal{P}_1(\mathfrak{G})$ it holds

$$\bigcap_{\alpha \in A} P_\alpha \cap - \bigcap_{\beta \in A} P_\beta = \bigcap_{\alpha, \beta \in A} (P_\alpha \cap -P_\beta) = \{0\}\,,$$

thus $\bigwedge_{\alpha \in A} {}_{\mathcal{P}(\mathfrak{G})} P_\alpha \in \mathcal{P}_1(\mathfrak{G})$.

Corollary 2.4.1. *The set $\mathcal{Q}_1(\mathfrak{G})$ of all orders of a group $\mathfrak{G}$ is a closed $\wedge$-subsemilattice of the lattice $\mathcal{Q}(\mathfrak{G})$.*

Theorem 2.5. *Let $\mathcal{Q}_d(\mathfrak{G})$ be the set of all directed orders of a group $\mathfrak{G}$ and let $\mathcal{Q}_d(\mathfrak{G}) \neq \emptyset$. Then the following conditions are equivalent:*

(a) $\mathfrak{G} = \{0\}$.

(b) *$\mathcal{Q}_d(\mathfrak{G})$ is a sublattice of the lattice $\mathcal{Q}(\mathfrak{G})$.*

(c) *$\mathcal{Q}_d(\mathfrak{G})$ is an $\wedge$-subsemilattice of the lattice $\mathcal{Q}(\mathfrak{G})$.*

(d) *$\mathcal{Q}_d(\mathfrak{G})$ is a $\vee$-subsemilattice of the lattice $\mathcal{Q}(\mathfrak{G})$.*

Proof. (c) $\Rightarrow$ (a): Let $R \in \mathcal{Q}_d(\mathfrak{G})$ and let P be the positive cone of R. Then $-P$ is the positive cone of the dual order of the group $\mathfrak{G}$ and $P \cap -P = \{0\}$. Thus $\{0\}$ is the positive cone of a directed order of $\mathfrak{G}$, and so $\mathfrak{G} = \{0\}$.

(d) $\Rightarrow$ (a): If P is the positive cone of a directed order of $\mathfrak{G}$, then

$$P \vee -P = P + (-P) = P - P = G \quad \text{and} \quad G \cap -G = G\,.$$

Therefore $\mathfrak{G} = \{0\}$.

(a) $\Rightarrow$ (b) $\Rightarrow$ (c) and (a) $\Rightarrow$ (d) are evident.

Similarly, we have

Theorem 2.6. *Let $\mathcal{Q}_1(\mathfrak{G})$ be the set of all lattice orders of a group $\mathfrak{G}$ and let $\mathcal{Q}_1(\mathfrak{G}) \neq \emptyset$. Then the following conditions are equivalent:*

(a) $\mathfrak{G} = \{0\}$.

(b) *$\mathcal{Q}_l(\mathfrak{G})$ is a sublattice of the lattice $\mathcal{Q}(\mathfrak{G})$.*

(c) *$\mathcal{Q}_l(\mathfrak{G})$ is an $\wedge$-subsemilattice of the lattice $\mathcal{Q}(\mathfrak{G})$.*

(d) *$\mathcal{Q}_l(\mathfrak{G})$ is a $\vee$-subsemilattice of the lattice $\mathcal{Q}(\mathfrak{G})$.*

Theorem 2.7. a) *If R is a directed order of a group $\mathfrak{G}$, then R has complements in the lattices $\mathcal{Q}(\mathfrak{G})$ and $\mathcal{Q}_0(G)$.*

b) *If R is an order of a group $\mathfrak{G}$, then its dual order is complement of R in $\mathcal{Q}(\mathfrak{G})$ (in $\mathcal{Q}_0(G)$) if and only if R is directed.*

Proof. Part a) is a consequence of part b).

b) Let us denote the positive cone of R by P. Then

$$P \cap -P = \{0\}\,, \quad P \bigvee\nolimits_{\mathscr{P}(\mathfrak{G})} -P = P + (-P) = P - P\,,$$

and $P - P = G$ if and only if R is directed. Thus, in this case, the dual order is a complement of R in $\mathcal{Q}(\mathfrak{G})$ and, by Corollary 1.2.2, in $\mathcal{Q}_0(G)$ as well.

Note. If $\mathfrak{G} \neq \{0\}$ is a group and if $R \in \mathcal{Q}_1(\mathfrak{G})$ has a complement in $\mathcal{Q}(\mathfrak{G})$, then there need not exist an element of $\mathcal{Q}_1(\mathfrak{G})$ among complements of R. Namely, if we can order $\mathfrak{G}$ only trivially, then $\{0\} \cap G = \{0\}$, $\{0\} + G = G$, thus G is a complement of $\{0\}$ in $\mathscr{P}(\mathfrak{G})$ and there exists no complement of $\{0\}$ that belongs to $\mathscr{P}_1(\mathfrak{G})$.

Theorem 2.8. *In general, the lattice $\mathcal{Q}(\mathfrak{G})$ is not modular.*